Laplace Table Derivations

- $L(t^n) = \frac{n!}{s^{1+n}}$
- $L(e^{at}) = \frac{1}{s-a}$
- $L(\cos bt) = \frac{s}{s^2 + b^2}$
- $L(\sin bt) = \frac{b}{s^2 + b^2}$
- $L(H(t-a)) = \frac{e^{-as}}{s}$
- $L(\delta(t-a)) = e^{-as}$
- $L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$
- $L(\text{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$
- $L(\text{trw}(t/a)) = \frac{1}{s^2} \tanh(as/2)$
- $L(t^\alpha) = \frac{\Gamma(1 + \alpha)}{s^{1+\alpha}}$
- $L(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$
Proof of $L(t^n) = n! / s^{1+n}$

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The first step is to evaluate $L(f(t))$ for $f(t) = t^0$ [$n = 0$ case]. The function $t^0$ is written as 1, but Laplace theory conventions require $f(t) = 0$ for $t < 0$, therefore $f(t)$ is technically the unit step function.

$$L(1) = \int_0^\infty (1)e^{-st} \, dt$$

Laplace integral of $f(t) = 1$.

$$= -(1/s)e^{-st}|_{t=0}^{t=\infty}$$

Evaluate the integral.

$$= 1/s$$

Assumed $s > 0$ to evaluate $\lim_{t \to \infty} e^{-st}$. 

$$\lim_{t \to \infty} e^{-st}.$$
Proof of \( L(t^n) = n!/s^{1+n} \)

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The value of \( L(f(t)) \) for \( f(t) = t \) can be obtained by \( s \)-differentiation of the relation \( L(1) = 1/s \), as follows. Technically, \( f(t) = 0 \) for \( t < 0 \), then \( f(t) \) is called the ramp function.

\[
\frac{d}{ds} L(1) = \frac{d}{ds} \int_0^\infty (1)e^{-st}dt
\]

Laplace integral for \( f(t) = 1 \).

\[
= \int_0^\infty \frac{d}{ds} (e^{-st}) dt
\]

Used \( \frac{d}{ds} \int_a^b F dt = \int_a^b \frac{dF}{ds} dt \).

\[
= \int_0^\infty (-t)e^{-st}dt
\]

Calculus rule \((e^u)' = u'e^u\).

\[
= -L(t)
\]

Definition of \( L(t) \).

Then

\[
L(t) = -\frac{d}{ds} L(1)
\]

Rewrite last display.

\[
= -\frac{d}{ds} (1/s)
\]

Use \( L(1) = 1/s \).

\[
= 1/s^2
\]

Differentiate.
Proof of $L(t^n) = n!/s^{1+n}$

This idea can be repeated to give

$$L(t^2) = -\frac{d}{ds}L(t)$$
$$= L(t^2)$$
$$= 2$$
$$= \frac{2}{s^3}.$$ 

The pattern is $L(t^n) = -\frac{d}{ds}L(t^{n-1})$, which implies the formula

$$L(t^n) = \frac{n!}{s^{1+n}}.$$ 

The proof is complete.
Proof of $L(e^{at}) = \frac{1}{s - a}$

The result follows from $L(1) = 1/s$, as follows.

\[ L(e^{at}) = \int_0^\infty e^{at}e^{-st}dt \]

\[ = \int_0^\infty e^{-(s-a)t}dt \]

\[ = \int_0^\infty e^{-St}dt \]

\[ = 1/S \]

\[ = 1/(s - a) \]

Direct Laplace transform.

Use $e^Ae^B = e^{A+B}$.

Substitute $S = s - a$.

Apply $L(1) = 1/s$.

