Determinant Theory

- Unique Solution of $Ax = b$
- College Algebra Definition of Determinant
- Diagram for Sarrus’ $3 \times 3$ Rule
- Transpose Rule
- How to Compute the Value of any Determinant
  - Four Rules to Compute any Determinant
  - Special Determinant Rules
  - Cofactor Expansion
- Adjugate Formula for the Inverse
- Determinant Product Theorem
Unique Solution of a $2 \times 2$ System

The $2 \times 2$ system

\[
\begin{align*}
ax + by &= e, \\
.cx + dy &= f,
\end{align*}
\]

(1)

has a unique solution provided $\Delta = ad - bc$ is nonzero, in which case the solution is given by

\[
\begin{align*}
x &= \frac{de - bf}{ad - bc}, \\
y &= \frac{af - ce}{ad - bc}.
\end{align*}
\]

(2)

This result is called Cramer’s Rule for $2 \times 2$ systems, learned in college algebra.
College algebra introduces matrix notation and determinant notation:

\[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{det}(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}. \]

Evaluation of \( \text{det}(A) \) is by **Sarrus’ 2 \( \times \) 2 Rule**:

\[ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \]

The first product \( ad \) is the product of the main diagonal entries and the other product \( bc \) is from the anti-diagonal.

Cramer’s 2 \( \times \) 2 rule in determinant notation is

\[
\begin{align*}
    x &= \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} , \\
    y &= \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}.
\end{align*}
\]
Relation to Inverse Matrices

System

\begin{align*}
ax + by &= e, \\
 cx + dy &= f,
\end{align*}

(4)

can be expressed as the vector-matrix system \( A\mathbf{u} = \mathbf{b} \) where

\[
A = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} e \\ f \end{pmatrix}.
\]

Inverse matrix theory implies

\[
A^{-1} = \frac{1}{ad - bc} \begin{pmatrix}
d & -b \\
-c & a
\end{pmatrix}, \quad \mathbf{u} = A^{-1}\mathbf{b} = \frac{1}{ad - bc} \begin{pmatrix} de - bf \\ af - ce \end{pmatrix}.
\]

Cramer’s Rule is a compact summary of the unique solution of system (4).
Unique Solution of an $n \times n$ System

System

\[
\begin{align*}
  a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
  a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
  \vdots & \quad \vdots \quad \cdots \quad \vdots \quad \vdots \\
  a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{align*}
\]

(5)

can be written as an $n \times n$ vector-matrix equation $A\vec{x} = \vec{b}$, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{b} = (b_1, \ldots, b_n)$. The system has a unique solution provided the determinant of coefficients $\Delta = \det(A)$ is nonzero, and then Cramer’s Rule for $n \times n$ systems gives

\[
\begin{align*}
  x_1 &= \frac{\Delta_1}{\Delta}, \\
  x_2 &= \frac{\Delta_2}{\Delta}, \\
  \vdots & \quad \vdots \\
  x_n &= \frac{\Delta_n}{\Delta}
\end{align*}
\]

(6)

Symbol $\Delta_j = \det(B)$, where matrix $B$ has the same columns as matrix $A$, except $\text{col}(B, j) = \vec{b}$. 

Determinants of Order $n$

Determinants will be defined shortly; intuition from the $2 \times 2$ case and Sarrus’ rule should suffice for the moment.
Determinant Notation for Cramer’s Rule

The determinant of coefficients for system $A\vec{x} = \vec{b}$ is denoted by

$$\Delta = \left| \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array} \right|.$$  \hspace{1cm} (7)

The other $n$ determinants in Cramer’s rule (6) are given by

$$\Delta_1 = \left| \begin{array}{cccc} b_{1} & a_{12} & \cdots & a_{1n} \\ b_{2} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n} & a_{n2} & \cdots & a_{nn} \end{array} \right|, \ldots, \Delta_n = \left| \begin{array}{cccc} a_{11} & a_{12} & \cdots & b_{1} \\ a_{21} & a_{22} & \cdots & b_{2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_{n} \end{array} \right|.$$  \hspace{1cm} (8)
College Algebra Definition of Determinant

Given an $n \times n$ matrix $A$, define

\begin{equation}
\text{det}(A) = \sum_{\sigma \in S_n} (-1)^{\text{parity}(\sigma)} a_{1\sigma_1} \cdots a_{n\sigma_n}.
\end{equation}

