Applied Differential Equations 2250
Midterm Exam 3
Exam date: Thursday, 4 Dec, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (ch4) Complete enough of the following to add to 100%.

   (1a) [100%] Let $V$ be the vector space of all twice-continuously differentiable functions defined on $0 \leq x \leq 2\pi$. Define $S$ to be the set of all functions $f(x)$ in $V$ such that $\int_0^{2\pi} x f(x) dx = 0$ and $f''(x) + 16 f(x) = 0$. Prove that $S$ is a subspace of $V$.

   (1b) [50%] If you solved (a), then skip (b) and (c). Let $V$ be the set of all $4 \times 1$ column vectors $\vec{x}$ with components $x_1, x_2, x_3$. Assume the usual $\mathbb{R}^3$ rules for addition and scalar multiplication. Let $S$ be the subset of $V$ defined by the equations

   \[
   \begin{align*}
   x_1 + x_2 &= 0, \\
   x_3 &= x_1 + 2x_2, \\
   \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
   \end{align*}
   \]

   Prove that $S$ is a subspace of $V$.

   (1c) [50%] If you solved (a), then skip (b) and (c). Solve for the unknowns $x_1, x_2, x_3, x_4$ in the system of equations below by showing all details of a frame sequence from the augmented matrix $C$ to $\text{rref}(C)$. Report the vector form of the general solution.

   \[
   \begin{align*}
   x_1 + 5x_2 - 4x_3 + x_4 &= 8, \\
   x_1 + 2x_2 + 2x_3 + x_4 &= 5, \\
   x_2 - 2x_3 &= 1, \\
   x_1 + 3x_2 + 4x_4 &= 6.
   \end{align*}
   \]

Use this page to start your solution. Attach extra pages as needed, then staple.
2. (ch5) Complete any combination to make 100%.

(2a) [60%] Find a particular solution \( y_p(x) \) for \( \frac{d^4 y}{dx^4} + 4 \frac{d^2 y}{dx^2} = 24x \).

(2b) [60%] Find the steady-state periodic solution for the forced spring-mass system \( x'' + 2x' + 5x = 10 \sin(t) \).

(2c) [40%] Given \( 4x''(t) + 10x'(t) + 2wx(t) = 0 \), which represents a damped spring-mass system with \( m = 4 \), \( c = 10 \), \( k = 2w \), determine all values of \( w \) such that the equation is under-damped. Do not solve for \( x(t) \)!

(2d) [40%] Write the solution of \( x'' + 9x = 30 \sin t \), \( x(0) = x'(0) = 0 \), as the sum of two harmonic oscillations of different natural frequencies. To save time, don't convert to phase-amplitude form.

(2a) \[ p = r^2, \quad q = r^4 + 4r^2, \quad p \bar{q} = r^2(r^4 + 4r^2) = r^4(4r^2 + 1) \]

\[ y = d_1x^2 + d_2x^3 + d_4x^2 + d_5 \cos 2x + d_6 \sin 2x \]

Substitute \( y_p \) into DE \( y'' + 9y = 24x \). Solve for \( d_3, d_4 \) from \( 8d_3 = 0, 24d_4 = 24 \) to get \( d_3 = 0, d_4 = 1 \) and \( \frac{d_3}{d_4} = x^2 \).

(2b) \[ x_{ss} = \text{undetermined coeff ans, by Eq Theory.} \]

Substitute into Eq DE, solve for \( d_1, d_6 \): \( 4d_1 + 2d_2 = 0 \Rightarrow d_1 = -2d_2 = 2 \) \( x_{ss} = \cos t + 2 \sin t \).

(2c) Discriminant = \( b^2 - 4ac \) for \( ax^2 + bx + c = 0 \). Under-damped case is oscillation or \( b^2 - 4ac < 0 \) \( \Rightarrow \) Appendix: quadratic formula. Now \( \boxed{100 - 32W < 0} \).

(2d) \[ x_p = d_1 \cos t + d_2 \sin t \]. Substitute into Eq DE, get \( d_1 = 0, d_2 = 15/4 \). \( x_p = \frac{15}{4} \sin t \). Then \( x_h = c_1 \cos 3t + c_2 \sin 3t \) from char. eq.

\[ y'' + 9y = 0 \]. Let \( x = x_h + x_p \). Use \( x(0) = x'(0) = 0 \) to solve for \( c_1, c_2 \).

Use \( c_1 = 0, c_2 = -5/4 \). Finally, \( x = -\frac{5}{4} \sin 3t + \frac{15}{4} \sin t \) is the sum of two harmonic oscillations of frequencies 3 and 1.

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3. (ch5) Complete enough parts to make 100%.

(3a) [50%] Assume $f(x)$ is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is $r^2(r + 1)(r^2 + 9) = 0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' - y' = f(x)$. To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$.

(3b) [50%] Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'' - y' = x + e^x$. To save time, do not evaluate the undetermined coefficients and do not find $y_p(x)$.

(3c) [25%] The general solution of a linear homogeneous differential equation with constant coefficients is

$$y = c_1 e^x \cos 2x + c_2 e^x \sin 2x + c_3 + c_4 x + c_5 x^2.$$ 

Find the factored form of the characteristic polynomial.

(3d) [25%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots $-1, -1, 0, 0, 2 + i, 2 - i$.

