

## **Eigenanalysis**

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## What's Eigenanalysis?

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Matrix eigenanalysis is a computational theory for the matrix equation

$$\mathbf{y} = \mathbf{A}\mathbf{x}.$$

## Fourier's Eigenanalysis Model

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For exposition purposes, we assume  $\mathbf{A}$  is a  $3 \times 3$  matrix.

$$(1) \quad \begin{aligned} \mathbf{x} &= \mathbf{c}_1 \mathbf{v}_1 + \mathbf{c}_2 \mathbf{v}_2 + \mathbf{c}_3 \mathbf{v}_3 \text{ implies} \\ \mathbf{y} &= \mathbf{A}\mathbf{x} \\ &= \mathbf{c}_1 \lambda_1 \mathbf{v}_1 + \mathbf{c}_2 \lambda_2 \mathbf{v}_2 + \mathbf{c}_3 \lambda_3 \mathbf{v}_3. \end{aligned}$$

## Eigenanalysis Notation

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The scale factors  $\lambda_1, \lambda_2, \lambda_3$  and independent vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  depend only on  $\mathbf{A}$ . Symbols  $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$  stand for arbitrary numbers. This implies variable  $\mathbf{x}$  exhausts all possible 3-vectors in  $\mathbf{R}^3$ .

## Fourier's Model is a Replacement Process

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$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$

To compute  $A\mathbf{x}$  from  $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ , replace each vector  $\mathbf{v}_i$  by its scaled version  $\lambda_i\mathbf{v}_i$ .

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Fourier's model is said to **hold** provided there exist scale factors and independent vectors satisfying (1). Fourier's model is known to fail for certain matrices  $A$ .

## Powers and Fourier's Model

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Equation (1) applies to compute powers  $A^n$  of a matrix  $A$  using only the basic vector space toolkit. To illustrate, only the vector toolkit for  $\mathbb{R}^3$  is used in computing

$$A^5 \mathbf{x} = x_1 \lambda_1^5 \mathbf{v}_1 + x_2 \lambda_2^5 \mathbf{v}_2 + x_3 \lambda_3^5 \mathbf{v}_3.$$

This calculation does not depend upon finding previous powers  $A^2$ ,  $A^3$ ,  $A^4$  as would be the case by using matrix multiply.

## Differential Equations and Fourier's Model

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Systems of differential equations can be solved using Fourier's model, giving a compact and elegant formula for the general solution. An example:

$$\begin{aligned}x_1' &= x_1 + 3x_2, \\x_2' &= \quad \quad 2x_2 - x_3, \\x_3' &= \quad \quad \quad - 5x_3.\end{aligned}$$

The general solution is given by the formula [Fourier's theorem, proved later]

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = c_1 e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-5t} \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix},$$

which is related to Fourier's model by the symbolic formula

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + c_3 e^{\lambda_3 t} \mathbf{v}_3.$$

## Fourier's model illustrated

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Let

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -5 \end{pmatrix}$$
$$\lambda_1 = 1, \quad \lambda_2 = 2, \quad \lambda_3 = -5,$$
$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}.$$

Then Fourier's model holds (details later) and

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix} \quad \text{implies}$$
$$A\mathbf{x} = c_1(1) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2(2) \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + c_3(-5) \begin{pmatrix} 1 \\ -2 \\ -14 \end{pmatrix}$$

Eigenanalysis might be called *the method of simplifying coordinates*. The nomenclature is justified, because Fourier's model computes  $\mathbf{y} = A\mathbf{x}$  by scaling independent vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ , which is a triad or **coordinate system**.

## What is Eigenanalysis?

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The subject of **eigenanalysis** discovers a coordinate system  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  and scale factors  $\lambda_1, \lambda_2, \lambda_3$  such that Fourier's model holds. Fourier's model simplifies the matrix equation  $\mathbf{y} = \mathbf{A}\mathbf{x}$ , through the formula

$$\mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3.$$



## What's an Eigenvalue? \_\_\_\_\_

It is a **scale factor**. An eigenvalue is also called a *proper value* or a *hidden value* or a *characteristic value*. Symbols  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$  used in Fourier's model are eigenvalues.

