Name KEY 2250 Midterm 2 [7:30, F2008]

Applied Differential Equations 2250

Exam date: Thursday, 6 November, 2008

Instructions: This in-class exam is 50 minutes. Up to 30 extra minutes will be given. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences and the 3 possibilities)**
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x &+& 2y &+& 2z &=& -2b \\ 3x &+& 2ay &+& 4z &=& -b \\ 4x &+& 8y &+& 6z &=& 2-b \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 3 & 2a & 4 & -b \\ 4 & 8 & 6 & 2-b \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 0 & 2a-6 & -2 & 5b \\ 4 & 8 & 6 & 2-b \end{array}\right) \quad \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 0 & 2a-6 & -2 & 5b \\ 0 & 0 & -2 & 2+7b \end{array}\right) \quad \text{combo}(1,3,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 0 & a-3 & -1 & 5b/2 \\ 0 & 0 & 1 & -1-\frac{7b}{2} \end{array}\right) \quad \begin{array}{l}\text{mult}(2,1/2) \\ \text{mult}(3,-1/2)\end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & -2b \\ 0 & a-3 & 0 & -1-b \\ 0 & 0 & 1 & -1-7b/2 \end{array}\right) \quad \text{Combo}(3,2,1)$$

NO Solution: $a-3=0$ $1+b\neq 0$

∞-many solutions: $a-3=0$, $1+b=0$

unique solution: $a-3\neq 0$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY 2250 Midterm 2 [7:30, F2008]

2. **(vector spaces)** Do all three parts.

(a) [30%] The vector space V is the set of all solutions $y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^x + c_5 e^{-x}$ of the differential equation $y^{(5)} - y^{(2)} = 0$. Find a basis for a subspace S of V such that $\dim(S) = 4$ and $x - e^x$ belongs to S. Don't justify anything.

③ typo

(b) [30%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 = 2x_3$, $x_2 + 2x_3 = 0$. Prove or disprove that S is a subspace of V.

(c) [40%] Find a basis of 4-vectors for the subspace of $\mathcal{R}^4$ given by the system of restriction equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 4x_2 & + & 2x_3 & + & 2x_4 & = & 0, \\ x_1 & + & 2x_2 & + & 3x_3 & + & 2x_4 & = & 0, \\ & & 4x_2 & - & 2x_3 & & & = & 0 \end{array}$$

2(a) $S = \text{span}\{x - e^x, 1, x, x^2\}$ $\dim(S) = 4$, $x - e^x$ is in S.
$= \text{span}\{1, x, x^2, e^x\}$, which consists of atoms, known independent

2(b) Apply the kernel Theorem: $S = \{\vec{x} : A\vec{x} = \vec{0}\}$ is a subspace.

Define $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A\vec{x} = \vec{0} \iff \left\{ \begin{array}{l} x_1 = 2x_3 \\ x_2 + 2x_3 = 0 \\ 0 = 0 \end{array} \right.$.

2(c)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 2 & 2 & 0 \\ 1 & 2 & 3 & 2 & 0 \\ 0 & 4 & -2 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 2 & 2 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 4 & -2 & 0 & 0 \end{array}\right) \text{combo}(1,2,-1)$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 2 & 2 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{comb}(2,3,2)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 0 \\ 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{combo}(2,1,2)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 4 & 2 & 0 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \text{mult}(2,-1/2)$$

$$\left\{ \begin{array}{l} x_1 + 4x_3 + 2x_4 = 0 \\ x_2 - \frac{1}{2}x_3 = 0 \\ 0 = 0 \end{array} \right.$$

$x_1 = -4t_1 - 2t_2$
$x_2 = \frac{1}{2}t_1$
$x_3 = t_1$
$x_4 = t_2$

Basis $= \frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}$

Basis $= \left\{ \begin{pmatrix} -4 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name ~~KEY~~ 2250 Midterm 2 [7:30, F2008]

3. **(independence)** Do **only two** of the three parts.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \\ 3 \end{pmatrix}$, State two tests that can decide independence or dependence of the list of three vectors [20%]. Apply one test and report the result [30%].

(b) [50%] State the pivot theorem [20%]. Then extract from the list below a largest set of independent vectors [30%].

$$\mathbf{a} = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 5 \end{pmatrix}, \mathbf{d} = \begin{pmatrix} 3 \\ 3 \\ 0 \\ 9 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!]

Assume that matrix A is 3×2 and that there exists an invertible matrix E such that $EA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Prove or disprove: the columns of A are independent.