Back-substitute $S = s - a$. 
Proof of $L(\cos bt) = \frac{s}{s^2 + b^2}$ and $L(\sin bt) = \frac{b}{s^2 + b^2}$

Use will be made of Euler’s formula

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

usually first introduced in trigonometry. In this formula, $\theta$ is a real number in radians and $i = \sqrt{-1}$ is the complex unit.

$$e^{ibt}e^{-st} = (\cos bt)e^{-st} + i(\sin bt)e^{-st}$$

Substitute $\theta = bt$ into Euler’s formula and multiply by $e^{-st}$.

$$\int_0^\infty e^{-ibt}e^{-st}dt = \int_0^\infty (\cos bt)e^{-st}dt$$

Integrate $t = 0$ to $t = \infty$. Then use properties of integrals.

$$\frac{1}{s - ib} = \int_0^\infty (\cos bt)e^{-st}dt$$

Evaluate the left hand side using $L(e^{at}) = 1/(s - a)$, $a = ib$. 
Proof of \( L(\cos bt) = \frac{s}{s^2 + b^2} \) and \( L(\sin bt) = \frac{b}{s^2 + b^2} \)

\[
\frac{1}{s - ib} = L(\cos bt) + iL(\sin bt)
\]

Direct Laplace transform definition.

\[
\frac{s + ib}{s^2 + b^2} = L(\cos bt) + iL(\sin bt)
\]

Use complex rule \( 1/z = \bar{z}/|z|^2 \), \( z = A + iB, \bar{z} = A - iB, |z| = \sqrt{A^2 + B^2} \).

\[
\frac{s}{s^2 + b^2} = L(\cos bt)
\]

Extract the real part.

\[
\frac{b}{s^2 + b^2} = L(\sin bt)
\]

Extract the imaginary part.
Proof of $L(H(t - a)) = e^{-as}/s$

$L(H(t - a)) = \int_{0}^{\infty} H(t - a)e^{-st}dt$

$= \int_{a}^{\infty} (1)e^{-st}dt$

Because $H(t - a) = 0$ for $0 \leq t < a$.

$= \int_{0}^{\infty} (1)e^{-s(x+a)}dx$

Change variables $t = x + a$.

$= e^{-as}\int_{0}^{\infty} (1)e^{-sx}dx$

Constant $e^{-as}$ moves outside integral.

$= e^{-as}(1/s)$

Direct Laplace transform. Assume $a \geq 0$.

Apply $L(1) = 1/s$. 

Proof of \( L(\delta(t - a)) = e^{-as} \)

The definition of the delta function is a formal one, in which every occurrence of symbol \( \delta(t - a) \) under an integrand is replaced by \( dH(t - a) \). The differential symbol \( dH(t - a) \) is taken in the sense of the Riemann-Stieltjes integral. This integral is defined in Rudin’s *Real analysis* for monotonic integrators \( \alpha(x) \) as the limit

\[
\int_{a}^{b} f(x) \, d\alpha(x) = \lim_{N \to \infty} \sum_{n=1}^{N} f(x_n)(\alpha(x_n) - \alpha(x_{n-1}))
\]

where \( x_0 = a, x_N = b \) and \( x_0 < x_1 < \cdots < x_N \) forms a partition of \([a, b] \) whose mesh approaches zero as \( N \to \infty \).

The steps in computing the Laplace integral of the delta function appear below. Admittedly, the proof requires advanced calculus skills and a certain level of mathematical maturity. The reward is a fuller understanding of the Dirac symbol \( \delta(x) \).
Proof of $L(\delta(t - a)) = e^{-as}$

Laplace integral, $a > 0$ assumed.

Replace $\delta(t - a)dt$ by $dH(t - a)$.

Definition of improper integral.

