Differential Equations and Linear Algebra

2250-2, F2008 7:30am, 15 December 2008

Instructions. The time allowed is 120 minutes. The examination consists of eight problems, one for each of chapters 3, 4, 5, 6, 7, 8, 9, 10, each problem with multiple parts. A chapter represents 15 minutes on the final exam.

Each problem on the final exam represents several textbook problems numbered (a), (b), (c), $\cdots$. Choose the problems to be graded by check-mark $\boxed{\text{X}}$. The credits should add to 100. Each chapter (3 to 10) adds at most 100 towards the maximum final exam score of 800. There may be replacement problems at reduced credit, in case the problem (a), (b), (c), $\cdots$ cannot be solved. The number of graded problems is fixed. For instance, if ch3(a) and ch3(a_1) are solved, then ch3(a) is ignored and its replacement ch3(a_1) will be graded instead. The final exam grade is reported as a percentage 0 to 100, as follows:

$$\text{Final Exam Grade} = \frac{\text{Sum of scores on eight chapters}}{8}.$$

- Calculators, books, notes and computers are not allowed.
- Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.
- Completely blank pages count 40% or less, at the whim of the grader.
- Answer checks are not expected or required. First drafts are expected, not complete presentations.
- Please submit **exactly eight** separately stapled packages of problems, one package per chapter.

Final Grade. The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\text{Exam Average} = \frac{90 + 91 + 92 + 89 + 89}{5} = 90.2.$$

Dailies count 30% of the final grade. The course average is computed from the formula

$$\text{Course Average} = \frac{70}{100}(\text{Exam Average}) + \frac{30}{100}(\text{Dailies Average}).$$

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Ch3. (Linear Systems and Matrices) Check the boxes ☐ on the **three problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [40%] Ch3(a): Let B be the matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Find the value of $\det(CB)$.

$$B = \begin{pmatrix} -2 & ? & -2 & 3 \\ ? & ? & 1 & -2 \\ ? & 1 & 2 & ? \\ ? & 0 & ? & ? \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

$\det(CB) = 1$

☐ [40%] Ch3(b): Determine which values of k correspond to a unique solution of the system $A\mathbf{x} = \mathbf{b}$ given by

$$A = \begin{pmatrix} 4 & 3 & k \\ 0 & -k-2 & -k-4 \\ 4 & 3 & -4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2k-6 \\ -k-3 \end{pmatrix}.$$

$k \neq -4,\ k \neq -2$

☐ [20%] Ch3(c): Find the value of x_4 by Cramer's Rule in the system $C\mathbf{x} = \mathbf{b}$, given C and $\mathbf{b}$ below. Evaluate 4×4 and 3×3 determinants **only** by the cofactor method.

$$C = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix}$$

$x_4 = \frac{5}{2}$

If you finished Ch3(a), Ch3(b) and Ch3(c), then 100% has been marked – go on to Ch4. **Otherwise,** replace Ch3(a) with Ch3(a_1) and/or Ch3(b) by Ch3(b_1). A maximum of three problems will be graded. There is reduced credit for each replacement.

☐ [30%] Ch3(a_1): Find the adjugate of A and the inverse of A.

$$A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{pmatrix}$$

$\text{adj}(A) = \begin{pmatrix} -12 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

$A^{-1} = \frac{1}{8}\,\text{adj}(A)$

☐ [30%] Ch3(b_1):
Part I [10%]: State the three possibilities for a linear system $A\mathbf{x} = \mathbf{b}$. unique, ∞-many, No sol
Part II [10%]: Give an example of two triangular 2×2 matrices A and B such that AB is **not** triangular.
Part III [10%]: Give an example of a matrix A with 3 rows and 2 columns such that $A\mathbf{x} = \mathbf{0}$ has a unique solution $\mathbf{x}$.

part II $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

part III $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

Staple this page to the top of all Ch3 work. Submit one package per chapter.