(3e) [25%] Let $f(x) = x^3 + x \sin 2x$. Find a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

3a

$$p = r^2(r + 1)(r^2 + 9)$$

$$q = r^3 - r = r(r - 1)(r + 1)$$

$$p_4 = r^3(r + 1)^2$$

$$y = 1, c_1, c_2 e^x, x e^x, x^2 e^x, c^3 e^x, \cos 3x, \sin 3x$$

$$y_p = d_1 x + d_2 x^2 + d_3 e^{-x} + d_4 x \cos 3x + d_5 x \sin 3x$$

$$y_h = c_1 + c_2 e^x + c_3 e^{-x}$$

3b

$$p = r^2(r - 1)$$

$$q = r^3 - r = r(r - 1)(r + 1)$$

$$p_4 = r^3(r - 1)^2$$

$$y = 1, c_1, c_2 e^x, x e^x, e^{-x}$$

$$y_h = 1, c_1, c_2 e^x, e^{-x}$$

$$y_p = d_1 x + d_2 x^2 + d_3 x e^{-x}$$

$$y_h = c_1 + c_2 e^x + c_3 e^{-x}$$

3c

Roots: $1 + 2i, 1 - 2i, -1, -1, 0, 0$ →

$$((r - 1)^2 + 4)^{3/2}$$

3d

$$e^x, x e^x, 1, x, e^{2x} \cos x, e^{2x} \sin x$$

3e

$$x^3 \rightarrow \text{root } 0, 0, 0, 0 \rightarrow \text{factor } x^4$$

$$x^2 \rightarrow \text{roots } 2i, -2i, 2i, -2i \rightarrow \text{factor } (r^2 + 4)^2$$

Characteristic polynomial = $r^4 (2i)^2 = r^4 (4x^2 + 16) = r^4 + 16r^2 + 16$.

DE: $y^{(4)} + 16y^{(2)} + 16y^{(4)} = 0$.

Use this page to start your solution. Attach extra pages as needed, then staple.
4. (ch10) Complete enough parts to make 100%. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don’t know a table entry, then leave the expression unevaluated for partial credit.

(4a) [50%] Apply Laplace’s method to solve the system for \( x(t) \). Don’t waste time solving for \( y(t) \):

\[
x' = 3y, \\
y' = 2x - y, \\
x(0) = 0, \quad y(0) = 1.
\]

(4b) [25%] Find \( f(t) \) by partial fraction methods, given

\[
\mathcal{L}(f(t)) = \frac{3s + 4}{s(s + 4)}.
\]

(4c) [25%] Solve for \( f(t) \), given

\[
\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}(te^{2t}) \bigg|_{s \to s+1}
\]

(4d) [25%] Solve for \( f(t) \), given

\[
\mathcal{L}(f(t)) = \left(\frac{s + 2}{s + 1}\right)^2 \frac{1}{s + 1}
\]

(4a) Resolvent method: \( \tilde{u} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \tilde{f}(\tilde{u}) = \left( s \mathbf{I} - \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \)

\[
= \left( \begin{pmatrix} s + 1 \\ 2 \\ -1 \\ 1 \end{pmatrix} \right) \frac{3}{s + 1}, \quad \Delta = \begin{vmatrix} s + 1 & 3 \\ 2 & 1 \end{vmatrix} = (s-2)(s+3), \quad \therefore \tilde{f}(\tilde{x}) = \frac{3}{\Delta} \tilde{x} = \frac{3}{s-2} \quad \frac{3}{s+3} = \mathcal{L}(\frac{3}{s} e^{2t} - \frac{3}{s} e^{3t}) \Rightarrow \tilde{x} = \frac{3}{s} (e^{2t} - e^{3t})\]

\[
\tilde{f}(\tilde{x}) = \frac{3s + 4}{s(s + 1)} = \frac{a}{s} + \frac{b}{s + 1} = \tilde{f}(a + b e^{-4t}) \Rightarrow \tilde{f} = a + b e^{-4t}
\]

\[
\frac{d}{dt} \tilde{f}(t) = \mathcal{L}(t e^{2t}) \bigg|_{s \to s+1} = \mathcal{L}(t e^{2t} e^{-t}) \quad \Rightarrow \quad \tilde{f}(t) = -e^{t}
\]

\[
\mathcal{L}(f(t)) = \frac{(s + 2)^2}{(s + 1)^2} = \frac{(s + 1)^2}{s^2} \left|_{s \to s+1} = \frac{s^2 + 2s + 1}{s^2} = \left( \frac{2}{s^2} + \frac{2}{s} + 1 \right) \right|_{s \to s+1} \]

\[
\tilde{f}(t) = \mathcal{L}(1 + 2t + t^2 e^{2t}) \bigg|_{s \to s+1} \tilde{f}(1) = \mathcal{L}(1 + 2t + t^2 e^{2t} e^{-t}) \]

\[
\tilde{f} = (1 + 2t + t^2) e^t
\]
5. (ch10) Complete enough parts to make 100%.

(5a) [25%] Fill in the blank spaces in the Laplace table:

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$t^3$</th>
<th>$\frac{1}{2}e^{-t}$</th>
<th>$e^{-t}\cos 3\frac{t}{2}$</th>
<th>$e^{t}\cos t$</th>
<th>$t^2e^{-t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}(f(t))$</td>
<td>$\frac{6}{s^4}$</td>
<td>$\frac{1}{2s+2}$</td>
<td>$\frac{s+1}{s^2+2s+10}$</td>
<td>$\frac{s-1}{(s-1)^2+1}$</td>
<td>$\frac{2}{(s+1)^3}$</td>
</tr>
</tbody>
</table>

(5b) [50%] Solve by Laplace’s method for the solution $x(t)$:

$$x''(t) + x'(t) = 9e^{-t}, \quad x(0) = x'(0) = 0.$$

(5c) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds} \left( \mathcal{L}(e^{2t}\sin 2t) \right) + \mathcal{L}(t\sin t) |_{s \to (s+2)}.$$

(5d) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{s+2}{(s+1)^2} + \frac{1+s}{s^2+5s},$$

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