Introduction to Linear Algebra 2270-3
Sample Midterm Exam 3 Fall 2008
Exam Date: Wednesday, 2 December 2008

Instructions. The exam is 50 minutes. Calculators are not allowed. Books and notes are not allowed. The sample exam has too many problems by perhaps a factor of three. Expect midterm 3 to have only problem types selected from the ones represented here.

1. (Orthogonality, Gram-Schmidt) Complete all.

(1a) [25%] Find the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto $V = \textbf{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

(1b) [25%] Find the QR-factorization of $A = \begin{pmatrix} 1 & 0 \\ 7 & 7 \\ 1 & 2 \end{pmatrix}$.

(1c) [25%] Prove that the product AB of two orthogonal matrices A and B is again orthogonal.

(1d) [25%] Fit $c_0 + c_1 x$ to the data points $(0, 2)$, $(1, 0)$, $(2, 1)$, $(3, 1)$ using least squares. Sketch the solution and the data points as an answer check. This is a 2×2 system problem, and should take only 1-3 minutes to complete.

(1e) [25%] Prove that $\textbf{im}(A^T B^T) = \textbf{ker}(BA)^{\perp}$, when matrix product BA is defined.

(1f) [25%] Prove that the span of the Gram-Schmidt vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ equals exactly the span of the independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ used to construct them.

2. (Determinants) Complete two.

(2a) [25%] Given a 7×7 matrix A with each entry either a zero or a one, then what is the least number of zero entries possible such that A is invertible?

(2b) [25%] Find A^{-1} by two methods: the classical adjoint method and the **rref** method applied to $\textbf{aug}(A, I)$:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

(2c) [25%] Let 4×4 matrix A be invertible and assume $\textbf{rref}(A) = E_3 E_2 E_2 A$. The elementary matrices E_1, E_2, E_3 represent `combo(1,3,-15)`, `swap(1,4)`, `mult(2,-1/4)`, respectively. Find $\det(A)$.

(2c) [25%] Assume given 3×3 matrices A, B. Suppose $E_5 E_4 B = E_3 E_2 E_1 A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a combination, a multiply by 3, a swap and a multiply by 7. Assume $\det(A) = 5$. Find $\det(5A^2 B)$.

(2c) [25%] Let 3×3 matrix A be invertible and assume $\textbf{rref}(A) = E_3 E_2 E_2 A$. The elementary matrices E_1, E_2, E_3 represent `combo(1,3,-5)`, `swap(1,3)`, `mult(2,-2)`, respectively. Find $\det(A^T A^2)$.

(2d) [25%] Let $C + B^2 + BA = A^2 + AB$. Assume $\det(A - B) = 4$ and $\det(C) = 5$. Find $\det(CA + CB)$.

(2e) [25%] Determine all values of x for which A^{-1} fails to exist: $A = \begin{pmatrix} 1 & 2x-1 & 0 \\ 2 & 3 & 0 \\ 5x & -44x & 64x^2 \end{pmatrix}$.

(25) [25%] Apply the adjugate formula for the inverse to find the value of the entry in row 2, column 3 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(2g) [25%] Let B be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Let A be a matrix such that $BC^T(A^2C^3 + B^2CA^T) = 0$. Find all possible values of $\det(A)$.
Notation: X^T is the transpose of X. And X^2 means XX.

$$B = \begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 8 & ? & 4 & 0 \\ ? & ? & -4 & 0 \\ -4 & ? & 8 & ? \\ ? & -3 & ? & ? \end{pmatrix}$$

(2h) [25%] Solve for z in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 4 \\ 5 & 6 & 8 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$