ch3(a) $\det(CB) = \det(BC) = \det(B)\,I$. Then $\mathrm{row}(B,3)\,\mathrm{col}(C,3) = \det(B)$, because of $CB = BC = \det(B)\,I$, implies $\det(B) = 1$. Therefore $\boxed{\det(CB) = 1}$

ch3(b) A unique solution occurs $\Leftrightarrow$ A^{-1} exists $\Leftrightarrow$ $\det(A) \neq 0$. Then

$$\begin{vmatrix} 4 & 3 & k \\ 0 & -k-2 & -k-4 \\ 4 & 3 & -4 \end{vmatrix} = \begin{vmatrix} 4 & 3 & k \\ 0 & -k-2 & -k-4 \\ 0 & 0 & -k-4 \end{vmatrix} = (-k-4)(-k-2)(4) \Rightarrow \boxed{k \neq -4,\ k \neq -2}$$

ch3(c) $x_4 = \dfrac{\Delta_4}{\Delta}$, $\Delta = \begin{vmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 2 & 1 \end{vmatrix} = (-1)(4)\begin{vmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 8$

$\boxed{x_4 = \dfrac{5}{2}}$ $\Delta_4 = \begin{vmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 1 & 3 & -2 & 2 \\ 0 & 0 & 2 & 3 \end{vmatrix} = (+1)(1)\begin{vmatrix} 2 & -1 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & 3 \end{vmatrix} = (+1)(1)(2)(12-2) = 20$

ch3(a_1) $\mathrm{adj}(A)$ = transpose of matrix of cofactors $\det(A) = \begin{vmatrix} 0 & 2 & -1 \\ 0 & 0 & 4 \\ 1 & 3 & -2 \end{vmatrix} = (-1)(4)(-2) = 8$

$$= \begin{pmatrix} -12 & 4 & 0 \\ 1 & 1 & 2 \\ 8 & 0 & 0 \end{pmatrix}^T$$

$$= \boxed{\begin{pmatrix} -12 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}}$$

$$\boxed{A^{-1} = \frac{1}{8}\begin{pmatrix} -12 & 1 & 8 \\ 4 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix}}$$

ch3(b_1)

part I. (1) unique sol, (2) ∞-many sols, (3) no sol.

part II. $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$

part III. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$

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Ch4. (Vector Spaces) Check the boxes ☐ on the **four problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [30%] Ch4(a): Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^5$ such that $x_1 + 4x_5 = x_3$ and $x_3 = 0$. Prove that S is a subspace of $\mathcal{R}^5$. $A = \begin{pmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

☐ [10%] Ch4(b): Independence of 4 fixed vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ can be decided by counting the pivot columns of their augmented matrix. State a **different** test which can decide upon independence of four vectors in $\mathcal{R}^n$, where $n > 1$ is an arbitrary integer. Rank Test

☐ [30%] Ch4(c): Consider the four vectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

The subspaces $S_1 = \mathbf{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and $S_2 = \mathbf{span}\{\mathbf{w}_1, \mathbf{w}_2\}$ each have dimension 2 and share a common vector $\mathbf{v}_2 = \mathbf{w}_1$. Explain why $S_1 = S_2$. See attached

☐ [30%] Ch4(d): The 5×5 matrix A below has some independent columns. Report the maximum number of independent columns, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 & 1 \\ 0 & -1 & -2 & 0 & 1 \\ 0 & 6 & 6 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 \end{pmatrix}$$

Max = 3

If you marked Ch4(a) through Ch4(d), then 100% has been marked – go on to Ch5. **Otherwise**, mark replacement problems from the possibilities Ch4(a) → Ch4(a_1), Ch4(c) → Ch4(c_1). A maximum of four problems will be graded. Replacements have reduced credit.

