Second Order Systems

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Coupled Spring-Mass Systems

Three masses are attached to each other by four springs as in Figure 1. A model will be developed for the positions of the three masses.

Figure 1. Three masses connected by springs. The masses slide along a frictionless horizontal surface.
## Variables

The analysis uses the following constants, variables and assumptions.

<table>
<thead>
<tr>
<th><strong>Mass Constants</strong></th>
<th>The masses $m_1$, $m_2$, $m_3$ are assumed to be point masses concentrated at their center of gravity.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spring Constants</strong></td>
<td>The mass of each spring is negligible. The springs operate according to Hooke’s law: Force = $k$(elongation). Constants $k_1$, $k_2$, $k_3$, $k_4$ denote the Hooke’s constants. The springs restore after compression and extension.</td>
</tr>
<tr>
<td><strong>Position Variables</strong></td>
<td>The symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the mass positions along the horizontal surface, measured from their equilibrium positions, plus right and minus left.</td>
</tr>
<tr>
<td><strong>Fixed Ends</strong></td>
<td>The first and last spring are attached to fixed walls.</td>
</tr>
</tbody>
</table>
Derivation

The competition method is used to derive the equations of motion. In this case, the law is

Newton’s Second Law Force = Sum of the Hooke’s Forces.

The model equations are

\[
\begin{align*}
    m_1 x_1''(t) &= -k_1 x_1(t) + k_2 [x_2(t) - x_1(t)], \\
    m_2 x_2''(t) &= -k_2 [x_2(t) - x_1(t)] + k_3 [x_3(t) - x_2(t)], \\
    m_3 x_3''(t) &= -k_3 [x_3(t) - x_2(t)] - k_4 x_3(t).
\end{align*}
\]

(1)

• The equations are justified in the case of all positive variables by observing that the first three springs are elongated by \( x_1, x_2 - x_1, x_3 - x_2 \), respectively. The last spring is compressed by \( x_3 \), which accounts for the minus sign.

• Another way to justify the equations is through mirror-image symmetry: interchange \( k_1 \leftrightarrow k_4, k_2 \leftrightarrow k_3, x_1 \leftrightarrow x_3 \), then equation 2 should be unchanged and equation 3 should become equation 1.
Vector-Matrix form $x'' = Ax$

In vector-matrix form, this system is a second order system

$$M x''(t) = K x(t)$$

where the displacement $x$, mass matrix $M$ and stiffness matrix $K$ are defined by the formulas

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

Because $M$ is invertible, the system can always be re-written using $A = M^{-1}K$ as the second-order system

$$x'' = Ax.$$