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Applied Differential Equations 2250
Sample Exam 3

Exam date: Thursday, 4 Dec, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. The sample exam has extra problems to show different problem types. On exam day, the problems will be shortened to fit into the 50-minute exam time: approximately 10 minutes for each of the five problems.

1. **(ch4)** Complete enough of the following to add to 100%.

(1a) [100%] Let V be the vector space of all continuous functions defined on $0 \le x \le 1$. Define S to be the set of all twice-continuously differentiable functions $f(x)$ in V such that $f'(0) = f(0)$ and $f''(x) + 3f'(x) + 2f(x) = 0$. Prove that S is a subspace of V.

(1b) [50%] **If you solved (a), then skip (b) and (c).** Let V be the set of all 4×1 column vectors $\vec{x}$ with components x_1, x_2, x_3, x_4. Assume the usual $\mathcal{R}^4$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations

$$x_1 + x_2 = 0, \quad x_3 = x_4, \quad \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Prove that S is a subspace of V.

(1c) [50%] **If you solved (a), then skip (b) and (c).** Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by showing all details of a frame sequence from the augmented matrix C to **rref**(C). Report the **vector form** of the general solution.

$$\begin{array}{rcrcrcrcl} x_1 & + & 10x_2 & + & 4x_3 & + & x_4 & = & 8 \\ x_1 & + & 4x_2 & - & 2x_3 & + & x_4 & = & 5 \\ & & 2x_2 & + & 2x_3 & & & = & 1 \\ x_1 & + & 6x_2 & & & + & x_4 & = & 6 \end{array}$$

(1a) apply The Subspace criterion: (A) zero is in S; (B) f, g in $S \Rightarrow f+g$ in S; (C) c constant and f in $S \Rightarrow cf$ in S.

(1b) apply The kernel Theorem: $S = \{\vec{x} : A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$.
Choose $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & -2 \\ 0 & 0 & 2 & -2 \end{pmatrix}$ applied in $\mathbb{R}^4$.

(1c) $\vec{x} = \begin{pmatrix} 3 \\ 1/2 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 6 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

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2. **(ch5)** Complete any combination of three parts to make 100%. **Do not do all four!**

(2a) [30%] Given $4x''(t) + 20x'(t) + 4wx(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 20$, $k = 4w$, determine all values of w such that the equation is over-damped , critically damped or under-damped. **Do not solve for** $x(t)$!

(2b) [40%] Find a particular solution $y_p(x)$ and the homogeneous solution $y_h(x)$ for $\dfrac{d^4y}{dx^4} + 16\dfrac{d^2y}{dx^2} = 96x$.

(2b) [40%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 10x = 170\sin(t)$.

(2c) [30%] Find by variation of parameters an integral formula for a particular solution x_p for the equation $x'' + 4x' + 20x = e^{t^2}\ln(t^2+1)$. To save time, don't try to evaluate integrals (it's impossible).

(2d) [30%] **If you did (2a), (2b) and (2c), then skip this one!** Write the general solution of $x'' + 4x = 10\sin t$ as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form.**

(2a) $c^2 - 4mk =$ Discriminant $= 400 - 64w$ over-damped: $400-64w > 0$, Critically damped: $400-64w = 0$, under-damped: $400-64w < 0$.

(2b) $y_p = x^3$, $y_h =$ linear combination of atoms $1, x, \cos 4x, \sin 4x$

(2b) $x_{ss}(t) = 18\sin(t) - 4\cos(t)$

(2c) $x_1 = e^{-2t}\cos(4t)$, $x_2 = e^{-2t}\sin(4t)$, $W = \text{wronskian}(x_1,x_2) = 4e^{-4t}$

$x_p = \left(-\int \frac{x_2 f}{W}\right)x_1 + \left(\int \frac{x_1 f}{W}\right)x_2$ where $f = e^{t^2}\ln(t^2+1)$

(2d) $x(t) = \frac{10}{3}\sin(t) + c_1\cos(2t) + c_2\sin(2t)$

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3. **(ch5)** Complete all parts below.

