Name KEY

Differential Equations and Linear Algebra 2250

Midterm Exam 1 [7:30]

Thursday, 2 October 2008

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. **(Quadrature Equations)**

(a) [25%] Solve $y' = \dfrac{1-x^3}{1+x^2}$.

(b) [25%] Solve $y' = (\sin x + \cos x)^2$.

(c) [25%] Solve $y' = e^x \sin(e^x)$, $y(0) = 2$.

(d) [25%] Find the position $x(t)$ from the velocity model $\dfrac{\frac{d}{dt}(e^{-t}v)}{e^{-t}} = e^{2t}$ and the position model $\dfrac{dx}{dt} = v(t)$.

(a)

$$\begin{array}{r|l} & -x \\ \hline 1+x^2 & 1-x^3 \\ & -x-x^3 \\ \hline & 1+x \end{array}$$

Do long division. Then use standard integral calculus.

$$y = c + \int \frac{1-x^3}{1+x^2}dx = c + \int -x\,dx + \int \frac{1+x}{1+x^2}dx = \boxed{c - \frac{x^2}{2} + \tan^{-1}(x) + \frac{1}{2}\ln(1+x^2)}$$

(b) $y' = \sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + 2\sin x \cos x$

$\boxed{y = c + x + \sin^2 x}$ also $y' = 1 + \sin 2x$, $\boxed{y = c + x - \frac{\cos 2x}{2}}$

(c) $y' = \sin(u)\,du$, where $u = e^x$.

$y = c - \cos(e^x)$, $2 = c - \cos(1)$, $\boxed{y = 2 + \cos(1) - \cos(e^x)}$

(d) $(e^{-t}v)' = e^t \Rightarrow e^{-t}v = c_1 + e^t \Rightarrow v = c_1 e^t + e^{2t}$

$x(t) = c_2 + \int v\,dt = c_2 + \int (c_1 e^t + e^{2t})\,dt$

$\boxed{x(t) = c_2 + c_1 e^t + \frac{1}{2}e^{2t}}$

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Name. KEY

2. (Classification of Equations)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{X}$) the problems that can be put into separable form, but don't supply any details.

X	$y' + y = y(2x + x^2) + xy$		$y' = (x-1)(y+1) - xy$
X	$y' = 2e^{2x}e^{2y} + e^{3x+2y}$		$y' + e^y = x$

(b) [10%] Is $y' + 2y = xy$ linear? No details expected.

(c) [10%] Give an example of $y' = f(x,y)$ which is separable but not quadrature and not linear. No details expected.

(d) [40%] Apply a separable equation test to show that $y' = e^x + 2y$ is not separable.

(a) $y' + y = (3x + x^2)y \longrightarrow y' + (1 - 3x - x^2)y = 0$ homog linear DE separable

$y' = xy + x - y + 1 - xy \longrightarrow y' = -1 + x - y$ linear non-homog DE not separable

$y' = (2e^{2x} + e^{3x})e^{2y}$ Separable

$y' = x - e^y$ not separable

(b) $y' + (2-x)y = 0$ is homog linear DE $\boxed{\text{yes}}$

(c) $y' = y^2$ is separable, not linear, not quadrature

(d) $f(x,y) = e^x + 2y$

$f_x = e^x$

$\dfrac{f_x}{f} = \dfrac{e^x}{e^x + 2y}$ depends on y $\Longrightarrow$ $y' = f(x,y)$ not separable

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Other Tests: II. $\dfrac{f_y}{f}$ depends on x $\Longrightarrow$ $y' = f(x,y)$ not separable

III. Let $c = f(0,0) = 1$

Let $F(x) = \dfrac{f(x,0)}{c}$ $\quad G(y) = f(0,y)$

Then $F(x)G(y) = e^x(1+2y) = e^x + 2e^xy \neq e^x + 2y \equiv f$

So $y' = f(x,y)$ is not separable.

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Name. KEY

3. (Solve a Separable Equation)
Given $(3x+9)yy' = \left((x+3)e^{-x+3} + 2x^2 + 1\right)(y+1)(y-2)$.