In the formula, $a_{ij}$ denotes the element in row $i$ and column $j$ of the matrix $A$. The symbol $\sigma = (\sigma_1, \ldots, \sigma_n)$ stands for a rearrangement of the subscripts 1, 2, ..., $n$ and $S_n$ is the set of all possible rearrangements. The nonnegative integer $\text{parity}(\sigma)$ is determined by counting the minimum number of pairwise interchanges required to assemble the list of integers $\sigma_1, \ldots, \sigma_n$ into natural order 1, ..., $n$. 
College Algebra Definition and Sarrus’ Rule

For a $3 \times 3$ matrix, the College Algebra formula reduces to Sarrus’ $3 \times 3$ Rule

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13}. \quad (10)$$
The number $\det(A)$, in the $3 \times 3$ case, can be computed by the algorithm in Figure 1, which parallels the one for $2 \times 2$ matrices. The $5 \times 3$ array is made by copying the first two rows of $A$ into rows 4 and 5.

**Warning**: *there is no Sarrus’ rule diagram for $4 \times 4$ or larger matrices!*

**Figure** 1. Sarrus’ rule diagram for $3 \times 3$ matrices, which gives

$$\det(A) = (a + b + c) - (d + e + f).$$
**Transpose Rule**

A consequence of the college algebra definition of determinant is the relation

\[ \det(A) = \det(A^T) \]

where \( A^T \) means the transpose of \( A \), obtained by swapping rows and columns. This relation implies the following.

All determinant theory results for rows also apply to columns.
How to Compute the Value of any Determinant

- **Four Rules.** These are the *Triangular Rule, Combination Rule, Multiply Rule* and the *Swap Rule*.

- **Special Rules.** These apply to evaluate a determinant as zero.

- **Cofactor Expansion.** This is an iterative scheme which reduces computation of a determinant to a number of smaller determinants.

- **Hybrid Method.** The four rules and the cofactor expansion are combined.
# Four Rules

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Triangular</strong></td>
<td>The value of $\det(A)$ for either an upper triangular or a lower triangular matrix $A$ is the product of the diagonal elements: $\det(A) = a_{11}a_{22}\cdots a_{nn}$.</td>
</tr>
<tr>
<td><strong>Swap</strong></td>
<td>If $B$ results from $A$ by swapping two rows, then $\det(A) = (-1) \det(B)$.</td>
</tr>
<tr>
<td><strong>Combination</strong></td>
<td>The value of $\det(A)$ is unchanged by adding a multiple of a row to a different row.</td>
</tr>
<tr>
<td><strong>Multiply</strong></td>
<td>If one row of $A$ is multiplied by constant $c$ to create matrix $B$, then $\det(B) = c \det(A)$.</td>
</tr>
</tbody>
</table>
1 Example (Four Properties) Apply the four properties of a determinant to justify the formula

\[
\det \begin{pmatrix}
12 & 6 & 0 \\
11 & 5 & 1 \\
10 & 2 & 2
\end{pmatrix} = 24.
\]
Solution: Let $D$ denote the value of the determinant. Then

\[ D = \begin{vmatrix} 12 & 6 & 0 \\ 11 & 5 & 1 \\ 10 & 2 & 2 \end{vmatrix} \quad \text{Given.} \]

\[ = \begin{vmatrix} 12 & 6 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{vmatrix} \quad \text{combo}(1,2,-1), \ combo(1,3,-1). \text{ Combination leaves the determinant unchanged.} \]

\[ = 6 \begin{vmatrix} 2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{vmatrix} \quad \text{Multiply rule } m = 1/6 \text{ on row 1 factors out a 6.} \]

\[ = 6 \begin{vmatrix} -1 & -1 & 2 \\ -1 & -1 & 1 \\ 0 & -3 & 2 \end{vmatrix} \quad \text{combo}(1,3,1), \ combo(2,1,2). \]

\[ = -6 \begin{vmatrix} -1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{vmatrix} \quad \text{swap}(1,2). \text{ Swap changes the sign of the determinant.} \]

\[ = 6 \begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -3 & 2 \end{vmatrix} \quad \text{Multiply rule } m = -1 \text{ on row 1.} \]

\[ = 6 \begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -4 \end{vmatrix} \quad \text{combo}(2,3,-3). \]

\[ = 6(1)(-1)(-4) = 24 \quad \text{Triangular rule. Formula verified.} \]
Elementary Matrices and the Four Rules

The four rules can be stated in terms of elementary matrices as follows.

**Triangular**

The value of \( \det(A) \) for either an upper triangular or a lower triangular matrix \( A \) is the product of the diagonal elements:

\[
\det(A) = a_{11}a_{22}\cdots a_{nn}.
\]

This is a one-arrow Sarrus’ rule valid for dimension \( n \).