(3a) [30%] The general solution of a linear homogeneous differential equation with constant coefficients is

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 e^x + c_4 x e^x + c_5.$$

Find the factored form of the characteristic polynomial.

(3b) [20%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots -1, 0, 0, 0, $2+3i$, $2-3i$.

(3c) [30%] The function $f(x) = 3\cos x$ is a solution of $y'' + y = 0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' + y = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(3c) [30%] Assume $f(x)$ is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is $(r-1)(r^2+1)r^3 = 0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y''' - y' = f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(3d) [20%] Let $f(x) = 4e^x - \sinh x + x \sin^2 4x$. Find a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

(3d) [20%] Let $f(x) = 4e^x - \cosh x + e^x \cos^2 2x$. Find the characteristic polynomial of a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

(3a) Roots $= 2i, -2i, 1, 1, 0$; $(r^2+4)(r-1)^2 r$

(3b) $e^{-x}, 1, x, x^2, e^{2x}\cos 3x, e^{2x}\sin 3x$

(3c) using Laplace: $\mathcal{L}(y) = \dfrac{\mathcal{L}(f)}{s^2+1} = \dfrac{3}{(s^2+1)^2} \Rightarrow y_p = d_1 x\cos x + d_2 x \sin x$

using classical U.C. Theory: multiply r^2+1 and r^2+1 to get $(r^2+1)^2$. atoms $=$ $\cos x$, $\sin x$, $x\cos x$, $x\sin x$. Remove $\cos x$, $\sin x$ because they are sols of the homogeneous eq. Then $y_p = d_1 x\cos x + d_2 x \sin x$.

(3c) multiply $p(r) \equiv (r-1)(r^2+1)r^3$ by $q(r) \equiv r^3 - r = r(r-1)(r+1)$ to get $pq \equiv r^4(r-1)^2(r+1)(r^2+1)$. Roots of $pq=0$, by Euler's Theorem, imply atoms $1, x, x^2, x^3, e^x, xe^x, e^{-x}, \cos x, \sin x$. Remove atoms which arise from roots of $q=0$ $[0, 1, -1]$ To arrive at $y_p = d_2 x + d_3 x^2 + d_4 x^3 + d_6 x e^x + d_8 \cos x + d_9 \sin x$

(3d) Char eq is $(r-1)(r+1)r^2(r^2+64)^2 = 0$, because $\sinh(u) = \frac{1}{2}e^u - \frac{1}{2}e^{-u}$ and $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$. ans: $y^{(8)} + 127 y^{(6)} + 3968 y^{(4)} - 4096 y'' = 0$

(3d) Char polynomial is $(r-1)(r+1)((r-1)^2+16)$ because $\cosh(u) = \frac{1}{2}e^u + \frac{1}{2}e^{-u}$ and $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

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4. **(ch10)** Complete all of the items below. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(4a) [40%] Apply Laplace's method to solve the system. Find a 2×2 system for $\mathcal{L}(x)$, $\mathcal{L}(y)$ [10%]. Solve it for $\mathcal{L}(x)$, $L(y)$ [10%]. Find formulas for $x(t)$, $y(t)$ [10%].

$$\begin{aligned} x' &= 3y, \\ y' &= 2x - y, \\ x(0) &= 0, \quad y(0) = 1. \end{aligned}$$

(4a) [40%] Apply Laplace's resolvent method $L(\mathbf{u}) = (sI - A)^{-1}\mathbf{u}(0)$ to solve the system $\mathbf{u}' = A\mathbf{u}$, $\mathbf{u}(0) = \mathbf{u}_0$. Find explicit formulas for the components $x(t)$, $y(t)$ of the 2-vector $\mathbf{u}(t)$.

$$\begin{aligned} x'(t) &= 3x(t) - y(t), \\ y'(t) &= x(t) + y(t), \\ x(0) &= 0, \\ y(0) &= 2. \end{aligned}$$