☐ [25%] Ch4(a_1): State the subspace criterion and the kernel theorem, which are the two theorems in Chapter 4 which apply to prove that a given set S in a vector space V is a subspace of V. See attached

☐ [25%] Ch4(c_1): Apply an independence test and report for which values of x the four vectors are **dependent**.

$$\mathbf{v}_1 = \begin{pmatrix} 0 \\ 2 \\ 5 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 2x \\ 3 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} -2 \\ 2 \\ 9 \\ x \end{pmatrix}.$$

$x = 5/2$ or $x = 6$

Staple this page to the top of all Ch4 work. Submit one package per chapter.

Ch 4 (a) Define $A = \begin{pmatrix} 1 & 0 & -1 & 0 & 4 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Then $S = \{\vec{x} : A\vec{x} = \vec{0}\}$ is a subspace by the Kernel Theorem.

Ch 4 (b) v_1, v_2, v_3, v_4 independent $\iff$ rank (aug (v_1, v_2, v_3, v_4)) = 4

Determinant Test does not apply unless $n=4$.

Ch 4 (c) $v_1 - v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = w_2$ implies w_2 is in S_1. Hence w_1, w_2 are in S_1

$v_1 = v_2 + w_2 = w_1 + w_2$ implies v_1 is in S_2. Hence v_1, v_2 are in S_2

$\therefore S_1 = S_2$.

Ch4(d) rref(A) has leading ones in cols 2, 3, 5 $\Rightarrow$ $\boxed{\text{Max} = 3}$

Ch4(a$_1$) Subspace Criterion: S is a subspace of vector space V provided

1. $\vec{0}$ is in S
2. $\vec{x}, \vec{y}$ in $S \Rightarrow \vec{x} + \vec{y}$ in S
3. $\vec{x}$ in S, c = scalar $\Rightarrow c\vec{x}$ in S

Kernel Theorem: $S = \{\vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$, for any $m \times n$ matrix A.

Ch4(g) Test: v_1, v_2, v_3, v_4 in $\mathbb{R}^4$ indep $\iff$ det(aug(v_1, v_2, v_3, v_4)) $\neq 0$

Dependent $\iff$ det $= 0$

$\iff (2x-5)(-4)(x-6) = 0$

$\iff x = 5/2$ or $x = 6$

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Ch5. (Linear Equations of Higher Order) Solve all five problems Ch5(a) through Ch5(e).

☐ [20%] Ch5(a): Report the general solution $y(x)$ of the differential equation

$$5y'' + 26y' + 5y = 0.$$

$y = c_1 e^{-5t} + c_2 e^{-t/5}$

☐ [10%] Ch5(b): Given a damped spring-mass system $mx''(t) + cx'(t) + kx(t) = 0$ with $m = 1$, $c = 4$ and $k = 5$, classify the solution $x(t)$ as **over-damped**, **critically damped** or **under-damped**. Please, **do not solve** the differential equation!

under – damped

☐ [20%] Ch5(c): Find the characteristic equation of a higher order linear homogeneous differential equation with constant coefficients such that $y = x(x + e^{-x}) + 2\cos x$ is a solution.

$r^3 (r+1)^2 (r^2+1) = 0$

☐ [25%] Ch5(d): Determine the general solution $y(x)$ of the homogeneous constant-coefficient differential equation, given it has characteristic equation

$$r^2(r^2 + 2r + 5)^3(r^2 - 3) = 0.$$

10 atoms $= 1, x, e^{\sqrt{3}x}, e^{-\sqrt{3}x}$
$e^{-x}\cos 2x, e^{-x}\sin 2x$
$xe^{-x}\cos 2x, xe^{-x}\sin 2x$
$x^2e^{-x}\cos 2x, x^2e^{-x}\sin 2x$

$y =$ l.c. of 10 atoms

☐ [25%] Ch5(e): A particular solution of the differential equation $x'' + 2x' + 17x = 50\cos(3t)$ is $x(t) = 4\cos 3t + e^{-t}\sin 4t + 3\sin 3t$. Find the the steady-state periodic solution $x_{SS}(t)$.

