Name KEY

Applied Differential Equations 2250
Midterm Exam 3

Exam date: Thursday, 4 Dec, 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch4)** Complete enough of the following to add to 100%.

(1a) [100%] Let V be the vector space of all twice-continuously differentiable functions defined on $0 \le x \le 2\pi$. Define S to be the set of all functions $f(x)$ in V such that $\int_0^{2\pi} xf(x)dx = 0$ and $f''(x) + 16f(x) = 0$. Prove that S is a subspace of V.

(1b) [50%] **If you solved (a), then skip (b) and (c).** Let V be the set of all 4×1 column vectors $\vec{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations

$$x_1 + x_2 = 0, \quad x_3 = x_1 + 2x_2, \quad \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Prove that S is a subspace of V.

(1c) [50%] **If you solved (a), then skip (b) and (c).** Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by showing all details of a frame sequence from the augmented matrix C to **rref**(C). Report the **vector form** of the general solution.

$$\begin{array}{rcrcrcrcl} x_1 & + & 5x_2 & - & 4x_3 & + & x_4 & = & 8 \\ x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 5 \\ & & x_2 & - & 2x_3 & & & = & 1 \\ x_1 & + & 3x_2 & & & + & x_4 & = & 6 \end{array}$$

(1a) • Zero is in S: Define $f(x)=0$. Then both restriction equations hold, so $f \in S$.

• f, g in $S \Rightarrow f+g$ in S: Define $h = f+g$ and assume f, g satisfy both eqs. Then $\int_0^{2\pi} x\, h(x)dx = \int_0^{2\pi} x f(x) dx + \int_0^{2\pi} x g(x) dx = 0+0=0$ and $h''+16h = f''+16f + g''+16g = 0+0 = 0$ so $h \in S$.

• f in S, c = constant $\Rightarrow cf$ in S: Let $h = cf$. Then $\int_0^{2\pi} x\, h(x)dx = c\int_0^{2\pi} x f(x) dx = 0$ and $h''+16h = cf''+16cf = c(f''+16f) = 0$. So $h \in S$.

(1b) Define $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & -2 \\ 0 & 2 & 2 \end{pmatrix}$. Then $S = \{x : Ax = 0\}$ is a subspace by the Kernel Theorem.

(1c) rref: $\left(\begin{array}{cccc|c} 1 & 0 & 6 & 1 & 3 \\ 0 & -1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \cong \left(\begin{array}{cccc|c} 1 & 0 & 6 & 1 & 3 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ $\quad \vec{x} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} -6 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

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2. **(ch5)** Complete any combination to make 100%.

(2a) [60%] Find a particular solution $y_p(x)$ for $\frac{d^4y}{dx^4} + 4\frac{d^2y}{dx^2} = 24x$.

(2b) [60%] Find the steady-state periodic solution for the forced spring-mass system $x'' + 2x' + 5x = 10\sin(t)$.

(2c) [40%] Given $4x''(t) + 10x'(t) + 2wx(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 10$, $k = 2w$, determine all values of w such that the equation is under-damped. **Do not solve for $x(t)$!**

(2d) [40%] Write the solution of $x'' + 9x = 30\sin t$, $x(0) = x'(0) = 0$, as the sum of two harmonic oscillations of different natural frequencies. **To save time, don't convert to phase-amplitude form.**

(2a) $p = r^2$, $q = r^4 + 4r^2$, $pq = r^2(r^4+4r^2) = r^4(r^2+4)$

$y = d_1 + d_2x + d_3x^2 + d_4x^3 + d_5\cos 2x + d_6 \sin 2x$

$y_p = d_3x^2 + d_4x^3$ [rest of Terms make up $y_h = d_1 + d_2x + d_5\cos 2x + d_6\sin 2x$]

Substitute y_p into DE $y'''' + 4y'' = 24x$. Solve for d_3, d_4 from $8d_3 = 0, 24d_4 = 24$ to get $d_3 = 0$, $d_4 = 1$ and $\boxed{y_p = x^3}$

(2b) x_{ss} = undetermined coeff ans, by The Theory. Then $x_p = d_1\cos t + d_2 \sin t$

Substitute into The DE, Solve for d_1, d_2: $\begin{cases} 4d_1 + 2d_2 = 0 \\ -2d_1 + 4d_2 = 10 \end{cases} \Rightarrow d_1 = -1, d_2 = 2$

$x_{ss} = -\cos t + 2\sin t$.

(2c) Discriminant = $b^2 - 4ac$ for $ar^2 + br + c = 0$. Under-damped case is oscillation or $b^2 - 4ac < 0$ [$\sqrt{-1}$ appears in quadratic formula]. Then $\boxed{100 - 32w < 0}$.