(4b) [30%] Ch10(b): Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^3 + 30s^2 + 32s + 40}{(s+2)^2(s^2+4)}.$$

(4c) [30%] Ch10(c): Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \frac{d}{ds}\left(\mathcal{L}\left(t^2 e^{3t}\right)\Big|_{s \to (s+3)}\right).$$

(4c) [30%] Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+1}{s+2}\right)^2 \frac{1}{(s+2)^2}$$

4a
$$\begin{cases} s\mathcal{L}(x) = 3\mathcal{L}(y) \\ s\mathcal{L}(y) = 1 + 2\mathcal{L}(x) - \mathcal{L}(y) \end{cases} \quad \Bigg| \quad \mathcal{L}(x) = \frac{3}{(s-2)(s+3)}, \quad \mathcal{L}(y) = \frac{s}{(s-2)(s+3)} \quad \Bigg| \quad \begin{aligned} x &= \tfrac{3}{5}e^{2t} - \tfrac{3}{5}e^{-3t} \\ y &= \tfrac{1}{2}e^{2t} + \tfrac{1}{2}e^{-3t} - \tfrac{1}{10}e^{2t} + \tfrac{1}{10}e^{-3t} \\ &= \tfrac{2}{5}e^{2t} + \tfrac{3}{5}e^{-3t} \end{aligned}$$

4a
$$\begin{pmatrix} s-3 & 1 \\ -1 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{-2}{(s-2)^2} \\ \frac{2s-6}{(s-2)^2} \end{pmatrix} = \mathcal{L}\begin{pmatrix} -2te^{2t} \\ 2e^{2t} - 4te^{2t} \end{pmatrix} \quad \Bigg\| \quad \begin{aligned} x &= -2te^{2t} \\ y &= 2e^{2t} - 4te^{2t} \end{aligned}$$

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(4b) $\mathcal{L}(f) = \dfrac{8s^3+30s^2+32s+40}{(s+2)^2(s^2+4)}$

$= \dfrac{a}{s+2} + \dfrac{b}{(s+2)^2} + \dfrac{cs+d}{s^2+4}$

$= \mathcal{L}\left(ae^{-2t} + bte^{-2t} + c\cos(2t) + \frac{d}{2}\sin 2t\right)$

By partial fractions, $a=3$, $b=4$, $c=0$, $d=5$

(4c) $\mathcal{L}(f) = \frac{d}{ds}\left(\mathcal{L}(t^2e^{3t})\Big|_{s\to(s+3)}\right)$

$= \mathcal{L}((-t)t^2e^{3t})\Big|_{s\to s+3}$ s-diff Thm

$= \mathcal{L}(-t^3e^{3t}e^{-3t})$ shift Thm

$= \mathcal{L}(-t^3) \Longrightarrow \boxed{f=-t^3}$

(4c) $\mathcal{L}(f) = \left(\dfrac{s+1}{s+2}\right)^2\dfrac{1}{(s+2)}$

$= \dfrac{(s+1)^2}{(s+2)^3}$

$= \dfrac{(s-1)^2}{s^3}\Big|_{s\to s+2}$ Shift Thm

$= \left(\frac{1}{s} - \frac{2}{s^2} + \frac{1}{s^3}\right)\Big|_{s\to s+2}$

$= \mathcal{L}(1 - 2t + t^2/2)\Big|_{s\to s+2}$

$= \mathcal{L}(e^{-2t}(1-2t+t^2/2)) \longrightarrow \boxed{f = \left(1-2t+\frac{t^2}{2}\right)e^{-2t}}$

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5. **(ch10)** Complete all of the items below.

(5a) [30%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + 3x'(t) = 9e^{-3t}, \quad x(0) = x'(0) = 0.$$

(5a) [30%] Apply Laplace's method to find a formula for $\mathcal{L}(x(t))$. **Do not** solve for $x(t)$! Document steps by reference to tables and rules.

$$\frac{d^4x}{dt^4} + 4\frac{d^2x}{dt^2} = e^t(5t + 4e^t + 3\sin 3t), \quad x(0) = x'(0) = x''(0) = 0, \quad x'''(0) = -1.$$

(5b) [30%] Find $\mathcal{L}(f(t))$, given $f(t) = \sinh(2t)\dfrac{\sin(t)}{t}$.