$x_{ss}(t) = 4\cos 3t + 3\sin 3t$

Staple this page to the top of all Ch5 work. Submit one package per chapter.

Ch 5(a) $5r^2+26r+5=0 \rightarrow (5r+1)(r+5)=0$

$y = c_1 e^{-5t} + c_2 e^{-t/5}$

Ch5(b) $c^2-4mk = 16-20 = -4 < 0 \Rightarrow$ under-damped

Ch5(c) $y = x^2 + xe^{-x} + 2\cos x$

$r^3(r+1)^2(r^2+1) = 0$

Ch5(d) $r^2((r+1)^2+4)^3(r-\sqrt{3})(r+\sqrt{3}) = 0$

atoms = $1, x, e^{-x}\cos 2x, e^{-x}\sin 2x, xe^{-x}\cos 2x, xe^{-x}\sin 2x,$
$x^2e^{-x}\cos 2x, x^2e^{-x}\sin 2x, e^{\sqrt{3}x}, e^{-\sqrt{3}x}$

y = linear combination of these 10 atoms

Ch5(e) Set all transient terms to zero to obtain

$x_{ss}(t) = 4\cos 3t + 3\sin 3t$

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Ch6. (Eigenvalues and Eigenvectors) Check the boxes ☐ on the **three problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [40%] Ch6(a): Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -2 & -11 & 0 & 0 \\ 2 & 0 & -20 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 20 & 1 & 2 \end{pmatrix}$.

$\lambda = 2, 2, 2, 2i, -2i$

To save time, **do not** find eigenvectors!

☐ [30%] Ch6(b): Let A be a 3×3 matrix satisfying Fourier's model

$$A\left(c_1\begin{pmatrix}1\\-2\\0\end{pmatrix} + c_2\begin{pmatrix}-1\\3\\0\end{pmatrix} + c_3\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = 4c_3\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $P = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Find a diagonal matrix D and an invertible matrix P such that $AP = PD$.

☐ [30%] Ch6(c): Find a 2×2 matrix A with eigenpairs

$$\left(5, \begin{pmatrix}1\\2\end{pmatrix}\right), \quad \left(-4, \begin{pmatrix}1\\3\end{pmatrix}\right).$$

$A = PDP^{-1} = \begin{pmatrix} 23 & -9 \\ 54 & -22 \end{pmatrix}$

If you finished Ch6(a), Ch6(b) and Ch6(c), then 100% has been marked – go on to Ch4. **Otherwise,** replace **one** of problems Ch6(a), Ch6(b) or Ch6(c) with Ch6(d_1). A maximum of three problems will be graded. Replacement of Ch6(a) has reduced credit. The other two replacements have equal credit.

☐ [30%] Ch6(d_1): Assume two 3×3 matrices A, B have exactly the same characteristic equation. Let A have eigenvalues 3, 4, 5. Find the eigenvalues of $(1/2)B - 3I$, where I is the identity matrix.

$\lambda = -\frac{3}{2}, -1, -\frac{1}{2}$

Staple this page to the top of all Ch6 work. Submit one package per chapter.

Ch6(a) $\begin{vmatrix} -\lambda & -2 & -11 & 0 & 0 \\ 2 & -\lambda & -20 & 3 & 0 \\ 0 & 0 & 1-\lambda & -1 & 0 \\ 0 & 0 & 1 & 3-\lambda & 0 \\ 0 & 0 & 20 & 1 & 2-\lambda \end{vmatrix} \underset{\text{cofactor on col 5}}{=} (2-\lambda) \begin{vmatrix} -\lambda & -2 & -11 & 0 \\ 2 & -\lambda & -20 & 3 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & 0 & 1 & 3-\lambda \end{vmatrix}$

$$= (2-\lambda)\left[-\lambda \begin{vmatrix} -\lambda & -20 & 3 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{vmatrix} + (-1)2 \begin{vmatrix} -2 & -11 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 1 & 3-\lambda \end{vmatrix} \right]$$

$$= (2-\lambda)\left[-\lambda(-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} + (-2)(-2) \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} \right]$$

$$= (2-\lambda)\left[+\lambda^2 + 4 \right] \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= (2-\lambda)(\lambda^2+4)(\lambda^2 - 4\lambda + 4) \qquad \lambda = 2, 2, 2, 2i, -2i$$