Explained below.
Proof of $L(\delta(t-a)) = e^{-as}$

To explain the last step, apply the definition of the Riemann-Stieltjes integral:

$$
\int_0^M e^{-st} dH(t-a) = \lim_{N \to \infty} \sum_{n=0}^{N-1} e^{-st_n}(H(t_n-a) - H(t_{n-1} - a))
$$

where $0 = t_0 < t_1 < \cdots < t_N = M$ is a partition of $[0, M]$ whose mesh $\max_{1 \leq n \leq N}(t_n - t_{n-1})$ approaches zero as $N \to \infty$. Given a partition, if $t_{n-1} < a \leq t_n$, then $H(t_n-a) - H(t_{n-1} - a) = 1$, otherwise this factor is zero. Therefore, the sum reduces to a single term $e^{-st_n}$. This term approaches $e^{-sa}$ as $N \to \infty$, because $t_n$ must approach $a$. 
Proof of \( L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})} \)

The library function \textit{floor} present in computer languages C and Fortran is defined by \( \text{floor}(x) = \) greatest whole integer \( \leq x \), e.g., \( \text{floor}(5.2) = 5 \) and \( \text{floor}(-1.9) = -2 \). The computation of the Laplace integral of \text{floor}(t) \) requires ideas from infinite series, as follows.

\[
F(s) = \int_0^\infty \text{floor}(t)e^{-st}dt \\
= \sum_{n=0}^{\infty} \int_n^{n+1} (n)e^{-st}dt \\
= \sum_{n=0}^{\infty} \frac{n}{s}(e^{-ns} - e^{-ns-s}) \\
= \frac{1 - e^{-s}}{s} \sum_{n=0}^{\infty} ne^{-sn} \\
\]

Laplace integral definition.

On \( n \leq t < n + 1 \), \( \text{floor}(t) = n \).

Evaluate each integral.

Common factor removed.
Proof of $L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$

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\[= \frac{x(1 - x)}{s} \sum_{n=0}^{\infty} nx^{n-1} \]

Define $x = e^{-s}$.

\[= \frac{x(1 - x)}{s} \frac{d}{dx} \sum_{n=0}^{\infty} x^n \]

Term-by-term differentiation.

\[= \frac{x(1 - x)}{s} \frac{d}{dx} \frac{1}{1 - x} \]

Geometric series sum.

\[= \frac{x}{s(1 - x)} \frac{e^{-s}}{s(1 - e^{-s})} \]

Compute the derivative, simplify.

Substitute $x = e^{-s}$. 
Proof of $L(\text{floor}(t/a)) = \frac{e^{-as}}{s(1 - e^{-as})}$

To evaluate the Laplace integral of $\text{floor}(t/a)$, a change of variables is made.

$L(\text{floor}(t/a)) = \int_0^\infty \text{floor}(t/a)e^{-st}dt$  \hspace{1cm} \text{Laplace integral definition.}

= a \int_0^\infty \text{floor}(r)e^{-asr}dr  \hspace{1cm} \text{Change variables } t = ar.

= aF(as)  \hspace{1cm} \text{Apply the formula for } F(s).

= \frac{e^{-as}}{s(1 - e^{-as})}  \hspace{1cm} \text{Simplify.}$
Proof of \( L(\text{sqw}(t/a)) = \frac{1}{s} \tanh(as/2) \)

The square wave defined by \( \text{sqw}(x) = (-1)^{\text{floor}(x)} \) is periodic of period 2 and piecewise-defined. Let \( P = \int_0^2 \text{sqw}(t)e^{-st}dt \).

\[
P = \int_0^1 \text{sqw}(t)e^{-st}dt + \int_1^2 \text{sqw}(t)e^{-st}dt
= \int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt
= \frac{1}{s} \left(1 - e^{-s}\right) + \frac{1}{s} \left(e^{-2s} - e^{-s}\right)
= \frac{1}{s} \left(1 - e^{-s}\right)^2
\]

Apply \( \int_a^b = \int_a^c + \int_c^b \).

Use \( \text{sqw}(x) = 1 \) on \( 0 \leq x < 1 \) and \( \text{sqw}(x) = -1 \) on \( 1 \leq x < 2 \).

Evaluate each integral.

Collect terms.
Proof of $L(\text{sqw}(t/a)) = \frac{1}{s} \tanh(\frac{as}{2})$

Slide 2 of 3 – Compute $L(\text{sqw}(t))$

$L(\text{sqw}(t)) = \frac{\int_0^2 \text{sqw}(t)e^{-st}dt}{1 - e^{-2s}}$

Periodic function formula.