(2d) $x_p = d_1\cos t + d_2 \sin t$. Substitute into The DE, get $d_1 = 0$, $d_2 = 15/4$.

$x_p = \frac{15}{4}\sin t$. Then $x_h = c_1\cos 3t + c_2 \sin 3t$ from char. eq. $r^2 + 9 = 0$. Let $x = x_h + x_p$. Use $x(0) = x'(0) = 0$ to solve for c_1, c_2. Then $c_1 = 0$, $c_2 = -5/4$. Finally, $x = -\frac{5}{4}\sin 3t + \frac{15}{4}\sin t$ is The Sum of two harmonic oscillations of frequencies 3 and 1.

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3. (ch5) Complete enough parts to make 100%.

(3a) [50%] Assume $f(x)$ is a solution of a constant-coefficient linear homogeneous differential equation whose factored characteristic equation is $r^2(r+1)(r^2+9)=0$. Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'''-y'=f(x)$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(3b) [50%] Find the corrected trial solution in the method of undetermined coefficients for the differential equation $y'''-y'=x+e^x$. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$!

(3c) [25%] The general solution of a linear homogeneous differential equation with constant coefficients is

$$y=c_1e^x\cos 2x+c_2e^x\sin 2x+c_3+c_4x+c_5x^2.$$

Find the factored form of the characteristic polynomial.

(3d) [25%] Find six independent solutions of the homogeneous linear constant coefficient differential equation whose sixth order characteristic equation has roots -1, -1, 0, 0, $2+i$, $2-i$.

(3e) [25%] Let $f(x)=x^3+x\sin 2x$. Find a constant-coefficient linear homogeneous differential equation which has $f(x)$ as a solution.

3a $p=r^2(r+1)(r^2+9)$, $q=r^3-r=r(r-1)(r+1)$, $pq=r^3(r+1)^2(r-1)(r^2+9)$
$y=$ l.c. of atoms $1, x, x^2, e^{-x}, xe^{-x}, e^x, \cos 3x, \sin 3x$
$y_p = d_1x+d_2x^2+d_3xe^{-x}+d_4\cos 3x+d_5\sin 3x$ (5 terms)
$y_h = c_1+c_2e^x+c_3e^{-x}$ (3 terms)

3b $p=r^2(r-1)$, $q=r^3-r=r(r-1)(r+1)$, $pq=r^3(r-1)^2(r+1)$
$y=$ l.c. of atoms $1, x, x^2, e^x, xe^x, e^{-x}$
$y_h=$ l.c. of atoms $1, e^x, e^{-x}$ (3 terms)
$y_p = d_1x+d_2x^2+d_3xe^{-x}$ (3 terms)

3c roots $= 1+2i, 1-2i, 0, 0, 0 \longrightarrow \boxed{((r-1)^2+4)\,r^3}$

3d $e^{-x}, xe^{-x}, 1, x, e^{2x}\cos x, e^{2x}\sin x$

3e $x^3 \to$ root $0,0,0,0 \longrightarrow$ factor r^4
$x\sin 2x \to$ root $2i, -2i, 2i, -2i \to$ factor $((r-2i)(r+2i))^2=(r^2+4)^2$
Char polynomial $= r^4(r^2+4)^2 = r^4(r^4+4r^2+16) = r^8+4r^6+16r^4$
DE is $y^{(8)}+4y^{(6)}+16y^{(4)}=0$

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4. **(ch10)** Complete enough parts to make 100%. It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

(4a) [50%] Apply Laplace's method to solve the system for $x(t)$. Don't waste time solving for $y(t)$!

$$\begin{aligned} x' &= 3y, \\ y' &= 2x - y, \\ x(0) &= 0, \quad y(0) = 1. \end{aligned}$$

(4b) [25%] Find $f(t)$ by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{3s+4}{s(s+4)}.$$

(4c) [25%] Solve for $f(t)$, given

$$\frac{d}{ds}\mathcal{L}(f(t)) = \mathcal{L}\left(te^{2t}\right)\Big|_{s\to(s+1)}$$

(4d) [25%] Solve for $f(t)$, given

$$\mathcal{L}(f(t)) = \left(\frac{s+2}{s+1}\right)^2 \frac{1}{s+1}$$

(4a) Resolvent method: $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathcal{L}(\vec{u}) = \left(sI - \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix}\right)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} s+1 & 3 \\ 2 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{1}{\Delta}$, $\Delta = \begin{vmatrix} s+1 & 3 \\ 2 & s \end{vmatrix} = (s-2)(s+3)$. Then $\mathcal{L}(x) = \frac{3}{\Delta}$

$\mathcal{L}(x) = \frac{3}{(s-2)(s+3)} = \frac{3/5}{s-2} + \frac{-3/5}{s+3} = \mathcal{L}\left(\frac{3}{5}e^{2t} - \frac{3}{5}e^{-3t}\right) \Rightarrow \boxed{x = \frac{3}{5}\left(e^{2t} - e^{-3t}\right)}$