(5b) [30%] Find $\mathcal{L}(f(t))$, given $f(t) = u(t-\pi)\dfrac{\sin(t)}{t}$, where u is the unit step function.

(5c) [30%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3	e^{-2t}	$e^{-t}\cos(2t)$	$t\cos t$	t^2e^{2t}
$\mathcal{L}(f(t))$	$\dfrac{6}{s^4}$	$\dfrac{1}{s+2}$	$\dfrac{s+1}{s^2+2s+5}$	$-\dfrac{d}{ds}\left(\dfrac{s}{s^2+1}\right)$	$\dfrac{2}{(s-2)^3}$

(5d) [40%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \frac{s+1}{(s+2)^2} + \frac{2+s}{s^2+5s} + \mathcal{L}(t + \sin t)|_{s\to(s-2)}.$$

(5d) [40%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{8s^2 - 24}{(s-1)(s+3)(s+1)^2}.$$

5a $\quad \mathcal{L}(x) = \dfrac{9/(s+3)}{s^2+3s} = \dfrac{9}{(s+3)^2 s} \quad \Big| \quad x = 1 - e^{-3t} - 3te^{-3t}$

5a $\quad 1 + (s^4 + 4s^2)\mathcal{L}(x) = \mathcal{L}(5t + 4e^t + 3\sin 3t)\,|_{s\to s-1}$

$$\mathcal{L}(x) = \frac{-1 + \dfrac{5}{(s-1)^2} + \dfrac{4}{s-2} + \dfrac{9}{(s-1)^2+9}}{s^4 + 4s^2}$$

5c $\quad \dfrac{s+1}{s^2+2s+5} = \dfrac{s+1}{(s+1)^2+4} = \dfrac{s}{s^2+4}\Big|_{s\to s+1} = \mathcal{L}((\cos 2t)e^{-t})$

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(5d) $\mathcal{L}(x) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 2t) + \left.\frac{s-1}{s^2}\right|_{s\to s+2} + \frac{a}{s} + \frac{b}{s+5} + \mathcal{L}((t+\sin t)e^{2t})$

$= \mathcal{L}(-te^{2t}\sin 2t) + \mathcal{L}((1-t)e^{-2t}) + \mathcal{L}(a+be^{-5t}) + \mathcal{L}(te^{2t}+e^{2t}\sin t)$

$= \mathcal{L}(-te^{2t}\sin 2t + (1-t)e^{-2t} + a + be^{-5t} + te^{2t} + e^{2t}\sin t)$

$$\frac{2+s}{s(s+5)} = \frac{a}{s} + \frac{b}{s+5} \longrightarrow \quad a = \frac{2}{5}, \quad b = \frac{-3}{-5}$$

(5d) $\mathcal{L}(f) = \frac{8s^2-24}{(s-1)(s+3)(s+1)^2}$

$= \frac{a}{s-1} + \frac{b}{s+3} + \frac{c}{s+1} + \frac{d}{(s+1)^2}$

$= \mathcal{L}(ae^{t} + be^{-3t} + ce^{-t} + dte^{-t})$

Heaviside method $\quad a = \frac{-16}{16}, \quad b = \frac{8(9)-24}{(-4)(4)}, \quad c = \text{unknown}, \quad d = \frac{8-24}{(-2)(2)}$

$a = -1, \quad b = -3, \quad c = 4, \quad d = 4$

Get $c = 4$ from limit at $s = \infty$ after multiply by $s+1$;

$$\frac{8s^2-24}{(s-1)(s+3)(s+1)} = \frac{a(s+1)}{s-1} + \frac{b(s+1)}{s+3} + c + \frac{d}{s+1}$$

$$\Rightarrow \quad 0 = a + b + c + 0$$

$$0 = -1 - 3 + c$$

$$4 = c$$