Ch6(b) $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}, P = \begin{pmatrix} 1 & -1 & 1 \\ -2 & 3 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Ch6(c) $D = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix}, P = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \qquad AP = PD \Rightarrow A = PDP^{-1}$

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \frac{1}{1} = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 15 & -5 \\ 8 & -4 \end{pmatrix} = \begin{pmatrix} 23 & -9 \\ 54 & -22 \end{pmatrix}$$

Ch6(d_1) $\det\left(\frac{1}{2}B - 3I - \lambda I\right) = \det\left(\frac{1}{2}I\right)(B - 6I - 2\lambda I) = \det\left(\frac{1}{2}I\right)\det(B - (6+2\lambda)I)$

$= \det\left(\frac{1}{2}I\right)\det(A - (6+2\lambda)I)$ because $\det(A - \mu I) = \det(B - \mu I)$ for all μ. Then

$$6 + 2\lambda = 3, 4, 5$$

$$\Rightarrow \quad 2\lambda = -3, -2, -1$$

$$\Rightarrow \quad \lambda = -3/2, -1, -1/2$$

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Ch7. (Linear Systems of Differential Equations) Check the boxes ☐ on the **three problems** to be graded, which is 100%. Label worked problems accordingly.

☐ [40%] Ch7(a): Solve for the general solution $x(t)$, $y(t)$ in the system below. Use any method that applies, from the lectures or any chapter of the textbook.

$$\frac{dx}{dt} = x + 3y,$$
$$\frac{dy}{dt} = 18x + 4y.$$

$x(t) = c_1 e^{10t} + c_2 e^{-5t}$
$y(t) = 3c_1 e^{10t} - 2c_2 e^{-5t}$

☐ [40%] Ch7(b): Apply the eigenanalysis method to solve the differential system $\mathbf{u}' = A\mathbf{u}$, given

$$A = \begin{pmatrix} 1 & -4 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$$

$\vec{u} = c_1 e^{3t}\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

The term *eigenanalysis* refers to the process of finding eigenvalues and eigenvectors of the matrix A.

☐ [20%] Ch7(c): Assume A is a 3×3 matrix and the general solution of $\mathbf{u}' = A\mathbf{u}$ is given by

$$\mathbf{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

$A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}^{-1}$

Display a matrix-multiply formula for A [don't multiply it out].

If you marked Ch7(a), Ch7(b) and Ch7(c), then 100% has been marked – go on to Ch8. **Otherwise**, you may replace problem Ch7(c) by Ch7(c_1). A maximum of three problems will be graded. The replacement has reduced credit.

☐ [15%] Ch7(c_1): A 3×3 real matrix A has all eigenvalues equal to 0 and corresponding eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Display the general solution of the differential equation $\mathbf{u}' = A\mathbf{u}$.

$\vec{u} = c_1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Staple this page to the top of all Ch7 work. Submit one package per chapter.

Ch 7(a) $\quad A = \begin{pmatrix} 1 & 3 \\ 18 & 4 \end{pmatrix}$ $\quad$ Eigenvalues = roots of $\lambda^2 - 5\lambda - 50 = 0$

