1. (Matrices, bases and independence)

(a) Prove that the column positions of leading ones in \( \text{rref}(A) \) identify columns of \( A \) which form a basis for \( \text{im}(A) \).

(b) Find a basis for the image of any invertible \( n \times n \) matrix.

(c) Let \( T \) be the linear transformation on \( \mathbb{R}^3 \) defined by mapping the columns of the identity respectively into three independent vectors \( v_1, v_2, v_3 \). Define \( u_1 = v_1 + 2v_3, u_2 = v_1 + 3v_2, u_3 = v_2 + 4v_3 \). Verify that \( B = \{u_1, u_2, u_3\} \) is a basis for \( \mathbb{R}^3 \) and report the \( B \)-matrix of \( T \) (Otto Bretscher 3E, page 142).
2. (Kernel and similarity)

(a) Prove or disprove: $AB = I$ with $A$, $B$ possibly non-square implies $\ker(A) = \{0\}$.
(b) Prove or disprove: $\ker(\text{rref}(BA)) = \ker(A)$, for all invertible matrices $B$.
(c) Prove or disprove: $\text{im}(\text{rref}(BA)) = \text{im}(A)$, for all invertible matrices $B$.
(d) Prove or disprove: Similar matrices $A$ and $B$ satisfy $\text{nullity}(A) = \text{nullity}(B)$.
3. (Independence and bases)
   (a) Let $A$ be an $n \times m$ matrix. Report a condition on $A$ such that all possible finite sets of independent vectors $v_1, \ldots, v_k$ are mapped by $A$ into independent vectors $Av_1, \ldots, Av_k$. Prove that any matrix $A$ satisfying the condition maps independent sets into independent sets.

   (b) Let $V$ be the vector space of all polynomials $c_0 + c_1x + c_2x^2$ under function addition and scalar multiplication. Prove that $1 - x$, $2x + 1$, $(x - 1)^2$ form a basis of $V$. 

Please staple this page to the front of your submitted exam problem 3.
4. (Linear transformations)
(a) Let $L$ be a line through the origin in $\mathbb{R}^3$ with unit direction $u$. Let $T$ be a reflection through $L$. Define $T$ precisely. Compute and display its representation matrix $A$, i.e., the unique matrix $A$ such that $T(x) = Ax$.

(b) Let $T$ be a linear transformation from $\mathbb{R}^n$ into $\mathbb{R}^m$. Given a basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$, let $A$ be the matrix whose columns are $T(v_1), \ldots, T(v_n)$. Prove that $T(x) = Ax$.

(c) Consider the equations

$$
I = \frac{1}{3}(R + G + B) \\
L = R - G \\
S = B - \frac{1}{2}(R + G).
$$

On page 94 of Otto Bretscher 3E, these equations are discussed as representing the intensity $I$, long-wave signal $L$ and short-wave signal $S$ in terms of the amounts $R, G, B$ of red, green and blue light. Submit all parts of problem 86, page 94.

In the last part 86d, let $T$ be the eye-brain transformation with matrix $M$ and let $T_1$ be the transformation in 86a, having matrix $P$. Otto wants $T_1T$ to be the sunglass-eye-brain composite transformation of 86c. This explains why 86c and 86d are different questions. A class discussion will help to clarify the Bretscher statement of the problem.
5. (Vector spaces)
   
   (a) Show that the set of all $5 \times 4$ matrices $A$ which have exactly one element equal to 1, and all other elements zero, form a basis for the vector space of all $5 \times 4$ matrices.

   (b) Let $W$ be the set of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let $V$ be the set of all polynomials spanned by $1, x, x^2, x^3, x^4$. Assume $W$ is known to be a vector space. Prove that $V$ is a subspace of $W$.