**Swap**

If \( E \) is an elementary matrix for a swap rule, then

\[
\det(EA) = (-1) \det(A).
\]

**Combination**

If \( E \) is an elementary matrix for a combination rule, then

\[
\det(EA) = \det(A).
\]

**Multiply**

If \( E \) is an elementary matrix for a multiply rule with multiplier \( m \neq 0 \), then

\[
\det(EA) = m \det(A).
\]

Because \( \det(E) = 1 \) for a combination rule, \( \det(E) = -1 \) for a swap rule and \( \det(E) = c \) for a multiply rule with multiplier \( c \neq 0 \), it follows that for any elementary matrix \( E \) there is the **determinant multiplication rule**

\[
\det(EA) = \det(E) \det(A).
\]
### Special Determinant Rules

The results are stated for rows but also hold for columns, because $\det(A) = \det(A^T)$.

<table>
<thead>
<tr>
<th>Rule</th>
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<tbody>
<tr>
<td>Zero row</td>
<td>If one row of $A$ is zero, then $\det(A) = 0$.</td>
</tr>
<tr>
<td>Duplicate rows</td>
<td>If two rows of $A$ are identical, then $\det(A) = 0$.</td>
</tr>
<tr>
<td>RREF $\neq I$</td>
<td>If $\text{rref}(A) \neq I$, then $\det(A) = 0$.</td>
</tr>
<tr>
<td>Common factor</td>
<td>The relation $\det(A) = c \det(B)$ holds, provided $A$ and $B$ differ only in one row, say row $j$, for which $\text{row}(A, j) = c \text{row}(B, j)$.</td>
</tr>
<tr>
<td>Row linearity</td>
<td>The relation $\det(A) = \det(B) + \det(C)$ holds, provided $A$, $B$ and $C$ differ only in one row, say row $j$, for which $\text{row}(A, j) = \text{row}(B, j) + \text{row}(C, j)$.</td>
</tr>
</tbody>
</table>
Cofactor Expansion for $3 \times 3$ Matrices

This is a review the college algebra topic, where the dimension of $A$ is 3. **Cofactor row expansion** means the following formulas are valid:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(+1) \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(+1) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{21}(-1) \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{22}(+1) \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + a_{23}(-1) \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{31}(+1) \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} + a_{32}(-1) \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} + a_{33}(+1) \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

The formulas expand a $3 \times 3$ determinant in terms of $2 \times 2$ determinants, along a row of $A$. The attached signs $\pm 1$ are called the **checkerboard signs**, to be defined shortly. The $2 \times 2$ determinants are called **minors** of the $3 \times 3$ determinant $|A|$. The checkerboard sign together with a minor is called a **cofactor**.
Cofactor Expansion Illustration

Cofactor expansion formulas are generally used when a row has one or two zeros, making it unnecessary to evaluate one or two of the $2 \times 2$ determinants in the expansion. To illustrate, row 1 cofactor expansion gives

$$
\begin{vmatrix}
3 & 0 & 0 \\
2 & 1 & 7 \\
5 & 4 & 8 \\
\end{vmatrix} = 3(+1) \begin{vmatrix} 1 & 7 \\ 4 & 8 \end{vmatrix} + 0(-1) \begin{vmatrix} 2 & 7 \\ 5 & 8 \end{vmatrix} + 0(+1) \begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}
$$

$$= 3(+1)(8 - 28) + 0 + 0
= -60.
$$

What has been said for rows also applies to columns, due to the transpose formula

$$\det(A) = \det(A^T).$$
Minor

The \((n - 1) \times (n - 1)\) determinant obtained from \(\det(A)\) by striking out row \(i\) and column \(j\) is called the \((i, j)\)–minor of \(A\) and denoted \(\text{minor}(A, i, j)\). Literature might use \(M_{ij}\) for a minor.

Cofactor

The \((i, j)\)–cofactor of \(A\) is \(\text{cof}(A, i, j) = (-1)^{i+j} \text{minor}(A, i, j)\). Multiplicative factor \((-1)^{i+j}\) is called the \textbf{checkerboard sign}, because its value can be determined by counting \textit{plus}, \textit{minus}, \textit{plus}, etc., from location \((1, 1)\) to location \((i, j)\) in any checkerboard fashion.