The **eigenvalues** of a model are scale factors. Think of them as a system of units.

## What's an Eigenvector? \_\_\_\_\_

Symbols  $v_1$ ,  $v_2$ ,  $v_3$  in Fourier's model are called eigenvectors, or *proper vectors* or *hidden vectors* or *characteristic vectors*. They are assumed independent.

The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues).

## Data Conversion Example

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Let  $\mathbf{x}$  in  $\mathbf{R}^3$  be a data set variable with coordinates  $x_1$ ,  $x_2$ ,  $x_3$  recorded respectively in units of meters, millimeters and centimeters. We consider the problem of conversion of the mixed-unit  $\mathbf{x}$ -data into proper MKS units (meters-kilogram-second)  $\mathbf{y}$ -data via the equations

$$(2) \quad \begin{aligned} y_1 &= x_1, \\ y_2 &= 0.001x_2, \\ y_3 &= 0.01x_3. \end{aligned}$$

Equations (2) are a **model** for changing units. Scaling factors  $\lambda_1 = 1$ ,  $\lambda_2 = 0.001$ ,  $\lambda_3 = 0.01$  are the **eigenvalues** of the model.

## Data Conversion Example – Continued

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Problem (2) can be represented as  $\mathbf{y} = \mathbf{A}\mathbf{x}$ , where the diagonal matrix  $\mathbf{A}$  is given by

$$\mathbf{A} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad \lambda_1 = 1, \quad \lambda_2 = \frac{1}{1000}, \quad \lambda_3 = \frac{1}{100}.$$

Fourier's model for this matrix  $\mathbf{A}$  is

$$\mathbf{A} \left( c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = c_1 \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

## Summary

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The **eigenvalues** of a model are **scale factors**, normally represented by symbols

$$\lambda_1, \lambda_2, \lambda_3, \dots$$

The **eigenvectors** of a model are independent **directions of application** for the scale factors (eigenvalues). They are normally represented by symbols

$$\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$$

## History of Fourier's Model

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The subject of **eigenanalysis** was popularized by J. B. Fourier in his 1822 publication on the theory of heat, *Théorie analytique de la chaleur*. His ideas can be summarized as follows for the  $n \times n$  matrix equation  $\mathbf{y} = \mathbf{A}\mathbf{x}$ .

The vector  $\mathbf{y} = \mathbf{A}\mathbf{x}$  is obtained from eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  and eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  by replacing the eigenvectors by their scaled versions  $\lambda_1\mathbf{v}_1, \lambda_2\mathbf{v}_2, \dots, \lambda_n\mathbf{v}_n$ :

$$\begin{aligned}\mathbf{x} &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_n\mathbf{v}_n \quad \text{implies} \\ \mathbf{y} &= x_1\lambda_1\mathbf{v}_1 + x_2\lambda_2\mathbf{v}_2 + \dots + c_n\lambda_n\mathbf{v}_n.\end{aligned}$$

## Determining Equations

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The eigenvalues and eigenvectors are determined by homogeneous matrix–vector equations. In Fourier’s model

$$A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$$

choose  $c_1 = 1, c_2 = c_3 = 0$ . The equation reduces to  $A\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ . Similarly, taking  $c_1 = c_2 = 0, c_3 = 1$  implies  $A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ . Finally, taking  $c_1 = c_2 = 0, c_3 = 1$  implies  $A\mathbf{v}_3 = \lambda_3\mathbf{v}_3$ . This proves the following fundamental result.

### **Theorem 1 (Determining Equations in Fourier’s Model)**

Assume Fourier’s model holds. Then the eigenvalues and eigenvectors are determined by the three equations

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1,$$

$$A\mathbf{v}_2 = \lambda_2\mathbf{v}_2,$$

$$A\mathbf{v}_3 = \lambda_3\mathbf{v}_3.$$

## Determining Equations and Linear Algebra

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The three relations of the theorem can be distilled into one homogeneous matrix–vector equation

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Write it as  $A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0}$ , then replace  $\lambda\mathbf{x}$  by  $\lambda I\mathbf{x}$  to obtain the standard form<sup>a</sup>

$$(A - \lambda I)\mathbf{v} = \mathbf{0}, \quad \mathbf{v} \neq \mathbf{0}.$$

Let  $B = A - \lambda I$ . The equation  $B\mathbf{v} = \mathbf{0}$  has a nonzero solution  $\mathbf{v}$  if and only if there are infinitely many solutions. Because the matrix is square, infinitely many solutions occurs if and only if  $\text{rref}(B)$  has a row of zeros. Determinant theory gives a more concise statement:  $\det(B) = 0$  if and only if  $B\mathbf{v} = \mathbf{0}$  has infinitely many solutions. This proves the following result.