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{3yy'}{(y+1)(y-2)} = \frac{(x+3)e^{-x+3} + 2x^2+1}{x+3}$$

$$\int LHS = \int \frac{A\,du}{u+1} + \frac{B\,du}{u-2}$$

where $u = y(x)$

$$\frac{3y}{(y+1)(y-2)} = \frac{A}{y+1} + \frac{B}{y-2}$$

by partial fractions. Solving, $A = 1$, $B = 2$

$$= A\ln|u+1| + B\ln|u-2| + C_1$$
$$= \ln|y+1| + 2\ln|y-2| + C_1$$

$$\int RHS = \int e^{-x+3}dx + \int \frac{2x^2+1}{x+3}dx$$
$$= -e^{-x+3} + \int \frac{2(u-3)^2+1}{u}du$$

use $u = x+3$, $du = dx$

$$= -e^{-x+3} + \int \left(2u - 12 + \frac{19}{u}\right)du$$
$$= -e^{-x+3} + u^2 - 12u + 19\ln|u| + C_2$$
$$= -e^{-x+3} + (x+3)^2 - 12(x+3) + 19\ln|x+3| + C_2$$
$$= -e^{-x+3} + x^2 - 6x + 19\ln|x+3| + C_3 \leftarrow$$

absorb all constants

$$\boxed{\ln|y+1| + 2\ln|y-2| = -e^{-x+3} + x^2 - 6x + 19\ln|x+3| + C}$$

Remark: Maple finds an answer, not this one, but 7 screens long.

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However, not using dsolve, but int(LHS), int(RHS) gives the identical answer to the box above.

Name. KEY

4. (Linear Equations)

(a) [60%] Solve the linear model $5x'(t) = -110 + \frac{5}{2t+3}x(t)$, $x(0) = -66$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $4\frac{dy}{dx} + (2x)y = 0$.

(c) [20%] Solve $\frac{dy}{dx} + 5y = 4$ using the superposition principle $y = y_h + y_p$. Expected are answers for y_h and y_p.

(a) $x' + \left(\frac{-1}{2t+3}\right)x = -22$ std form $x' + px = q$

$\frac{(xW)'}{W} = -22$ $W =$ integrating factor

$W = e^{-\int dt/(2t+3)}$

$(xW)' = -22W$ $W = e^{-\frac{1}{2}\ln(2t+3)}$

use $W = (2t+3)^{-1/2}$ [defined near $t=0$]

$xW = -22\int W$

$xW = -22\int (2t+3)^{-1/2}dt$

$xW = -22\,(2t+3)^{1/2} + c$

$x = -22\,(2t+3) + c/W$

$-66 = -22\,(0+3) + c/W(0) \implies c = 0$

$\boxed{x = -44t - 66}$

(b) $y' + \frac{x}{2}y = 0$ $\boxed{y(x) = ce^{-x^2/4}}$

Explain: $\frac{(yW)'}{W} = 0 \implies y = c/W$ where $W = e^{\int \frac{x}{2}dx}$

(c) $y_p = 4/5$ is an equilibrium sol.

$y_h = ce^{-5x}$ solves $y' + 5y = 0$

$y = y_h + y_p$

$\boxed{y = ce^{-5x} + \frac{4}{5}}$

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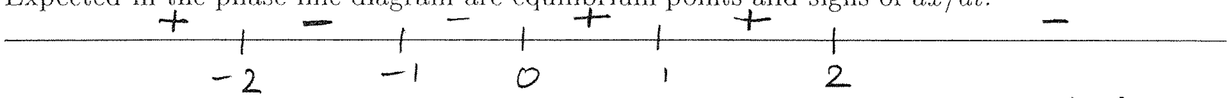
5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \ln(1+x^2)\,(1-|2x-1|)^3\,(1+x)(4-x^2)(1-x^2)^3.$$

$\ln(1+x^2)(1-|2x-1|)^3(1+x)^4(2-x)(1-x)^3(2+x)$

Expected in the phase line diagram are equilibrium points and signs of dx/dt.



