Applied Differential Equations 2250
Sample Exam 3

Exam date: Thursday, 4 Dec, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. The sample exam has extra problems to show different problem types. On exam day, the problems will be shortened to fit into the 50-minute exam time: approximately 10 minutes for each of the five problems.

1. (ch4) Complete enough of the following to add to 100%.

   (1a) [100%] Let $V$ be the vector space of all continuous functions defined on $0 \leq x \leq 1$. Define $S$ to be the set of all twice-continuously differentiable functions $f(x)$ in $V$ such that $f'(0) = f(0)$ and $f''(x) + 3f'(x) + 2f(x) = 0$. Prove that $S$ is a subspace of $V$.

   (1b) [50%] If you solved (a), then skip (b) and (c). Let $V$ be the set of all $4 \times 1$ column vectors $\vec{x}$ with components $x_1, x_2, x_3, x_4$. Assume the usual $\mathbb{R}^4$ rules for addition and scalar multiplication. Let $S$ be the subset of $V$ defined by the equations

   $\begin{align*}
   x_1 + x_2 &= 0, \\
   x_3 &= x_4,
   \end{align*}$

   $\begin{bmatrix}
   1 & 0 & -1 & -2 \\
   0 & 0 & 2 & -2
   \end{bmatrix}
   \begin{bmatrix}
   x_1 \\
   x_2 \\
   x_3 \\
   x_4
   \end{bmatrix}
   =
   \begin{bmatrix}
   0 \\
   0
   \end{bmatrix}.$

   Prove that $S$ is a subspace of $V$.

   (1c) [50%] If you solved (a), then skip (b) and (c). Solve for the unknowns $x_1, x_2, x_3, x_4$ in the system of equations below by showing all details of a frame sequence from the augmented matrix $C$ to $\text{rref}(C)$. Report the vector form of the general solution.

   $\begin{align*}
   x_1 + 10x_2 + 4x_3 + x_4 &= 8 \\
   x_1 + 4x_2 - 2x_3 + x_4 &= 5 \\
   2x_2 + 2x_3 &= 1 \\
   x_1 + 6x_2 + x_4 &= 6
   \end{align*}$

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (ch5) Complete any combination of three parts to make 100%. Do not do all four!

(2a) [30%] Given $4x''(t) + 20x'(t) + 4wx(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 20$, $k = 4w$, determine all values of $w$ such that the equation is over-damped, critically damped or under-damped. Do not solve for $x(t)$!

(2b) [40%] Find a particular solution $y_p(x)$ and the homogeneous solution $y_h(x)$ for $\frac{d^4y}{dx^4} + 16\frac{d^2y}{dx^2} = 96x$.

(2b) [40%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 10x = 170\sin(t)$.

(2c) [30%] Find by variation of parameters an integral formula for a particular solution $x_p$ for the equation $x'' + 4x' + 20x = e^{t^2}\ln(t^2 + 1)$. To save time, don’t try to evaluate integrals (it’s impossible).

(2d) [30%] If you did (2a), (2b) and (2c), then skip this one! Write the general solution of $x'' + 4x = 10\sin t$ as the sum of two harmonic oscillations of different natural frequencies. To save time, don’t convert to phase-amplitude form.

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3. (ch5) Complete all parts below.

(3a) [30%] The general solution of a linear homogeneous differential equation with constant coefficients is

\[ y = c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 xe^x + c_5. \]

Find the factored form of the characteristic polynomial.

(3b) [20%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots \(-1, 0, 0, 2 + 3i, 2 - 3i\).

(3c) [30%] The function \( f(x) = 3 \cos x \) is a solution of \( y'' + y = 0 \). Find the corrected trial solution in the method of undetermined coefficients for the differential equation \( y'' + y = f(x) \). To save time, **do not** evaluate the undetermined coefficients and **do not** find \( y_p(x) \)!