$= \frac{1}{s} (1 - e^{-s})^2 \frac{1}{1 - e^{-2s}}.$

Use the computation of $P$ above.

$= \frac{1}{s} \frac{1 - e^{-s}}{1 + e^{-s}}.$

Factor $1 - e^{-2s} = (1 - e^{-s})(1 + e^{-s}).$

$= \frac{1}{s} \frac{e^{s/2} - e^{-s/2}}{e^{s/2} + e^{-s/2}}.$

Multiply the fraction by $e^{s/2}/e^{s/2}.$

$= \frac{1}{s} \frac{\sinh(s/2)}{\cosh(s/2)}.$

Use $\sinh u = (e^u - e^{-u})/2$, $\cosh u = (e^u + e^{-u})/2$.

$= \frac{1}{s} \tanh(s/2).$

Use $\tanh u = \sinh u/\cosh u.$
Proof of $L(\text{sqw}(t/a)) = \frac{1}{s} \tanh(as/2)$

To complete the computation of $L(\text{sqw}(t/a))$, a change of variables is made:

\[
L(\text{sqw}(t/a)) = \int_0^\infty \text{sqw}(t/a)e^{-st}dt
\]

Direct transform.

\[
= \int_0^\infty \text{sqw}(r)e^{-asr}(a)dr
\]

Change variables $r = \frac{t}{a}$.

\[
= \frac{a}{as} \tanh(as/2)
\]

See $L(\text{sqw}(t))$ above.

\[
= \frac{1}{s} \tanh(as/2)
\]
Proof of $L(\text{trw}(t/a)) = \frac{1}{s^2} \tanh(\frac{as}{2})$

The triangular wave is defined by \( \text{trw}(t) = \int_0^t \text{sqw}(x) \, dx \).

\[
L(\text{trw}(t/a)) = \frac{f(0) + L(f'(t))}{s}
\]

Let \( f(t) = \text{trw}(t/a) \). Use \( L(f'(t)) = sL(f(t)) - f(0) \).

\[
= \frac{1}{s} L(\text{sqw}(t/a))
\]

Use \( f(0) = 0 \), then use

\[
(a \int_0^{t/a} \text{sqw}(x) \, dx)' = \text{sqw}(t/a).
\]

\[
= \frac{1}{s^2} \tanh(\frac{as}{2})
\]

Table entry for \( \text{sqw} \).
Proof of $L(t^\alpha) = \frac{\Gamma(1 + \alpha)}{s^{1+\alpha}}$

$L(t^\alpha) = \int_0^\infty t^\alpha e^{-st}dt$

= $\int_0^\infty (u/s)^\alpha e^{-u}du/s$  

= $\frac{1}{s^{1+\alpha}} \int_0^\infty u^\alpha e^{-u}du$  

= $\frac{1}{s^{1+\alpha}} \Gamma(1 + \alpha)$.  

Definition of Laplace integral.  

Change variables $u = st$, $du = sdt$.  

Because $s=\text{constant}$ for $u$-integration.  

Because $\Gamma(x) \equiv \int_0^\infty u^{x-1}e^{-u}du$.  

**Gamma Function**

The *generalized factorial function* $\Gamma(x)$ is defined for $x > 0$ and it agrees with the classical factorial $n! = (1)(2) \cdots (n)$ in case $x = n + 1$ is an integer. In literature, $\alpha!$ means $\Gamma(1 + \alpha)$. For more details about the Gamma function, see Abramowitz and Stegun or Maple documentation.

**Proof of** $L(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$

\[
L(t^{-1/2}) = \frac{\Gamma(1 + (-1/2))}{s^{1-1/2}} \quad \text{Apply the previous formula.}
\]

\[
= \frac{\sqrt{\pi}}{\sqrt{s}} \quad \text{Use } \Gamma(1/2) = \sqrt{\pi}.
\]