$x=0$ because of $\ln(1+x^2)$ and $(1-|2x-1|)^3$

$x=1$ because of $(1-|2x-1|)^3$ and $(1-x)^3$

$x=-1$ because of $(1+x)^4$

$x=2$ because of $4-x^2$

$x=-2$

$x'(3)<0$

$x'(1.5)>0$

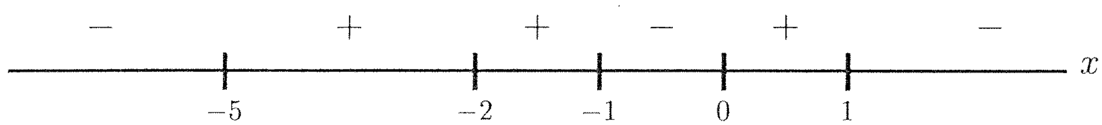
$x'(.5)>0$

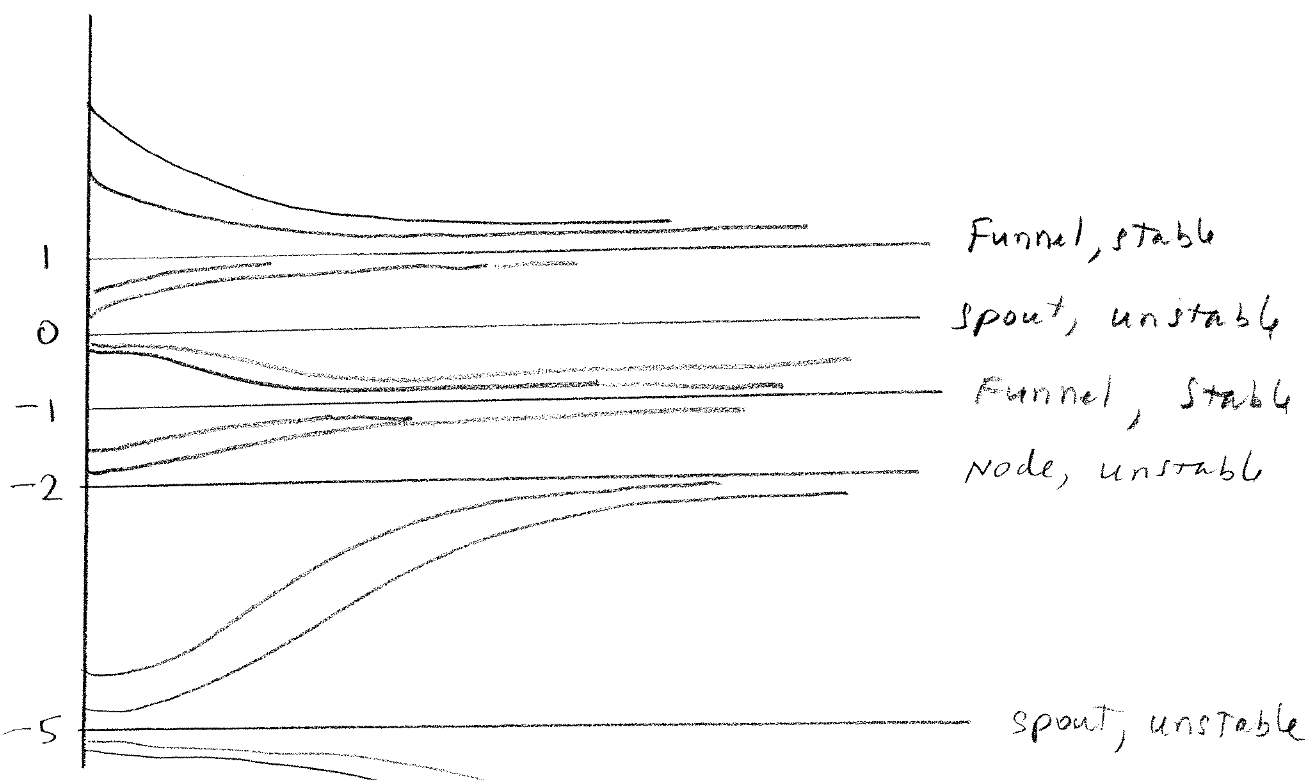
$x'(-.5)<0$

$x'(-1.5)<0$

$x'(-3)>0$

(b) [50%] Draw a phase diagram with at least 10 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, neither spout nor funnel [a node], stable, unstable. Assume an equation $x'(t) = f(x(t))$.





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