(3c) [30%] Assume \( f(x) \) is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is \( (r - 1)(r^2 + 1)r^3 = 0 \). Find the corrected trial solution in the method of undetermined coefficients for the differential equation \( y''' - y' = f(x) \). To save time, **do not** evaluate the undetermined coefficients and **do not** find \( y_p(x) \)!

(3d) [20%] Let \( f(x) = 4e^x - \sinh x + x \sin^2 4x \). Find a constant-coefficient linear homogeneous differential equation which has \( f(x) \) as a solution.

(3d) [20%] Let \( f(x) = 4e^x - \cosh x + e^x \cos^2 2x \). Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has \( f(x) \) as a solution.

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4. (ch10) Complete all of the items below. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

(4a) [40%] Apply Laplace’s method to solve the system. Find a $2 \times 2$ system for $L(x), L(y)$ [10%]. Solve it for $L(x), L(y)$ [10%]. Find formulas for $x(t), y(t)$ [10%].

$$x' = 3y,$$
$$y' = 2x - y,$$
$$x(0) = 0, \quad y(0) = 1.$$ 

Maple answer check:

```maple
dsolve({diff(x(t),t)=3*y(t),diff(y(t),t)=2*x(t)-y(t),x(0)=0,y(0)=1},{x(t),y(t)});
```

(4a) [40%] Apply Laplace’s resolvent method $L(u) = (sI - A)^{-1}u(0)$ to solve the system $u' = Au$, $u(0) = u_0$. Find explicit formulas for the components $x(t), y(t)$ of the 2-vector $u(t)$.

$$x'(t) = 3x(t) - y(t),$$
$$y'(t) = x(t) + y(t),$$
$$x(0) = 0,$$
$$y(0) = 2.$$ 

Maple answer check:

```maple
with(LinearAlgebra): A:=Matrix([[3,-1],[1,1]]); u0:=Vector([0,2]);
Lu:=(s*IdentityMatrix(2)-A)^(-1).u0; map(inttrans[invlaplace],Lu,s,t);
```

(4b) [30%] Ch10(b): Find $f(t)$ by partial fraction methods, given

$$L(f(t)) = \frac{8s^3 + 30s^2 + 32s + 40}{(s+2)^2(s^2+4)}.$$ 

(4c) [30%] Ch10(c): Solve for $f(t)$, given

$$L(f(t)) = \frac{d}{ds} \left( L(t^2e^{3t}) \bigg|_{s=(s+3)} \right).$$ 

(4c) [30%] Solve for $f(t)$, given

$$L(f(t)) = \left( \frac{s+1}{s+2} \right)^2 \frac{1}{(s+2)^2}.$$ 

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5. (ch10) Complete all of the items below.

(5a) [30%] Solve by Laplace’s method for the solution $x(t)$:

$$x''(t) + 3x'(t) = 9e^{-3t}, \quad x(0) = x'(0) = 0.$$  

(5a) [30%] Apply Laplace’s method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$\frac{d^4x}{dt^4} + 4 \frac{d^2x}{dt^2} = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$  

(5b) [30%] Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(2t)\frac{\sin(t)}{t}$.  

(5b) [30%] Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\frac{\sin(t)}{t}$, where $u$ is the unit step function.  

(5c) [30%] Fill in the blank spaces in the Laplace table:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$t^3$</th>
<th>$t\cos t$</th>
<th>$t^2e^{2t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}(f(t))$</td>
<td>$\frac{6}{s^4}$</td>
<td>$\frac{1}{s+2}$</td>
<td>$\frac{s+1}{s^2+2s+5}$</td>
</tr>
</tbody>
</table>

(5d) [40%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds} \left( \mathcal{L}(e^{2t}\sin 2t) \right) + \frac{s+1}{(s+2)^2} + \frac{2+s}{s^2+5s} + \mathcal{L}(t+\sin t)|_{s-(s-2)}.$$  

(5d) [40%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2-24}{(s-1)(s+3)(s+1)^2}.$$  

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