Expansion of Determinants by Cofactors

\[
\det(A) = \sum_{j=1}^{n} a_{kj} \text{cof}(A, k, j), \quad \det(A) = \sum_{i=1}^{n} a_{i\ell} \text{cof}(A, i, \ell),
\]

(11)

In (11), \(1 \leq k \leq n\), \(1 \leq \ell \leq n\). The first expansion is called a \textbf{cofactor row expansion} and the second is called a \textbf{cofactor column expansion}. The value \(\text{cof}(A, i, j)\) is the cofactor of element \(a_{ij}\) in \(\det(A)\), that is, the checkerboard sign times the minor of \(a_{ij}\).
2 Example (Hybrid Method) Justify by cofactor expansion and the four properties the identity
\[
\begin{vmatrix}
10 & 5 & 0 \\
11 & 5 & a \\
10 & 2 & b \\
\end{vmatrix}
= 5(6a - b).
\]

Solution: Let \( D \) denote the value of the determinant. Then
\[
D = \det \begin{pmatrix}
10 & 5 & 0 \\
11 & 5 & a \\
10 & 2 & b \\
\end{pmatrix}
= \det \begin{pmatrix}
10 & 5 & 0 \\
1 & 0 & a \\
0 & -3 & b \\
\end{pmatrix}
= \det \begin{pmatrix}
0 & 5 & -10a \\
1 & 0 & a \\
0 & -3 & b \\
\end{pmatrix}
= (1)(-1) \det \begin{pmatrix}
5 & -10a \\
-3 & b \\
\end{pmatrix}
= (1)(-1)(5b - 30a)
= 5(6a - b).
\]
3 Example (Cramer’s Rule) Solve by Cramer’s rule the system of equations

\[
\begin{align*}
2x_1 + 3x_2 + x_3 - x_4 &= 1, \\
x_1 + x_2 - x_4 &= -1, \\
3x_2 + x_3 + x_4 &= 3, \\
x_1 + x_3 - x_4 &= 0,
\end{align*}
\]

verifying \( x_1 = 1, \ x_2 = 0, \ x_3 = 1, \ x_4 = 2. \)
Solution: Form the four determinants $\Delta_1, \ldots, \Delta_4$ from the base determinant $\Delta$ as follows:

\[
\Delta = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},
\]

\[
\Delta_1 = \det \begin{pmatrix} 1 & 3 & 1 & -1 \\ -1 & 1 & 0 & -1 \\ 3 & 3 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad \Delta_2 = \det \begin{pmatrix} 2 & 1 & 1 & -1 \\ 1 & -1 & 0 & -1 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{pmatrix},
\]

\[
\Delta_3 = \det \begin{pmatrix} 2 & 3 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 0 & 3 & 3 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad \Delta_4 = \det \begin{pmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & -1 \\ 0 & 3 & 1 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.
\]

Five repetitions of the methods used in the previous examples give the answers $\Delta = -2$, $\Delta_1 = -2$, $\Delta_2 = 0$, $\Delta_3 = -2$, $\Delta_4 = -4$, therefore Cramer’s rule implies the solution

\[
x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta}, \quad x_3 = \frac{\Delta_3}{\Delta}, \quad x_4 = \frac{\Delta_4}{\Delta}.
\]

Then $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 2$. 
Maple Code for Cramer’s Rule

The details of the computation above can be checked in computer algebra system Maple as follows.

```maple
with(linalg):
A:=matrix([ [2, 3, 1, -1], [1, 1, 0, -1], [0, 3, 1, 1], [1, 0, 1, -1]])
Delta:= det(A);
b:=vector([1,-1,3,0]):
B1:=A: col(B1,1):=b:
Delta1:=det(B1);
x[1]:=Delta1/Delta;
```
The Adjugate Matrix

The adjugate \( \text{adj}(A) \) of an \( n \times n \) matrix \( A \) is the transpose of the matrix of cofactors,

\[
\text{adj}(A) = \begin{pmatrix}
\text{cof}(A, 1, 1) & \text{cof}(A, 1, 2) & \cdots & \text{cof}(A, 1, n) \\
\text{cof}(A, 2, 1) & \text{cof}(A, 2, 2) & \cdots & \text{cof}(A, 2, n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cof}(A, n, 1) & \text{cof}(A, n, 2) & \cdots & \text{cof}(A, n, n)
\end{pmatrix}^T.
\]

A cofactor \( \text{cof}(A, i, j) \) is the checkerboard sign \( (-1)^{i+j} \) times the corresponding minor determinant \( \text{minor}(A, i, j) \).

