Orthogonality

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Orthogonality

Definition 1 (Orthogonal Vectors)
Two vectors \( \mathbf{u}, \mathbf{v} \) are said to be orthogonal provided their dot product is zero:

\[
\mathbf{u} \cdot \mathbf{v} = 0.
\]

If both vectors are nonzero (not required in the definition), then the angle \( \theta \) between the two vectors is determined by

\[
\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = 0,
\]

which implies \( \theta = 90^\circ \). In short, orthogonal vectors form a right angle.
Unitization

Any nonzero vector $\mathbf{u}$ can be multiplied by $c = \frac{1}{\|\mathbf{u}\|}$ to make a unit vector $\mathbf{v} = c\mathbf{u}$, that is, a vector satisfying $\|\mathbf{v}\| = 1$.

This process of changing the length of a vector to 1 by scalar multiplication is called \textit{unitization}. 
Definition 2 (Orthogonal Set of Vectors)
A given set of nonzero vectors \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) that satisfies the orthogonality condition
\[
\mathbf{u}_i \cdot \mathbf{u}_j = 0, \quad i \neq j,
\]
is called an orthogonal set.

Definition 3 (Orthonormal Set of Vectors)
A given set of unit vectors \( \mathbf{u}_1, \ldots, \mathbf{u}_k \) that satisfies the orthogonality condition is called an orthonormal set.
Theorem 1 (Independence)
An orthogonal set of nonzero vectors is linearly independent.

Proof: Let $c_1, \ldots, c_k$ be constants such that nonzero orthogonal vectors $u_1, \ldots, u_k$ satisfy the relation

$$c_1 u_1 + \cdots + c_k u_k = 0.$$ 

Take the dot product of this equation with vector $u_j$ to obtain the scalar relation

$$c_1 u_1 \cdot u_j + \cdots + c_k u_k \cdot u_j = 0.$$ 

Because all terms on the left are zero, except one, the relation reduces to the simpler equation

$$c_j \|u_j\|^2 = 0.$$ 

This equation implies $c_j = 0$. Therefore, $c_1 = \cdots = c_k = 0$ and the vectors are proved independent.
Inner Product Spaces

An inner product on a vector space $V$ is a function that maps a pair of vectors $u, v$ into a scalar $\langle u, v \rangle$ satisfying the following four properties.

1. $\langle u, v \rangle = \langle v, u \rangle$ [symmetry]
2. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ [additivity]
3. $\langle cu, v \rangle = c \langle u, v \rangle$ [homogeneity]
4. $\langle u, u \rangle \geq 0$, $\langle u, u \rangle = 0$ if and only if $u = 0$ [positivity]

The length of a vector is then defined to be $\|u\| = \sqrt{\langle u, u \rangle}$.

A vector space $V$ with inner product defined is called an inner product space.
Theorem 2 (Cauchy-Schwartz Inequality)
In any inner product space $V$,

$$|\langle u, v \rangle| \leq \|u\| \|v\|.$$

Equality holds if and only if $u$ and $v$ are linearly dependent.

Theorem 3 (Triangle Inequality)
In any inner product space $V$,

$$\|u + v\| \leq \|u\| + \|v\|.$$
Theorem 4 (Pythagorean Identity)
In any inner product space $V$,

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

if and only if $u$ and $v$ are orthogonal.