3(a) pivot Theorem: u_1, u_2, u_3 are independent $\iff$ 1, 2, 3 are pivot columns of $A = \text{aug}(u_1, u_2, u_3)$

Rank Theorem: " $\iff \text{rank}(A) = 3$

Det Test does not apply — Not square.

$$A = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Dependent by either Test

3(b) pivot Thm

- The pivot cols of A are independent
- Each non-pivot col of A is a linear combination of pivot cols of A.

$$A = \begin{pmatrix} 2 & 0 & 3 & 3 & 1 \\ -2 & 2 & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 5 & 9 & 1 \end{pmatrix} \to \text{rref}(A) = \begin{pmatrix} 1 & 0 & 3/2 & 3/2 & 1/2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Largest set of independent vectors $= \boxed{\vec{a}, \vec{b}}$

3(c) proof: Let $A = \text{aug}(\vec{v}_1, \vec{v}_2)$. If $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$, then $A\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$ which implies $EA\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$ and then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Then $\begin{cases} c_1 = 0 \\ c_2 = 0 \\ 0 = 0 \end{cases} \Rightarrow c_1 = c_2 = 0 \Rightarrow \vec{v}_1, \vec{v}_2$ are independent.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY

4. **(determinants and elementary matrices)** Do both parts.
(a) [50%] Assume given 3×3 matrices A, B. Suppose $E_5E_4B = E_3E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a combination, a multiply by 3, a swap, and a multiply by 4. Assume $\det(A) = 2$. Find $\det(5(AB)^2)$.
(b) [50%] Let A, B and C be 3×3 matrices such that $C + AB + BA = A^2 + B^2$. Suppose $AB = BA$. Assume $E_2E_1C = 2I$ where E_1, E_2 are elementary swap matrices. Find all possible values of $\det(A - B)$.

4(a) $\det(E_1) = -1$, $\det(E_2) = 1$, $\det(E_3) = 3$, $\det(E_4) = -1$, $\det(E_5) = 4$
by theory of elementary matrices and determinant rules.
By the product theorem [$\det(CD) = \det(C)\det(D)$],

$$\det(E_5)\det(E_4)\det(B) = \det(E_3)\det(E_2)\det(E_1)\det(A)$$
$$\Rightarrow (4)(-1)\det(B) = (3)(1)(-1)(2)$$
$$\Rightarrow \det(B) = 3/2$$

Then $\det(5(AB)^2) = \det(5I)\det(A)^2\det(B)^2$
$$= (125)(2)^2(3/2)^2$$
$$= \boxed{(125)(9)}$$

4(b) $C = A^2 + B^2 - AB - BA$
$C = (A - B)^2$
$\det(E_1) = \det(E_2) = -1$ by theory of elementary matrices and determinant theory.

The product theorem [$\det(FG) = \det(F)\det(G)$] implies

$$\det(E_2)\det(E_1)\det(C) = \det(2I)$$
$$\Rightarrow (-1)(-1)\det(C) = 8$$
$$\Rightarrow (\det(A-B))^2 = \det((A-B)^2)$$
$$= \det(C)$$
$$= 8$$
$$\Rightarrow \det(A-B) = \boxed{\pm\sqrt{8}}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY 2250 Midterm 2 [7:30, F2008]

5. **(inverse, adjugate and Cramer's rule)** Do all three parts.

(a) [20%] Determine all values of x for which B^{-1} exists, where B is the transpose of A and matrix A is given by $A = \begin{pmatrix} 1 & 2x-1 & 3x-5 \\ 2 & 3 & 0 \\ 5x & 10x & 0 \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 3, column 2 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} -1 & 0 & -1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 2 \end{pmatrix}$$

(c) [40%] Solve for y in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$A = \begin{pmatrix} 1 & 2 & 0 \\ 5 & 6 & 8 \\ 3 & 0 & 4 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

5(a) Theorem: $\det(B) = \det(B^T)$

Theorem: B^{-1} exists $\iff \det(B) \neq 0$

$\det(A) = \det(B^T) = \det(B)$

$\det(A) = (3x-5)\begin{vmatrix} 2 & 3 \\ 5x & 10x \end{vmatrix} = (3x-5)(5x)$

$\boxed{B^{-1} \text{ exists} \iff x \neq 0 \text{ and } x \neq 5/3}$

5(b) Entry of A^{-1} in row 3, col 2 $= \dfrac{\text{adj}(A) \text{ entry in row 3, col 2}}{\det(A)}$

$= \text{cofactor}(A, 2, 3) / \det(A)$

$= (-1)\begin{vmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{vmatrix} / \det(A)$

$= -1/\det(A) = \boxed{\dfrac{-1}{-1}} = \boxed{1}$

5(c) $\boxed{y = -1/8}$

Details: $y = \dfrac{\Delta_2}{\Delta}$

$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 5 & 6 & 8 \\ 3 & 0 & 4 \end{vmatrix} = 32$

$\Delta_2 = \begin{vmatrix} 1 & 1 & 0 \\ 5 & 0 & 8 \\ 3 & 1 & 4 \end{vmatrix} = -4$

Use this page to start your solution. Attach extra pages as needed, then staple.