Adjugate of a \( 2 \times 2 \)

\[
\text{adj} \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} = \begin{pmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{pmatrix}
\]

In words: swap the diagonal elements and change the sign of the off–diagonal elements.
Adjugate Formula for the Inverse

For any $n \times n$ matrix

$$A \cdot \text{adj}(A) = \text{adj}(A) \cdot A = \det(A) I.$$  

The equation is valid even if $A$ is not invertible. The relation suggests several ways to find $\det(A)$ from $A$ and $\text{adj}(A)$ with one dot product.

For an invertible matrix $A$, the relation implies $A^{-1} = \text{adj}(A) / \det(A)$:

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix}
\text{cof}(A, 1, 1) & \text{cof}(A, 1, 2) & \cdots & \text{cof}(A, 1, n) \\
\text{cof}(A, 2, 1) & \text{cof}(A, 2, 2) & \cdots & \text{cof}(A, 2, n) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cof}(A, n, 1) & \text{cof}(A, n, 2) & \cdots & \text{cof}(A, n, n)
\end{pmatrix}^T$$
Application: Adjugate Shortcut

Given $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, then we can compute $\text{adj}(A) = \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix}$.

Suppose that we mark some unknown entries in $\text{adj}(A)$ by $\square$ and write $|A|$ for $\det(A)$. Then the formula $A \text{adj}(A) = \text{adj}(A)A = \det(A)I$ becomes

$$
\begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \square & 3 & \square \\ \square & 1 & \square \\ \square & -1 & \square \end{pmatrix} = \begin{pmatrix} \square & 3 & \square \\ \square & 1 & \square \\ \square & -1 & \square \end{pmatrix} \begin{pmatrix} 1 & 3 & -2 \\ -2 & 1 & 4 \\ 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}.
$$

While the second product $\text{adj}(A)A$ contains useless information, the first product gives $\text{row}(A, 2) \text{col}(\text{adj}(A), 2) = \det(A)$. Because the values are known, then $\det(A) = 6 + 1 + 0 = 7$.

Knowing $A$ and $\text{adj}(A)$ gives the value of $\det(A)$ in one dot product.
Theorem 1 (Determinants and Elementary Matrices)
Let $E$ be an $n \times n$ elementary matrix. Then

- Combination: $\det(E) = 1$
- Multiply: $\det(E) = m$ for multiplier $m$.
- Swap: $\det(E) = -1$
- Product: $\det(EX) = \det(E) \det(X)$ for all $n \times n$ matrices $X$.

Theorem 2 (Determinants and Invertible Matrices)
Let $A$ be a given invertible matrix. Then

$$\det(A) = \frac{(-1)^s}{m_1 m_2 \cdots m_r}$$

where $s$ is the number of swap rules applied and $m_1, m_2, \ldots, m_r$ are the nonzero multipliers used in multiply rules when $A$ is reduced to $\text{rref}(A)$. 
Determinant Products

Theorem 3 (Determinant Product Rule)
Let \( A \) and \( B \) be given \( n \times n \) matrices. Then

\[
\det(AB) = \det(A) \det(B).
\]

Proof

Assume one of \( A \) or \( B \) has zero determinant. Then \( \det(A) \det(B) = 0 \). If \( \det(B) = 0 \), then \( Bx = 0 \) has infinitely many solutions, in particular a nonzero solution \( x \). Multiply \( Bx = 0 \) by \( A \), then \( ABx = 0 \) which implies \( AB \) is not invertible. Then the identity \( \det(AB) = \det(A) \det(B) \) holds, because both sides are zero. If \( \det(B) \neq 0 \) but \( \det(A) = 0 \), then there is a nonzero \( y \) with \( Ay = 0 \). Define \( x = AB^{-1}y \). Then \( ABx = Ay = 0 \), with \( x \neq 0 \), which implies the identity holds. This completes the proof when one of \( A \) or \( B \) is not invertible.

Assume \( A, B \) are invertible and then \( C = AB \) is invertible. In particular, \( \text{rref}(A^{-1}) = \text{rref}(B^{-1}) = I \). Write \( I = \text{rref}(A^{-1}) = E_1E_2\cdots E_k A^{-1} \) and \( I = \text{rref}(B^{-1}) = F_1F_2\cdots F_m B^{-1} \) for elementary matrices \( E_i, F_j \). Then

(12) \[
AB = E_1E_2\cdots E_k F_1F_2\cdots F_m.
\]

The theorem follows from repeated application of the basic identity \( \det(EX) = \det(E) \det(X) \) to relation (12), because

\[
\det(A) = \det(E_1) \cdots \det(E_k), \quad \det(B) = \det(F_1) \cdots \det(F_m).
\]