Applied Differential Equations 2250
Sample Final Exam Chapters 8, 9 and 10

Exam date: Monday, 15 Dec, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. The sample exam has extra problems to show different problem types. On exam day, the problems will be shortened to fit into the 120-minute final exam time.

1. **(ch8)** Complete enough of the following to add to 100%.

(8a) [100%] Find the fundamental matrix e^{At} and report the solution $\mathbf{u} = e^{At}\mathbf{u}(0)$ for the initial value problem

$$\mathbf{u}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u}(t), \quad \mathbf{u}(0) = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

(8b) [100%] Solve for the general solution $\mathbf{u} = \mathbf{u}_h + \mathbf{u}_p$, finding the particular solution $\mathbf{u}_p$ by variation of parameters

$$\mathbf{u}_p(t) = e^{At} \int_0^t e^{-Au}\mathbf{F}(u)du,$$

for the special 2×2 system

$$\mathbf{u}'(t) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{u}(t) + \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

(8c) [50%] Find e^{At} by the Laplace resolvent method $\mathcal{L}(e^{At}) = (sI - A)^{-1}$ for the 2×2 system

$$\mathbf{u}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u}(t)$$

(8d) [50%] Find e^{At} by Putzer's formula

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$$

for the 2×2 system

$$\mathbf{u}'(t) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{u}(t).$$

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2. **(ch9)** Complete enough of the following to add to 100%.

(9a) [50%] Determine whether the equilibrium $\mathbf{u} = \mathbf{0}$ is stable or unstable. Then classify the equilibrium point as a saddle, center, spiral or node.

$$(1) \quad \mathbf{u}' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \mathbf{u}$$

$$(2) \quad \mathbf{u}' = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix} \mathbf{u}$$

$$(3) \quad \mathbf{u}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \mathbf{u}$$

$$(4) \quad \mathbf{u}' = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \mathbf{u}$$

Answers: unstable node, unstable saddle, stable spiral, stable center.

(9b) [50%] Find the equilibrium points of the nonlinear system and determine, via the linearized system, the stability of each.

$$\begin{aligned} x' &= xy - 2, \\ y' &= x - 2y. \end{aligned}$$

Answer: $(2, 1)$ and $(-2, -1)$ are the equilibria. Stability is determined by the eigenvalues of the linearization

$$\mathbf{u}' = \begin{pmatrix} y & x \\ 1 & -2 \end{pmatrix} \mathbf{u}$$

at each equilibria. Then $(2, 1)$ is an unstable saddle and $(-2, -1)$ is a stable spiral.

(9c) [50%] Identify the predator and the prey variables in the predator-prey system. Find the equilibrium points and identify the unique equilibrium which corresponds to coexistence with periodic populations oscillating about the two carrying capacities.

$$\begin{aligned} x' &= 0.005x(40 - y), \\ y' &= 0.01y(-50 + x). \end{aligned}$$

Answer: Because removal of the interaction terms (those containing xy) gives a growth equation $x' = 0.2x$ and a decay equation $y' = -0.5y$, then x is the prey and y is the predator. The equilibria are $(0, 0)$ and $(50, 40)$. The carrying capacities $x = 50$, $y = 40$ are from the second equilibrium, which which corresponds to coexistence of the predator and prey.

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3. **(ch10)** Complete enough of the following to add to 100%. These are sample midterm 3 problems, plus some new problem types added from 10.4, 10.5. Delta functions appear only in the dailies, not on exams.

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(4a) [20%] Apply Laplace's method to solve the system. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [10%]. Solve it for $\mathcal{L}(x)$, $L(y)$ [10%]. Find formulas for $x(t)$, $y(t)$ [10%].

$$\begin{aligned} &x' = 3y, \\ &y' = 2x - y, \\ &x(0) = 0, \quad y(0) = 1. \end{aligned}$$

Answer: $x(t) = -3/5\,e^{-3t} + 3/5\,e^{2t}, y(t) = 3/5\,e^{-3t} + 2/5\,e^{2t}$

(4a) [20%] Apply Laplace's resolvent method $L(\mathbf{u}) = (sI - A)^{-1}\mathbf{u}(0)$ to solve the system $\mathbf{u}' = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$. Find explicit formulas for the components $x(t)$, $y(t)$ of the 2-vector $\mathbf{u}(t)$.

$$\begin{aligned} x'(t) &= 3x(t) - y(t), \\ y'(t) &= x(t) + y(t), \\ x(0) &= 0, \\ y(0) &= 2. \end{aligned}$$

Maple answer check:

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with(LinearAlgebra): A:=Matrix([[3,-1],[1,1]]);u0:=Vector([0,2]);
Lu:=(s*IdentityMatrix(2)-A)^(-1).u0; map(inttrans[invlaplace],Lu,s,t);
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Answer: $x(t) = -2\,te^{2t}$, $y(t) = -2\,(t-1)\,e^{2t}$

(4b) [20%] Ch10(b): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^3 + 30s^2 + 32s + 40}{(s+2)^2(s^2+4)}.$$

(4c) [20%] Ch10(c): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \frac{d}{ds}\left(\mathcal{L}\left(t^2 e^{3t}\right)\Big|_{s \to (s+3)}\right).$$

(4d) [20%] Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+1}{s+2}\right)^2 \frac{1}{(s+2)^2}$$

(5a) [20%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 3x'(t) = 9e^{-3t}, \quad x(0) = x'(0) = 0.$$

(5b) [20%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$\frac{d^4x}{dt^4} + 4\frac{d^2x}{dt^2} = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(5c) [20%] Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(2t)\dfrac{\sin(t)}{t}$.

(5d) [20%] Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\dfrac{\sin(t)}{t}$, where u is the unit step function.

(5e) [20%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$t\cos t$	t^2e^{2t}
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{s+2}$	$\frac{s+1}{s^2+2s+5}$		

(5f) [30%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \frac{s+1}{(s+2)^2} + \frac{2+s}{s^2+5s} + \mathcal{L}(t+\sin t)|_{s\to(s-2)}.$$

(5g) [20%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2-24}{(s-1)(s+3)(s+1)^2}.$$

(5h) [30%] Solve for $f(t)$ using the convolution theorem $\mathcal{L}(f_1(t))\mathcal{L}(f_2(t)) = \mathcal{L}\left(\int_0^t f_1(u)f_2(t-u)du\right)$:

$$\mathcal{L}(f(t)) = \frac{2}{(s-1)(s^2+4)}.$$

Answer: $\frac{2}{5}e^t - \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$.

(5i) [40%] Let $f(t)$ equal the pulse defined by t on $1 \le t < 2$ and zero elsewhere. Find $\mathcal{L}(f(t))$.

Answer: Write $f(t) = t(u(t-1) - u(t-2))$ where u is the unit step. Then apply the second shifting theorem $\mathcal{L}(u(t-a)g(t-a)) = e^{-as}\mathcal{L}(g(t))$.

(5j) [40%] Let $f(t)$ equal the half-wave rectification of $\sin t$, defined by $f(t) = \sin t$ on $0 \le t \le \pi$, $f(t) = 0$ on $\pi < t \le 2\pi$, with $f(t)$ periodic of period 2π. Find $\mathcal{L}(f(t))$.

Answer: Use the periodic function formula $\mathcal{L}(f(t)) = \int_0^T f(t)e^{-st}dt/(1-e^{-sT})$ with period $T = 2\pi$ to obtain $L(f) = 1/((s^2+1)(1-e^{-\pi s}))$.

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