$= 10, -5$

By Cayley-Hamilton Shortcut, $x(t)$ = l.c. of atoms e^{10t}, e^{-5t}

$$x = c_1 e^{10t} + c_2 e^{-5t}$$

$$y = \frac{x' - x}{3} \quad \text{from the first DE}$$

$$y = (10c_1 e^{10t} - 5c_2 e^{-5t} - c_1 e^{10t} - c_2 e^{-5t})/3$$

$$y = 3c_1 e^{10t} - 2 e^{-5t}$$

Ch 7(b) $\begin{vmatrix} 1-\lambda & -4 & 2 \\ 0 & -1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda) \begin{vmatrix} 1-\lambda & -4 \\ 0 & -1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)(-1-\lambda)$

eigenvalues are $\lambda = 3, 1, -1$

$A - 3I = \begin{pmatrix} -2 & -4 & 2 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = 0,\ x_2 = \frac{1}{2}t_1,\ x_3 = t_1.$ $\quad v_1 = \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}$

$A - 1 \cdot I = \begin{pmatrix} 0 & -4 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & -4 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = t_1,\ x_2 = 0,\ x_3 = 0$ $\quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$A - (-1)I = \begin{pmatrix} 2 & -4 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 2 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = 2t_1,\ x_2 = t_1,\ x_3 = 0$ $\quad v_3 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$\vec{u}' = A\vec{u}$ has sol $\quad \vec{u} = c_1 e^{3t}\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} + c_2 e^{t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{-t}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Ch 7(c) $\quad AP = PD \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$A = PDP^{-1}$

Ch 7(c_1) $\quad \vec{u} = c_1 e^{0t}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{0t}\begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} + c_3 e^{0t}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

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Ch8. (Matrix Exponential) Complete both problems.

☐ [50%] Ch8(a): Display the matrix exponential for the 2×2 system [3/4 credit]. Use any method in the lectures or the textbook to find the matrix exponential. Then go on to solve the system for $\mathbf{u}$ in its matrix form $\mathbf{u}' = A\mathbf{u}$.

$$x' = 4x,$$
$$y' = y,$$
$$x(0) = 2, \quad y(0) = 1.$$

$$e^{At} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{t} \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} 2e^{4t} \\ e^{t} \end{pmatrix}$$

☐ [50%] Ch8(b): Display the matrix form of variation of parameters for the 2×2 system [3/4 credit]. Then find a particular solution.

$$x' = 4x + 4,$$
$$y' = y.$$

$$\vec{u} = e^{At} \int_0^t e^{-Au} \vec{F}(u)\,du, \qquad \vec{F}(u) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

$$\vec{u} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{t} \end{pmatrix} \int_0^t \begin{pmatrix} e^{-4u} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 4 \\ 0 \end{pmatrix} du$$

$$= \begin{pmatrix} -1 + e^{4t} \\ 0 \end{pmatrix}$$

Staple this page to the top of all Ch8 work. Submit one package per chapter.

Ch 8(a) $x = c_1 e^{4t}$ by Ch1 methods $\Rightarrow e^{At} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}$

$y = c_2 e^t$

$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad e^{At} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}$

can also solve $\mathcal{L}(\Phi) = (sI - A)^{-1}$ for $\Phi = e^{At} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}$

$\vec{u}(t) = e^{At}\vec{u}(0)$

$\vec{u}(t) = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{4t} \\ e^t \end{pmatrix}$

Ch 8(b)

$$\vec{u} = e^{At}\int_0^t e^{-Au}\vec{F}(u)\,du$$

$e^{At} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}$ from Ch 8(a)

$$\vec{u} = \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}\int_0^t \begin{pmatrix} e^{-4u} & 0 \\ 0 & e^{-u} \end{pmatrix}\begin{pmatrix} 4 \\ 0 \end{pmatrix} du$$

$$= \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}\int_0^t \begin{pmatrix} 4e^{-4u} \\ 0 \end{pmatrix} du$$

$$= \begin{pmatrix} e^{4t} & 0 \\ 0 & e^t \end{pmatrix}\begin{pmatrix} \frac{4e^{-4t}}{-4} - \frac{4}{-4} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 + e^{4t} \\ 0 \end{pmatrix}$$

chk: $x = -1 + e^{4t} \rightarrow x' = 4e^{4t} = 4(x+1)$

$y = 0 \longrightarrow y' = 0 = y$

Don't check initial data, only that it is a sol.

