Systems of Differential Equations and Laplace's Method

- Solving $x' = Cx$
- The Resolvent
- An Illustration for $\mathrm{x}' = C\mathrm{x}$
- Solving $x'' = Ax$

Solving $x' = Cx$

Apply $\boldsymbol{L}$ to each side to obtain $\boldsymbol{L}(\mathrm{x}') = \boldsymbol{C}\boldsymbol{L}(\mathrm{x})$. Use the parts rule

$$\boldsymbol{L}(\mathrm{x}') = \boldsymbol{s}\boldsymbol{L}(\boldsymbol{x}) - \mathrm{x}(0)$$

to obtain

$$\begin{aligned}
\boldsymbol{s}\boldsymbol{L}(\mathrm{x}) - \mathrm{x}(0) &= \boldsymbol{L}(\boldsymbol{C}\mathrm{x}) \\
\boldsymbol{s}\boldsymbol{L}(\mathrm{x}) - \boldsymbol{L}(\boldsymbol{C}\mathrm{x}) &= \mathrm{x}(0) \\
\boldsymbol{s}\boldsymbol{I}\,\boldsymbol{L}(\mathrm{x}) - \boldsymbol{C}\,\boldsymbol{L}(\mathrm{x}) &= \mathrm{x}(0) \\
(\boldsymbol{s}\boldsymbol{I} - \boldsymbol{C})\boldsymbol{L}(\mathrm{x}) &= \mathrm{x}(0).
\end{aligned}$$

Resolvent

The inverse of $\boldsymbol{s}\boldsymbol{I} - \boldsymbol{C}$ is called the **resolvent**, a term invented to describe the equation

$$\boldsymbol{L}(\mathrm{x}(\boldsymbol{t})) = (\boldsymbol{s}\boldsymbol{I} - \boldsymbol{C})^{-1}\mathrm{x}(0).$$

An Illustration for $\mathrm{x}' = C\mathrm{x}$

Define $\boldsymbol{C} = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$, $\mathrm{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\mathrm{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, which gives a scalar initial value problem

$$\begin{cases} x_1'(t) &= 2x_1(t) + 3x_2(t), \\ x_2'(t) &= 4x_2(t), \\ x_1(0) &= 1, \\ x_2(0) &= 2. \end{cases}$$

Then the adjugate formula $\boldsymbol{A}^{-1} = \dfrac{\mathsf{adj}(A)}{\det(A)}$ gives the resolvent

$$(sI - C)^{-1} = \frac{1}{(s-2)(s-4)} \begin{pmatrix} s-4 & -3 \\ 0 & s-2 \end{pmatrix}.$$

The Laplace transform of the solution is then

$$\boldsymbol{L}(\mathbf{x}(\boldsymbol{t})) = (\boldsymbol{s}\boldsymbol{I} - \boldsymbol{C})^{-1}\begin{pmatrix} \mathbf{1} \\ \mathbf{2} \end{pmatrix} = \begin{pmatrix} \dfrac{\boldsymbol{s} - \mathbf{10}}{(\boldsymbol{s} - \mathbf{2})(\boldsymbol{s} - \mathbf{4})} \\ \dfrac{\mathbf{2}}{\boldsymbol{s} - \mathbf{4}} \end{pmatrix}.$$

Partial fractions and use of the backward Laplace table imply

$$\mathbf{x}(\boldsymbol{t}) = \begin{pmatrix} \mathbf{4}\boldsymbol{e}^{\mathbf{2}\boldsymbol{t}} - \mathbf{3}\boldsymbol{e}^{\mathbf{4}\boldsymbol{t}} \\ \mathbf{2}\boldsymbol{e}^{\mathbf{4}\boldsymbol{t}} \end{pmatrix}.$$

Laplace Operations on $x'' = Ax$

Apply $\boldsymbol{L}$ to each side to obtain $\boldsymbol{L}(\mathrm{x}'') = \boldsymbol{A}\boldsymbol{L}(\mathrm{x})$. Use the parts rule to obtain

$$
\begin{aligned}
\boldsymbol{L}(\mathrm{x}'') &= \boldsymbol{s}(\boldsymbol{s}\boldsymbol{L}(\mathrm{x}) - \mathrm{x}(0)) - \mathrm{x}'(0) \\
\boldsymbol{s}^2\boldsymbol{L}(\mathrm{x}) - \boldsymbol{s}\mathrm{x}(0) - \mathrm{x}'(0) &= \boldsymbol{A}\boldsymbol{L}(\mathrm{x}) \\
(\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{A})\boldsymbol{L}(\mathrm{x}) &= \boldsymbol{s}\mathrm{x}(0) + \mathrm{x}'(0) \\
\boldsymbol{L}(\mathrm{x}(\boldsymbol{t})) &= (\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}\mathrm{x}(0) + \mathrm{x}'(0)) \,.
\end{aligned}
$$

Resolvent Formula

The inverse of $\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{C}$ is also called a **resolvent**, and we have the equation

$$\boldsymbol{L}(\mathbf{x}(\boldsymbol{t})) = (\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}\mathbf{x}(0) + \mathbf{x}'(0)).$$

Formal Solution of $\mathbf{x}'' = \boldsymbol{A}\mathbf{x}$

$$\mathbf{x}(\boldsymbol{t}) = \boldsymbol{L}^{-1}\left((\boldsymbol{s}^2\boldsymbol{I} - \boldsymbol{A})^{-1}(\boldsymbol{s}\mathbf{x}(0) + \mathbf{x}'(0))\right).$$