Second Order Systems

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Coupled Spring-Mass Systems

Three masses are attached to each other by four springs as in Figure 1. A model will be developed for the positions of the three masses.

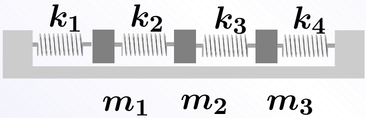

Figure 1. Three masses connected by springs. The masses slide along a frictionless horizontal surface.

Variables

The analysis uses the following constants, variables and assumptions.

Mass Constants The masses m_1, m_2, m_3 are assumed to be point masses concentrated at their center of gravity.

Spring Constants The mass of each spring is negligible. The springs operate according to Hooke's law: Force = k(elongation). Constants k_1, k_2, k_3, k_4 denote the Hooke's constants. The springs restore after compression and extension.

Position Variables The symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the mass positions along the horizontal surface, measured from their equilibrium positions, plus right and minus left.

Fixed Ends The first and last spring are attached to fixed walls.

Derivation

The **competition method** is used to derive the equations of motion. In this case, the law is

Newton's Second Law Force = Sum of the Hooke's Forces.

The model equations are

$$\begin{aligned} m_1 x_1''(t) &= -k_1 x_1(t) + k_2[x_2(t) - x_1(t)], \\ m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\ m_3 x_3''(t) &= -k_3[x_3(t) - x_2(t)] - k_4 x_3(t). \end{aligned} \tag{1}$$

- The equations are justified in the case of all positive variables by observing that the first three springs are elongated by x_1, $x_2 - x_1$, $x_3 - x_2$, respectively. The last spring is compressed by x_3, which accounts for the minus sign.
- Another way to justify the equations is through mirror-image symmetry: interchange $k_1 \longleftrightarrow k_4$, $k_2 \longleftrightarrow k_3$, $x_1 \longleftrightarrow x_3$, then equation 2 should be unchanged and equation 3 should become equation 1.

Vector-Matrix form $\mathbf{x}'' = A\mathbf{x}$

In vector-matrix form, this system is a **second order system**

$$M\mathbf{x}''(t) = K\mathbf{x}(t)$$

where the **displacement** $\mathbf{x}$, **mass matrix** M and **stiffness matrix** K are defined by the formulas

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

Because M is invertible, the system can always be re-written using $A = M^{-1}K$ as the second-order system

$$\mathbf{x}'' = A\mathbf{x}.$$