Name KEY 7:30am

Differential Equations and Linear Algebra 2250-2, F2008

Final Exam 7:30am, 15 December 2008

Ch9. (Nonlinear Systems) Complete both problems.

☐ [50%] Ch9(a): Determine whether the equilibrium $\mathbf{u} = \mathbf{0}$ is stable or unstable [1/2 credit]. Then classify the equilibrium point as a saddle, center, spiral or node.

$$\mathbf{u}' = \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix} \mathbf{u}$$

asymptotically stable
spiral

☐ [50%] Ch9(b): Find the equilibrium points of the nonlinear system [1/4 credit]. Classify from the linearized system the equilibrium point $x = -1$, $y = -2$ as a saddle, center, spiral or node for the nonlinear system.

$$\begin{aligned} x' &= y - 2x, \\ y' &= xy - 2. \end{aligned}$$

equilibria = $(1,2), (-1,-2)$
Eigenvalues $-\frac{3}{2} \pm \frac{\sqrt{7}}{2} i$
asymptotically stable
spiral for nonlinear DE

Staple this page to the top of all Ch9 work. Submit one package per chapter.

Ch 9(a) $A = \begin{pmatrix} -2 & -2 \\ 2 & -2 \end{pmatrix}$ $\lambda^2 + 4\lambda + 8 = 0$ $(\lambda+2)^2 + 4 = 0$

$\lambda = -2 \pm 2i$

Eigenvalues have negative real parts $\Longrightarrow$ asymp. stable

classify as a stable spiral (attractor).

Ch 9(b) $0 = y - 2x$ $\Longrightarrow$ $0 = x(2x) - 2$ $\Longrightarrow$ $\begin{matrix} x = 1 \\ y = 2 \end{matrix}$ $\begin{matrix} x = -1 \\ y = -2 \end{matrix}$

$0 = xy - 2$

$(1,2)$ and $(-1,-2)$ are the equilibrium points

Jacobian $= \text{aug}\left(\frac{\partial \vec{F}}{\partial x} \;\; \frac{\partial \vec{F}}{\partial y} \right)$ $\vec{F} = \begin{pmatrix} y - 2x \\ xy - 2 \end{pmatrix}$

$$J = \begin{pmatrix} -2 & 1 \\ y & x \end{pmatrix}$$

$$J(-1,-2) = \begin{pmatrix} -2 & 1 \\ -2 & -1 \end{pmatrix}$$

$\lambda^2 + 3\lambda + 4 = 0$

$\lambda = -\frac{3}{2} \pm \frac{\sqrt{7}}{2} i$

spiral for linearized $\Longrightarrow$ spiral for nonlinear

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Differential Equations and Linear Algebra 2250-2, F2008

Final Exam 7:30am, 15 December 2008

Ch10. (Laplace Transform Methods) Check the boxes $\square$ on the **four problems** to be graded, which is 100%. Label worked problems accordingly.

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 8 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

Ch10(a) [10%]: State five Laplace rules. None of these can be table entries. For example, one of them could be the shifting rule. See attached

Ch10(b) [20%]: Show how to solve for $f(t)$ using the convolution theorem. To save time, **don't evaluate any integrals.** See attached

$$\mathcal{L}(f(t)) = \frac{4}{(s+5)(s^2+16)}.$$

Ch10(c) [20%]: Let $f(t) = |\sin t|$ on $0 \le t \le 2\pi$, with $f(t)$ periodic of period 2π. Display the formula for $\mathcal{L}(f(t))$ according to the periodic function rule. To save time, **don't evaluate any integrals.** See attached

