Contents

Definition (Separable Equation). An equation $y' = f(x, y)$ is called **separable** provided there exists functions $F(x)$ and $G(y)$ such that

$$f(x, y) = F(x)G(y).$$

Definition (Separated Form of a Separable Equation). This is the equation

$$y'/G(y) = F(x).$$

It is obtained from the separable equation $y' = F(x)G(y)$ by dividing by $G(y)$. Such an equation is said to be *prepared for quadrature*, because the LHS is independent of x and the RHS is independent of y.

Finding a Separable Form

Given differential equation $y' = f(x, y)$, invent values x_0, y_0 such that $f(x_0, y_0) \neq 0$. Define F, G by the formulas

$$F(x) = \frac{f(x, y_0)}{f(x_0, y_0)}, \quad G(y) = f(x_0, y). \tag{1}$$

Because $f(x_0, y_0) \neq 0$, then (1) makes sense. Test I *infra* implies the following test.

Theorem 1 (Separability Test)

Let F and G be defined by (1). Multiply FG. Then

(a) If $F(x)G(y) = f(x, y)$, then $y' = f(x, y)$ is **separable**.

(b) If $F(x)G(y) \neq f(x, y)$, then $y' = f(x, y)$ is **not separable**.

Invention and Application

Initially, let (x_0, y_0) be $(0, 0)$ or $(1, 1)$ or some suitable pair, for which $f(x_0, y_0) \neq 0$; then define F and G by (1). Multiply to test the equation $FG = f$.

The algebra will discover a factorization $f = F(x)G(y)$ without having to know algebraic tricks like factorizing multi-variable equations. But if $FG \neq f$, then the algebra *proves* the equation is not separable.

Non-Separability Tests

Test I. Equation $y' = f(x, y)$ is not separable provided for some pair of points (x_0, y_0), (x, y) in the domain of f, (2) holds:

$$f(x, y_0)f(x_0, y) - f(x_0, y_0)f(x, y) \neq 0. \tag{2}$$

Test II. The equation $y' = f(x, y)$ is not separable if either of the following conditions hold:

- $f_x(x, y)/f(x, y)$ is non-constant in y

  or

- $f_y(x, y)/f(x, y)$ is non-constant in x.

Test I details

Assume $f(x, y) = F(x)G(y)$, then equation (2) fails because each term on the left side of (2) equals $F(x)G(y_0)F(x_0)G(y)$ for all choices of (x_0, y_0) and (x, y) (hence contradiction $0 \neq 0$).

Test II details

Assume $f(x, y) = F(x)G(y)$ and suppose F, G are sufficiently differentiable. Then

- $\dfrac{f_x(x, y)}{f(x, y)} = \dfrac{F'(x)}{F(x)}$ is independent of y

  and
- $\dfrac{f_y(x, y)}{f(x, y)} = \dfrac{G'(y)}{G(y)}$ is independent of x.

Illustration

Consider $y' = xy + y^2$.

Test I implies it is not separable, because the left side of the relation is

$$\begin{aligned} \text{LHS} &= f(x,1)f(0,y) - f(0,1)f(x,y) \\ &= (x+1)y^2 - (xy + y^2) \\ &= x(y^2 - y) \\ &\neq 0. \end{aligned}$$

Test II implies it is not separable, because

$$\frac{f_x}{f} = \frac{1}{x+y}$$

is not constant as a function of y.

Variables-Separable Method

The method determines two kinds of solution formulas.

Equilibrium Solutions.

They are the constant solutions $y = c$ of $y' = f(x, y)$. For any equation, find them by substituting $y = c$ into the differential equation.

Non-Equilibrium Solutions.

For a separable equation

$$y' = F(x)G(y),$$

a non-equilibrium solution y is a solution with $G(y) \neq 0$. It is found by dividing by $G(y)$, then applying the method of quadrature.

Theory of Non-Equilibrium Solutions

A given solution $y(x)$ satisfying $G(y(x)) \neq 0$ throughout its domain of definition is called a non-equilibrium solution. Then division by $G(y(x))$ is allowed.

The *method of quadrature* applies to the separated equation $y'/G(y(x)) = F(x)$. Some details:

$\int_{x_0}^{x} \frac{y'(t)dt}{G(y(t))} = \int_{x_0}^{x} F(t)dt$	Integrate both sides of the separated equation over $x_0 \leq t \leq x$.
$\int_{y_0}^{y(x)} \frac{du}{G(u)} = \int_{x_0}^{x} F(t)dt$	Apply on the left the change of variables $u = y(t)$. Define $y_0 = y(x_0)$.
$y(x) = W^{-1}\left(\int_{x_0}^{x} F(t)dt\right)$	Define $W(y) = \int_{y_0}^{y} du/G(u)$. Take inverses to isolate $y(x)$.

In practise, the last step with W^{-1} is never done. The preceding formula is called the *implicit solution*. Some work is done to find algebraically an *explicit solution*, as is given by W^{-1}.

Explicit and Implicit Solutions

Definition 1 (Explicit Solution)
A solution of $y' = f(x, y)$ is called **explicit** provided it is given by an equation

$$y = \text{an expression independent of } y.$$

To elaborate, on the left side must appear exactly the symbol y followed by an equal sign. Symbols y and $=$ are followed by an expression which does not contain the symbol y.

Definition 2 (Implicit Solution)
A solution of $y' = f(x, y)$ is called **implicit** provided it is not explicit.

Examples

- Explicit solutions: $y = 1$, $y = x$, $y = f(x)$, $y = 0$, $y = -1 + x^2$
- Implicit Solutions: $2y = 2$, $y^2 = x$, $y + x = 0$, $y = xy^2 + 1$, $y + 1 = x^2$, $x^2 + y^2 = 1$, $F(x, y) = c$

The General Solution of $y' = 2x(y-3)$

- The variables-separable method gives equilibrium solutions $y = c$, which are already *explicit*. In this case, $y = 3$ is an equilibrium solution.
- Because $F = 2x$, $G = y - 3$, then division by G gives the quadrature-prepared equation $y'/(y-3) = 2x$. A quadrature step gives the implicit solution

$$\ln|y-3| = x^2 + C.$$

- The non-equilibrium solutions may be left in *implicit* form, giving the **general solution** as the list

$$L_1 = \{y = 3, \ln|y-3| = x^2 + C\}.$$

- Algebra can be applied to $\ln|y-3| = x^2 + C$ to write it as $y = 3 + ke^{x^2}$ where $k \neq 0$. Then general solution L_1 can be re-written as

$$L_2 = \{y = 3, y = 3 + ke^{x^2}\}.$$

List L_2 can be distilled to the single formula $y = 3 + ce^{x^2}$, but L_1 has no simpler expression.