

# **Applications of Systems of Differential Equations**

- Brine Tank Cascade
- Cascade Model
- Recycled Brine Tank Cascade
- Recycled Cascade Model

## Brine Tank Cascade

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Let brine tanks  $A$ ,  $B$ ,  $C$  be given of volumes 20, 40, 60, respectively, as in Figure 1.

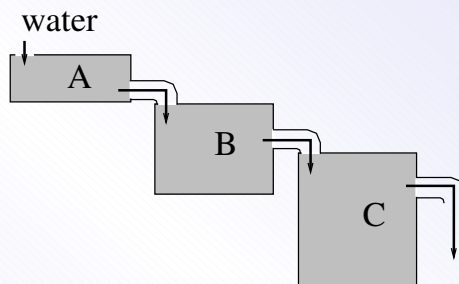


Figure 1. Three brine tanks in cascade.

## Assumptions and Notation

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- It is supposed that fluid enters tank  $A$  at rate  $r$ , drains from  $A$  to  $B$  at rate  $r$ , drains from  $B$  to  $C$  at rate  $r$ , then drains from tank  $C$  at rate  $r$ . Hence the volumes of the tanks remain constant. Let  $r = 10$ , to illustrate the ideas.
- Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.
- Let  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  denote the amount of salt at time  $t$  in each tank. We suppose added to tank  $A$  **water containing no salt**. Therefore, the salt in all the tanks is eventually lost from the drains.

## Cascade Model

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The cascade is modeled by the **chemical balance law**

$$\text{rate of change} = \text{input rate} - \text{output rate.}$$

Application of the balance law results in the triangular differential system

$$\begin{aligned}x_1' &= -\frac{1}{2}x_1, \\x_2' &= \frac{1}{2}x_1 - \frac{1}{4}x_2, \\x_3' &= \frac{1}{4}x_2 - \frac{1}{6}x_3.\end{aligned}$$

## Cascade Model Solution

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The solution is justified by the integrating factor method for first order scalar differential equations.

$$\begin{aligned}x_1(t) &= x_1(0)e^{-t/2}, \\x_2(t) &= -2x_1(0)e^{-t/2} + (x_2(0) + 2x_1(0))e^{-t/4}, \\x_3(t) &= \frac{3}{2}x_1(0)e^{-t/2} - 3(x_2(0) + 2x_1(0))e^{-t/4} \\&\quad + (x_3(0) - \frac{3}{2}x_1(0) + 3(x_2(0) + 2x_1(0)))e^{-t/6}.\end{aligned}$$

## Recycled Brine Tank Cascade

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Let brine tanks  $A$ ,  $B$ ,  $C$  be given of volumes  $60$ ,  $30$ ,  $60$ , respectively, as in Figure 2.

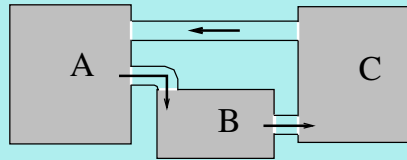


Figure 2. Three brine tanks in cascade with recycling.

## Assumptions and Notation

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- Suppose that fluid drains from tank  $A$  to  $B$  at rate  $r$ , drains from tank  $B$  to  $C$  at rate  $r$ , then drains from tank  $C$  to  $A$  at rate  $r$ . The tank volumes remain constant due to constant recycling of fluid. For purposes of illustration, let  $r = 10$ .
- Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.
- Let  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  denote the amount of salt at time  $t$  in each tank. No salt is lost from the system, due to recycling.

## Recycled Cascade Model

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Using compartment analysis, the recycled cascade is modeled by the non-triangular system

$$\begin{aligned}x_1' &= -\frac{1}{6}x_1 && + \frac{1}{6}x_3, \\x_2' &= \frac{1}{6}x_1 - \frac{1}{3}x_2, \\x_3' &= \frac{1}{3}x_2 - \frac{1}{6}x_3.\end{aligned}$$

## Recycled Cascade Solution

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$$x_1(t) = c_1 + (c_2 - 2c_3)e^{-t/3} \cos(t/6) + (2c_2 + c_3)e^{-t/3} \sin(t/6),$$

$$x_2(t) = \frac{1}{2}c_1 + (-2c_2 - c_3)e^{-t/3} \cos(t/6) + (c_2 - 2c_3)e^{-t/3} \sin(t/6),$$

$$x_3(t) = c_1 + (c_2 + 3c_3)e^{-t/3} \cos(t/6) + (-3c_2 + c_3)e^{-t/3} \sin(t/6).$$

- At infinity,  $x_1 = x_3 = c_1$ ,  $x_2 = c_1/2$ . The meaning is that the total amount of salt is uniformly distributed in the tanks, in the ratio **2 : 1 : 2**.
- The solution of the system was found by the eigenanalysis method. It can also be found by the resolvent method in Laplace theory.