Name ____________________________ 2250 Midterm 3 Sample [7:30, F2008]

Applied Differential Equations 2250
Sample Exam 3

Exam date: Thursday, 4 Dec, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. The sample exam has extra problems to show different problem types. On exam day, the problems will be shortened to fit into the 50-minute exam time: approximately 10 minutes for each of the five problems.

1. **(ch4)** Complete enough of the following to add to 100%.

(1a) [100%] Let V be the vector space of all continuous functions defined on $0 \le x \le 1$. Define S to be the set of all twice-continuously differentiable functions $f(x)$ in V such that $f'(0) = f(0)$ and $f''(x) + 3f'(x) + 2f(x) = 0$. Prove that S is a subspace of V.

(1b) [50%] **If you solved (a), then skip (b) and (c)**. Let V be the set of all 4×1 column vectors $\vec{x}$ with components x_1, x_2, x_3, x_4. Assume the usual $\mathcal{R}^4$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations

$$x_1 + x_2 = 0, \quad x_3 = x_4, \quad \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Prove that S is a subspace of V.

(1c) [50%] **If you solved (a), then skip (b) and (c)**. Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by showing all details of a frame sequence from the augmented matrix C to **rref**(C). Report the **vector form** of the general solution.

$$\begin{array}{rcrcrcrcl} x_1 & + & 10x_2 & + & 4x_3 & + & x_4 & = & 8 \\ x_1 & + & 4x_2 & - & 2x_3 & + & x_4 & = & 5 \\ & & 2x_2 & + & 2x_3 & & & = & 1 \\ x_1 & + & 6x_2 & & & + & x_4 & = & 6 \end{array}$$

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2. **(ch5)** Complete any combination of three parts to make 100%. **Do not do all four**!

(2a) [30%] Given $4x''(t) + 20x'(t) + 4wx(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 20$, $k = 4w$, determine all values of w such that the equation is over-damped , critically damped or under-damped. **Do not solve for** $x(t)$!

(2b) [40%] Find a particular solution $y_p(x)$ and the homogeneous solution $y_h(x)$ for $\dfrac{d^4y}{dx^4} + 16\dfrac{d^2y}{dx^2} = 96x$.

(2b) [40%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 10x = 170\sin(t)$.

(2c) [30%] Find by variation of parameters an integral formula for a particular solution x_p for the equation $x'' + 4x' + 20x = e^{t^2}\ln(t^2 + 1)$. To save time, don't try to evaluate integrals (it's impossible).

(2d) [30%] **If you did (2a), (2b) and (2c), then skip this one**! Write the general solution of $x'' + 4x = 10\sin t$ as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form**.

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3. **(ch5)** Complete all parts below.

(3a) [30%] The general solution of a linear homogeneous differential equation with constant coefficients is

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 x e^x + c_5.$$

Find the factored form of the characteristic polynomial.

(3b) [20%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots -1, 0, 0, 0, $2 + 3i$, $2 - 3i$.

(3c) [30%] The function $f(x) = 3\cos x$ is a solution of $y'' + y = 0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' + y = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(3c) [30%] Assume $f(x)$ is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is $(r - 1)(r^2 + 1)r^3 = 0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(3d) [20%] Let $f(x) = 4e^x - \sinh x + x \sin^2 4x$. Find a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

(3d) [20%] Let $f(x) = 4e^x - \cosh x + e^x \cos^2 2x$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

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4. **(ch10)** Complete all of the items below. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(4a) [40%] Apply Laplace's method to solve the system. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [10%]. Solve it for $\mathcal{L}(x)$, $L(y)$ [10%]. Find formulas for $x(t)$, $y(t)$ [10%].

$$\begin{aligned} x' &= 3y, \\ y' &= 2x - y, \\ x(0) &= 0, \quad y(0) = 1. \end{aligned}$$

Maple answer check:

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dsolve({diff(x(t),t)=3*y(t),diff(y(t),t)=2*x(t)-y(t),x(0)=0,y(0)=1},{x(t),y(t)});
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(4a) [40%] Apply Laplace's resolvent method $L(\mathbf{u}) = (sI - A)^{-1}\mathbf{u}(0)$ to solve the system $\mathbf{u}' = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$. Find explicit formulas for the components $x(t)$, $y(t)$ of the 2-vector $\mathbf{u}(t)$.

$$\begin{array}{rcrcl} x'(t) & = & 3x(t) & - & y(t), \\ y'(t) & = & x(t) & + & y(t), \\ x(0) & = & 0, & & \\ y(0) & = & 2. & & \end{array}$$

Maple answer check:

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with(LinearAlgebra): A:=Matrix([[3,-1],[1,1]]);u0:=Vector([0,2]);
Lu:=(s*IdentityMatrix(2)-A)^(-1).u0; map(inttrans[invlaplace],Lu,s,t);
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(4b) [30%] Ch10(b): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^3 + 30s^2 + 32s + 40}{(s+2)^2(s^2+4)}.$$

(4c) [30%] Ch10(c): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \frac{d}{ds}\left(\mathcal{L}\left(t^2 e^{3t}\right)\Big|_{s \to (s+3)}\right).$$

(4c) [30%] Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+1}{s+2}\right)^2 \frac{1}{(s+2)^2}$$

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5. **(ch10)** Complete all of the items below.

(5a) [30%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 3x'(t) = 9e^{-3t}, \quad x(0) = x'(0) = 0.$$

(5a) [30%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$\frac{d^4x}{dt^4} + 4\frac{d^2x}{dt^2} = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(5b) [30%] Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(2t)\dfrac{\sin(t)}{t}$.

(5b) [30%] Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\dfrac{\sin(t)}{t}$, where u is the unit step function.

(5c) [30%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3			$t\cos t$	t^2e^{2t}
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{s+2}$	$\dfrac{s+1}{s^2+2s+5}$		

(5d) [40%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \frac{s+1}{(s+2)^2} + \frac{2+s}{s^2+5s} + \mathcal{L}(t + \sin t)|_{s\to(s-2)}.$$

(5d) [40%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

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