Ch10(d) [50%]: Apply Laplace's resolvent method $L(\mathbf{u}) = (sI - A)^{-1}\mathbf{u}(0)$ to solve the system $\mathbf{u}' = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$. Find explicit formulas for the components $x(t)$, $y(t)$ of the 2-vector $\mathbf{u}(t)$.

$$\begin{aligned} x'(t) &= 2x(t) + 5y(t), \\ y'(t) &= 2y(t), \\ x(0) &= 0, \\ y(0) &= 2. \end{aligned}$$

$x = 10te^{2t}$
$y = 2e^{2t}$

If you finished Ch10(a), Ch10(b), Ch10(c) and Ch10(d), then 100% has been marked – the exam is finished. **Otherwise**, replace Ch10(d) with Ch10(d_1). A maximum of four problems will be graded.

Ch10(d_1) [50%]: Solve by Laplace's method for the solution $x(t)$:

$$x''(t) - 3x'(t) = 9e^{3t}, \quad x(0) = x'(0) = 0.$$

$x(t) = (-1)e^{3t} + (3)te^{3t} + 1$

Staple this page to the top of all Ch10 work. Submit one package per chapter.

ch10(a) $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ parts

$\mathcal{L}(e^{at}f(t)) = \mathcal{L}(f(t))\,|_{s\to s-a}$ shift

$\mathcal{L}(-t\,f(t)) = \frac{d}{ds}\mathcal{L}(f(t))$ s-drift

$\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$ linearity

$\mathcal{L}(cf) = c\,\mathcal{L}(f)$

$\mathcal{L}(f)\mathcal{L}(g) = \mathcal{L}(f*g)$ convolution

$\mathcal{L}(f) = \int_0^T f(t)e^{-st}dt \big/ (1-e^{-Ts})$ periodic

ch 10(b)

$$\mathcal{L}(f) = \frac{1}{s+5}\;\frac{4}{s^2+16}$$

$$= \mathcal{L}(e^{-5t})\,\mathcal{L}(\sin 4t)$$

$$= \mathcal{L}\left(\int_0^t e^{-5(t-u)}\sin(4u)\,du\right)$$

$$\Rightarrow f(t) = \int_0^t e^{-5(t-u)}\sin(4u)\,du$$

ch10(c) $$\mathcal{L}(f) = \frac{\int_0^{2\pi}|\sin t|e^{-st}dt}{1-e^{-2\pi s}}$$

ch10(d) $$(sI-A)^{-1} = \begin{pmatrix} s-2 & -5 \\ 0 & s-2\end{pmatrix}^{-1} = \begin{pmatrix} s-2 & 5 \\ 0 & s-2\end{pmatrix}\frac{1}{\Delta}, \quad \Delta = (s-2)^2$$

$u(0) = \begin{pmatrix}0\\2\end{pmatrix}$

$$\mathcal{L}(\vec{u}) = \frac{1}{\Delta}\begin{pmatrix} s-2 & 5 \\ 0 & s-2\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}\frac{10}{\Delta}\\ \frac{2(s-2)}{\Delta}\end{pmatrix} = \begin{pmatrix}\frac{10}{(s-2)^2}\\ \frac{2}{s-2}\end{pmatrix}$$

$$= \begin{pmatrix}\mathcal{L}(10te^{2t})\\ \mathcal{L}(2e^{2t})\end{pmatrix} \Rightarrow \boxed{\begin{aligned} x &= 10te^{2t}\\ y &= 2e^{2t}\end{aligned}}$$

ch10(d$_1$) $$\mathcal{L}(x) = \frac{\mathcal{L}(9e^{3t})}{s^2-3s} = \frac{9}{(s-3)^2 s} = \frac{a}{s-3} + \frac{b}{(s-3)^2} + \frac{c}{s}$$

$$= \mathcal{L}(ae^{3t} + bte^{3t} + c)$$

$$x = ae^{3t} + bte^{3t} + c$$

$a = -1 \quad b = 3 \quad c = 1 \leftarrow$ by modified Heaviside

$0 = a + c$