An RREF Method for Finding Inverses

An efficient method to find the inverse $B$ of a square matrix $A$, should it happen to exist, is to form the augmented matrix $C = \text{aug}(A, I)$ and then read off $B$ as the package of the last $n$ columns of $\text{rref}(C)$. This method is based upon the equivalence

$$\text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \quad \text{if and only if} \quad AB = I.$$
Main Results

Theorem 1 (Inverse Test)
If $A$ and $B$ are square matrices such that $AB = I$, then also $BA = I$. Therefore, only one of the equalities $AB = I$ or $BA = I$ is required to check an inverse.

Theorem 2 (The \texttt{rref} Inversion Method)
Let $A$ and $B$ denote square matrices. Then

(a) If $\text{rref}(\text{aug}(A, I)) = \text{aug}(I, B)$, then $AB = BA = I$ and $B$ is the inverse of $A$.

(b) If $AB = BA = I$, then $\text{rref}(\text{aug}(A, I)) = \text{aug}(I, B)$.

(c) If $\text{rref}(\text{aug}(A, I)) = \text{aug}(C, B)$ and $C \neq I$, then $A$ is not invertible.
Finding inverses

The \texttt{rref} inversion method will be illustrated for the matrix

\[
A = \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 1 & 1
\end{pmatrix}.
\]

Define the first frame of the sequence to be \( C_1 = \text{aug}(C, I) \), then compute the frame sequence to \( \text{rref}(C) \) as follows.
<table>
<thead>
<tr>
<th>Frame</th>
<th>Matrix</th>
<th>Comment</th>
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</thead>
<tbody>
<tr>
<td>First Frame</td>
<td>$C_1 = \begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; -1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 &amp; 1 \end{pmatrix}$</td>
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<td>$C_2 = \begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; -1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 2 &amp; 0 &amp; -1 &amp; 1 \end{pmatrix}$ combo $(3, 2, -1)$</td>
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<td>$C_3 = \begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; -1 &amp; 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 &amp; 0 &amp; -1/2 &amp; 1/2 \end{pmatrix}$ mult $(3, 1/2)$</td>
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<td>$C_4 = \begin{pmatrix} 1 &amp; 0 &amp; 1 &amp; 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 &amp; 0 &amp; 1/2 &amp; 1/2 \ 0 &amp; 0 &amp; 1 &amp; 0 &amp; -1/2 &amp; 1/2 \end{pmatrix}$ combo $(3, 2, 1)$</td>
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<td>$C_5 = \begin{pmatrix} 1 &amp; 0 &amp; 0 &amp; 1 &amp; 1/2 &amp; -1/2 \ 0 &amp; 1 &amp; 0 &amp; 0 &amp; 1/2 &amp; 1/2 \ 0 &amp; 0 &amp; 1 &amp; 0 &amp; -1/2 &amp; 1/2 \end{pmatrix}$ combo $(3, 1, -1)$</td>
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<td>Last Frame</td>
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The theory

\[ \text{rref}(\text{aug}(A, I)) = \text{aug}(I, B) \quad \text{if and only if} \quad AB = I \]

implies that the inverse of \( A \) is the matrix in the right half of the last frame:

\[
A^{-1} = \begin{pmatrix}
1 & 1/2 & -1/2 \\
0 & 1/2 & 1/2 \\
0 & -1/2 & 1/2
\end{pmatrix}
\]