How to Solve Linear Differential Equations

- Atoms
- Independence of Atoms
- Construction of the General Solution from a List of Distinct Atoms
- Euler’s Theorem
- The Atom List and Euler’s Method
- Explanation of Euler’s Method
- Main Theorems on Atoms and Linear Differential Equations
Atoms

An atom is a term with coefficient 1 obtained by taking the real and imaginary parts of

\[ x^j e^{ax} (\cos cx + i \sin cx), \quad j = 0, 1, 2, \ldots, \]

where \( a \) and \( c \) represent real numbers and \( c \geq 0 \). By definition, zero is not an atom.

Details and Remarks

- Euler’s formula \( e^{i\theta} = \cos \theta + i \sin \theta \) implies that an atom is constructed from the expression \( x^j e^{zx} \) where \( z = a + ic \).
- An atom is a term of one of the following types:

\[ x^n, \quad x^n e^{ax}, \quad x^n e^{ax} \cos bx, \quad x^n e^{ax} \sin bx. \]

The symbol \( n \) is an integer 0, 1, 2, \ldots and \( a, b \) are real numbers with \( b > 0 \).
- In particular, the powers 1, \( x, x^2, \ldots, x^k \) are atoms.
- The term that makes up an atom has coefficient 1, therefore \( 2e^x \) is not an atom, but the 2 can be stripped off to create the atom \( e^x \). Linear combinations like \( 2x + 3x^2 \) are not atoms, but the individual terms \( x \) and \( x^2 \) are indeed atoms. Terms like \( e^{x^2}, \ln |x| \) and \( x/(1 + x^2) \) are not atoms, nor are they constructed from atoms.
Independence

Linear algebra defines a list of functions $f_1, \ldots, f_k$ to be **linearly independent** if and only if the representation of the zero function as a linear combination of the listed functions is uniquely represented, that is,

$$0 = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x)$$

for all $x$ implies $c_1 = c_2 = \cdots = c_k = 0$.

Independence and Atoms

**Theorem 1 (Atoms are Independent)**
A list of finitely many distinct atoms is linearly independent.

**Theorem 2 (Powers are Independent)**
The list of distinct atoms $1, x, x^2, \ldots, x^k$ is linearly independent. And all of its sublists are linearly independent.
Construction of the General Solution from a List of Distinct Atoms

- **Picard’s theorem** says that the homogeneous constant-coefficient linear differential equation

\[ y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y = 0 \]

has solution space \( S \) of dimension \( n \). Picard’s theorem reduces the general solution problem to finding \( n \) linearly independent solutions.

- **Euler’s theorem infra** says that the required \( n \) independent solutions can be taken to be atoms. The theorem explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation [characteristic polynomial=left side]

\[ r^n + p_{n-1}r^{n-1} + \cdots + p_1r + p_0 = 0. \]

- The **Fundamental Theorem of Algebra** states that there are exactly \( n \) roots \( r \), real or complex, for an \( n \)th order polynomial equation. The result implies that the characteristic equation has exactly \( n \) roots, counting multiplicities.

- **General Solution.** Because the list of atoms constructed by Euler’s theorem has \( n \) distinct elements, then this list of independent atoms forms a **basis** for the general solution of the differential equation, giving

\[ y = c_1(\text{atom 1}) + \cdots + c_n(\text{atom } n). \]

Symbols \( c_1, \ldots, c_n \) are arbitrary coefficients. In particular, each atom listed is itself a solution of the differential equation.
Euler’s Theorem

Theorem 3 (L. Euler)
The function $y = x^j e^{r_1 x}$ is a solution of a constant-coefficient linear homogeneous differential of the $n$th order if and only if $(r - r_1)^{j+1}$ divides the characteristic polynomial.

The Atom List

1. If $r_1$ is a real root, then the atom list for $r_1$ begins with $e^{r_1 x}$. The revised atom list is

$$e^{r_1 x}, x e^{r_1 x}, \ldots, x^{k-1} e^{r_1 x}$$

provided $r_1$ is a root of multiplicity $k$. This means that factor $(r - r_1)^k$ divides the characteristic polynomial, but factor $(r - r_1)^{k+1}$ does not.

2. If $r_1 = \alpha + i\beta$, with $\beta > 0$ and its conjugate $r_2 = \alpha - i\beta$ are roots of the characteristic equation, then the atom list for this pair of roots (both $r_1$ and $r_2$ counted) begins with

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x.$$ 

For a root of multiplicity $k$, these real atoms are multiplied by atoms $1, \ldots, x^{k-1}$ to obtain a list of $2k$ atoms

$$e^{\alpha x} \cos \beta x, \quad x e^{\alpha x} \cos \beta x, \quad \ldots, \quad x^{k-1} e^{\alpha x} \cos \beta x,$$

$$e^{\alpha x} \sin \beta x, \quad x e^{\alpha x} \sin \beta x, \quad \ldots, \quad x^{k-1} e^{\alpha x} \sin \beta x.$$
Explanation of steps 1 and 2

1. Root $r_1$ always produces atom $e^{r_1 x}$, but if the multiplicity is $k > 1$, then $e^{r_1 x}$ is multiplied by the list of atoms $1, x, \ldots, x^{k-1}$.

2. The expected first terms $e^{r_1 x}$ and $e^{r_2 x}$ [$e^{\alpha x + i\beta x}$ and $e^{\alpha x - i\beta x}$] are not atoms, but they are linear combinations of atoms:

$$e^{\alpha x \pm i\beta x} = e^{\alpha x} \cos \beta x \pm ie^{\alpha x} \sin \beta x.$$ 

The atom list for a complex conjugate pair of roots $r_1 = \alpha + i\beta, r_2 = \alpha - i\beta$ is obtained by multiplying the two real atoms

$$e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x$$

by the powers

$$1, x, \ldots, x^{k-1}$$

to obtain the $2k$ distinct real atoms in item 2 above.
Theorem 4 (Homogeneous Solution $y_h$ and Atoms)
Linear homogeneous differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of atoms.

Theorem 5 (Particular Solution $y_p$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients a having forcing term $f(x)$ equal to a linear combination of atoms has a particular solution $y_p(x)$ which is a linear combination of atoms.

Theorem 6 (General Solution $y$ and Atoms)
A linear non-homogeneous differential equation with constant coefficients having forcing term

$$f(x) = \text{a linear combination of atoms}$$

has general solution

$$y(x) = y_h(x) + y_p(x) = \text{a linear combination of atoms}.$$ 

Proofs
The first theorem follows from Picard’s theorem, Euler’s theorem and independence of atoms. The second follows from the method of undetermined coefficients, infra. The third theorem follows from the first two.