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<sup>a</sup>Identity  $I$  is required to factor out the matrix  $A - \lambda I$ . It is wrong to factor out  $A - \lambda$ , because  $A$  is  $3 \times 3$  and  $\lambda$  is  $1 \times 1$ , incompatible sizes for matrix addition.

## College Algebra and Eigenanalysis

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### **Theorem 2 (Characteristic Equation)**

If Fourier's model holds, then the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  are roots  $\lambda$  of the polynomial equation

$$\det(A - \lambda I) = 0.$$

The equation  $\det(A - \lambda I) = 0$  is called the **characteristic equation**. The **characteristic polynomial** is the polynomial on the left,  $\det(A - \lambda I)$ , normally obtained by cofactor expansion or the triangular rule.



## Eigenvectors and Frame Sequences

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### Theorem 3 (Finding Eigenvectors of $A$ )

For each root  $\lambda$  of the characteristic equation, write the frame sequence for  $B = A - \lambda I$  with last frame  $\text{rref}(B)$ , followed by solving for the general solution  $\mathbf{v}$  of the homogeneous equation  $B\mathbf{v} = \mathbf{0}$ . Solution  $\mathbf{v}$  uses invented parameter names  $t_1, t_2, \dots$ . The vector basis answers  $\partial_{t_1}\mathbf{v}, \partial_{t_2}\mathbf{v}, \dots$  are independent **eigenvectors** of  $A$  paired to eigenvalue  $\lambda$ .

**Proof:** The equation  $A\mathbf{v} = \lambda\mathbf{v}$  is equivalent to  $B\mathbf{v} = \mathbf{0}$ . Because  $\det(B) = 0$ , then this system has infinitely many solutions, which implies the frame sequence starting at  $B$  ends with  $\text{rref}(B)$  having at least one row of zeros. The general solution then has one or more free variables which are assigned invented symbols  $t_1, t_2, \dots$ , and then the vector basis is obtained by from the corresponding list of partial derivatives. Each basis element is a nonzero solution of  $A\mathbf{v} = \lambda\mathbf{v}$ . By construction, the basis elements (eigenvectors for  $\lambda$ ) are collectively independent. The proof is complete.

## Eigenpairs of a Matrix

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### Definition 1 (Eigenpair)

An **eigenpair** is an eigenvalue  $\lambda$  together with a matching eigenvector  $\mathbf{v} \neq \mathbf{0}$  satisfying the equation  $A\mathbf{v} = \lambda\mathbf{v}$ . The pairing implies that scale factor  $\lambda$  is applied to direction  $\mathbf{v}$ .

An applied view of an eigenpair is a coordinate axis  $\mathbf{v}$  and a unit system along this axis where 1 unit is the dimensionless value  $\lambda$ .

A  $3 \times 3$  matrix  $A$  for which Fourier's model holds has eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  and corresponding eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ . The **eigenpairs** of  $A$  are

$$(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), (\lambda_3, \mathbf{v}_3).$$

## Eigenvectors are Independent

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### **Theorem 4 (Independence of Eigenvectors)**

If  $(\lambda_1, \mathbf{v}_1)$  and  $(\lambda_2, \mathbf{v}_2)$  are two eigenpairs of  $A$  and  $\lambda_1 \neq \lambda_2$ , then  $\mathbf{v}_1, \mathbf{v}_2$  are independent.

More generally, if  $(\lambda_1, \mathbf{v}_1), \dots, (\lambda_k, \mathbf{v}_k)$  are eigenpairs of  $A$  corresponding to distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ , then  $\mathbf{v}_1, \dots, \mathbf{v}_k$  are independent.