Elementary Matrices and Frame Sequences

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Elementary Matrices

An elementary matrix $\boldsymbol{E}$ is the result of applying a combination, multiply or swap rule to the identity matrix. The computer algebra system `maple` displays typical 4×4 elementary matrices (C=Combination, M=Multiply, S=Swap) as follows.

```
with(linalg):
Id:=diag(1,1,1,1);
C:=addrow(Id,2,3,c);
M:=mulrow(Id,3,m);
S:=swaprow(Id,1,4);
```

```
with(LinearAlgebra):
Id:=IdentityMatrix(4);
C:=RowOperation(Id,[3,2],c);
M:=RowOperation(Id,3,m);
S:=RowOperation(Id,[4,1]);
```

The answers:

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\boldsymbol{S} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Constructing elementary matrices E

Mult	Change a one in the identity matrix to symbol $m \neq 0$.
Combo	Change a zero in the identity matrix to symbol c.
Swap	Interchange two rows of the identity matrix.

Constructing E^{-1} from elementary matrix E

Mult	Change diagonal multiplier $m \neq 0$ in E to $1/m$.
Combo	Change multiplier c in E to $-c$.
Swap	The inverse of E is E itself.

Fundamental Theorem on Elementary Matrices

Theorem 1 (The rref and elementary matrices)
Let $\boldsymbol{A}$ be a given matrix of row dimension $\boldsymbol{n}$. Then there exist $\boldsymbol{n} \times \boldsymbol{n}$ elementary matrices $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\dots$, $\boldsymbol{E}_k$ such that

$$\mathbf{rref}(\boldsymbol{A}) = \boldsymbol{E}_k \cdots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}.$$

Details of Proof

The result is the observation that left multiplication of matrix $\boldsymbol{A}$ by elementary matrix $\boldsymbol{E}$ gives the answer $\boldsymbol{E}\boldsymbol{A}$ for the corresponding multiply, combination or swap operation. The matrices $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\dots$ represent the multiply, combination and swap operations performed in the frame sequence which take the First Frame into the Last Frame, or equivalently, original matrix $\boldsymbol{A}$ into $\mathbf{rref}(\boldsymbol{A})$.

A certain 6-frame sequence ____.

$$A_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 1, original matrix.

$$A_2 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & -6 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 2, combo(1,2,-2).

$$A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 3 & 6 & 3 \end{pmatrix}$$ Frame 3, mult(2,-1/6).

$$A_4 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & -6 \end{pmatrix}$$ Frame 4, combo(1,3,-3).

$$A_5 = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ Frame 5, combo(2,3,-6).

$$A_6 = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ Frame 6, combo(2,1,-3). Found $\mathrm{rref}(A_1)$.

Continued

The corresponding 3×3 elementary matrices are

$$\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 2, combo(1,2,-2) applied to $\boldsymbol{I}$.

$$\boldsymbol{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 3, mult(2,-1/6) applied to $\boldsymbol{I}$.

$$\boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$ Frame 4, combo(1,3,-3) applied to $\boldsymbol{I}$.

$$\boldsymbol{E}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix}$$ Frame 5, combo(2,3,-6) applied to $\boldsymbol{I}$.

$$\boldsymbol{E}_5 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ Frame 6, combo(2,1,-3) applied to $\boldsymbol{I}$.

Frame Sequence Details

$A_2 = E_1 A_1$	Frame 2, E_1 equals combo(1,2,-2) on I.
$A_3 = E_2 A_2$	Frame 3, E_2 equals mult(2,-1/6) on I.
$A_4 = E_3 A_3$	Frame 4, E_3 equals combo(1,3,-3) on I.
$A_5 = E_4 A_4$	Frame 5, E_4 equals combo(2,3,-6) on I.
$A_6 = E_5 A_5$	Frame 6, E_5 equals combo(2,1,-3) on I.
$A_6 = E_5 E_4 E_3 E_2 E_1 A_1$	Summary frames 1-6.

Then

$$\mathbf{rref}(\boldsymbol{A}_1) = \boldsymbol{E}_5 \boldsymbol{E}_4 \boldsymbol{E}_3 \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{A}_1,$$

which is the result of the Theorem.

Fundamental Theorem Illustrated

The summary:

$$A_6 = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_1$$

Because $A_6 = \mathbf{rref}(A_1)$, the above equation gives the inverse relationship

$$A_1 = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} \,\mathbf{rref}(A_1).$$

Each inverse matrix is simplified by the rules for constructing E^{-1} from elementary matrix E, the result being

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathrm{rref}(A_1)$$

Theorem 2 (RREF Inverse Method)

$$\operatorname{rref}(\operatorname{aug}(\boldsymbol{A},\boldsymbol{I})) = \operatorname{aug}(\boldsymbol{I},\boldsymbol{B}) \quad \text{if and only if} \quad \boldsymbol{AB} = \boldsymbol{I}.$$

Proof: For *any* matrix $\boldsymbol{E}$ there is the matrix multiply identity

$$\boldsymbol{E}\operatorname{aug}(\boldsymbol{C},\boldsymbol{D}) = \operatorname{aug}(\boldsymbol{EC},\boldsymbol{ED}).$$

This identity is proved by arguing that each side has identical columns. For example, $\mathbf{col}(\text{LHS},1) = \boldsymbol{E}\,\mathbf{col}(\boldsymbol{C},1) = \mathbf{col}(\text{RHS},1)$.

Assume $\boldsymbol{C} = \operatorname{aug}(\boldsymbol{A},\boldsymbol{I})$ satisfies $\operatorname{rref}(\boldsymbol{C}) = \operatorname{aug}(\boldsymbol{I},\boldsymbol{B})$. The fundamental theorem of elementary matrices implies $\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{C} = \operatorname{rref}(\boldsymbol{C})$. Then

$$\operatorname{rref}(\boldsymbol{C}) = \operatorname{aug}(\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{A},\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{I}) = \operatorname{aug}(\boldsymbol{I},\boldsymbol{B})$$

implies that $\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{A} = \boldsymbol{I}$ and $\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{I} = \boldsymbol{B}$. Together, $\boldsymbol{AB} = \boldsymbol{I}$ and then $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$.

Conversely, assume that $\boldsymbol{AB} = \boldsymbol{I}$. Then $\boldsymbol{A}$ has inverse $\boldsymbol{B}$. The fundamental theorem of elementary matrices implies the identity $\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{A} = \operatorname{rref}(\boldsymbol{A}) = \boldsymbol{I}$. It follows that $\boldsymbol{B} = \boldsymbol{E}_k\cdots\boldsymbol{E}_1$. Then $\operatorname{rref}(\boldsymbol{C}) = \boldsymbol{E}_k\cdots\boldsymbol{E}_1\operatorname{aug}(\boldsymbol{A},\boldsymbol{I}) = \operatorname{aug}(\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{A},\boldsymbol{E}_k\cdots\boldsymbol{E}_1\boldsymbol{I}) = \operatorname{aug}(\boldsymbol{I},\boldsymbol{B})$.