(4b) $\mathcal{L}(f) = \frac{3s+4}{s(s+4)} = \frac{a}{s} + \frac{b}{s+4} = \mathcal{L}(a + be^{-4t}) \Rightarrow \boxed{f = a + be^{-4t}, \; a = 1, \; b = 2}$

(4c) $\frac{d}{ds}\mathcal{L}(f) = \mathcal{L}(te^{2t})\big|_{s\to s+1} = \mathcal{L}(te^{2t}e^{-t})$

$\mathcal{L}((-t)f(t)) = \mathcal{L}(te^{2t}e^{-t}) \Rightarrow \boxed{f(t) = -e^{t}}$

(4d) $\mathcal{L}(f) = \frac{(s+2)^2}{(s+1)^3} = \frac{(s+1)^2}{s^3}\Big|_{s\to s+1} = \left(\frac{s^2}{s^3} + \frac{2s}{s^3} + \frac{1}{s^3}\right)\Big|_{s\to s+1}$

$\mathcal{L}(f) = \mathcal{L}(1 + 2t + t^2/2)\big|_{s\to s+1}$

$\mathcal{L}(f) = \mathcal{L}\left((1+2t+t^2/2)e^{-t}\right)$

$\boxed{f = (1 + 2t + t^2/2)e^{-t}}$

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5. **(ch10)** Complete enough parts to make 100%.

(5a) [25%] Fill in the blank spaces in the Laplace table:

$f(t)$	t^3	$\frac{1}{2}e^{-t}$	$e^{-t}\cos 3t$	$e^t \cos t$	$t^2 e^{-t}$
$\mathcal{L}(f(t))$	$\frac{6}{s^4}$	$\frac{1}{2s+2}$	$\frac{s+1}{s^2+2s+10}$	$\frac{s-1}{(s-1)^2+1}$	$\frac{2}{(s+1)^3}$

(5b) [50%] Solve by Laplace's method for the solution $x(t)$:

$$x''(t) + x'(t) = 9e^{-t}, \quad x(0) = x'(0) = 0.$$

(5c) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{d}{ds}\left(\mathcal{L}(e^{2t}\sin 2t)\right) + \mathcal{L}(t \sin t)|_{s\to(s+2)}.$$

(5d) [25%] Solve for $x(t)$, given

$$\mathcal{L}(x(t)) = \frac{s+2}{(s+1)^2} + \frac{1+s}{s^2+5s}$$

5a $\frac{1}{2s+2} = \frac{1}{2}\frac{1}{s+1} = \frac{1}{2}\mathcal{L}(e^{-t})$; $\frac{s+1}{(s+1)^2+9} = \frac{s}{s^2+9}\Big|_{s\to s+1} = \mathcal{L}(e^{-t}\cos 3t)$;

$\mathcal{L}(e^t\cos t) = \mathcal{L}(\cos t)|_{s\to s-1} = \frac{s}{s^2+1}\Big|_{s\to s-1} = \frac{s-1}{(s-1)^2+1}$; $\mathcal{L}(t^2e^{-t}) = \mathcal{L}(t^2)$

with shift $s \to s-(-1)$ $= \frac{2}{s^3}\Big|_{s\to s+1} = \frac{2}{(s+1)^3}$

5b $\mathcal{L}(x) = (\text{Tranfer func})\,\mathcal{L}(9e^{-t}) = \frac{9}{(s^2+s)(s+1)} = \frac{9}{s(s+1)^2} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{(s+1)^2}$

$\mathcal{L}(x) = \mathcal{L}(a + be^{-t} + cte^{-t}) \Rightarrow$ $\boxed{x = a + be^{-t} + cte^{-t},\ a = 9,\ b = -9,\ c = -9}$

5c $\mathcal{L}(x) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 2t) + \mathcal{L}(t\sin t)|_{s\to s+2}$

$= \mathcal{L}((-t)e^{2t}\sin 2t) + \mathcal{L}(e^{-2t}t\sin t)$ $\boxed{x = -te^{2t}\sin 2t + te^{-2t}\sin t}$

5d $\mathcal{L}(x) = \frac{s+2}{(s+1)^2} + \frac{1+s}{s^2+5s} = \frac{s+1}{s^2}\Big|_{s\to s+1} + \frac{a}{s} + \frac{b}{s+5}$

$\mathcal{L}(x) = \mathcal{L}(1+t)|_{s\to s+1} + \mathcal{L}(a + be^{-5t})$

$\boxed{x = (1+t)e^{-t} + a + be^{-5t},\ a = 1/5,\ b = 4/5}$

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