Name ____________________ Class Time _____

Math 2250 Extra Credit Problems
Chapter 7
August 2008

Due date: Submit these problems before the last day of classes. Records are locked on that date and only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Kindly label problems $\boxed{\textbf{Extra Credit}}$. Label each problem with its corresponding problem number, e.g., $\boxed{\text{Xc7.1-8}}$. You may attach this printed sheet to simplify your work.

Problem Xc7.1-8. (Transform to a first order system)

Use the position-velocity substitution $u_1 = x(t)$, $u_2 = x'(t)$, $u_3 = y(t)$, $u_4 = y'(t)$ to transform the system below into vector-matrix form $\mathbf{u}'(t) = A\mathbf{u}(t)$. Do not attempt to solve the system.

$$x'' - 2x' + 5y = 0, \quad y'' + 2y' - 5x = 0.$$

Problem Xc7.1-20a. (Dynamical systems)

Prove this result for system

(1)
$$\begin{array}{rcrcr} x' &=& ax &+& by, \\ y' &=& cx &+& dy. \end{array}$$

Theorem. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $\textbf{trace}(A) = a + d$. Then $p_1 = -\textbf{trace}(A)$, $p_2 = \det(A)$ are the coefficients in the determinant expansion

$$\det(A - rI) = r^2 + p_1 r + p_2$$

and $x(t)$ and $y(t)$ in equation (1) are both solutions of the differential equation $u'' + p_1 u' + p_2 u = 0$.

Problem xC7.1-20b. (Solve dynamical systems)

(a) Apply the previous problem to solve

$$\begin{array}{rcrcr} x' &=& 2x &-& y, \\ y' &=& x &+& 2y. \end{array}$$

(b) Use first order methods to solve the system

$$\begin{array}{rcrcr} x' &=& 2x &-& y, \\ y' &=& & & 2y. \end{array}$$

Problem Xc7.2-12. (General solution answer check)

(a) Verify that $\mathbf{x}_1(t) = e^{3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\mathbf{x}_2(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are solutions of $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}.$$

(b) Apply the Wronskian test $\det(\textbf{aug}(\mathbf{x}_1, \mathbf{x}2)) \neq 0$ to verify that the two solutions are independent.

(c) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

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Problem Xc7.2-14. **(Particular solution)**
(a) Find the constants c_1, c_2 in the general solution

$$\mathbf{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

satisfying the initial conditions $x_1(0) = 4$, $x_2(0) = -1$.
(b) Find the matrix A in the equation $\mathbf{x}' = A\mathbf{x}$. Use the formula $AP = PD$ and Fourier's model for A, which is given implicitly in (a) above.

Problem Xc7.3-8. **(Eigenanalysis method** 2×2**)**
(a) Find λ_1, λ_2, $\mathbf{v}_1$, $\mathbf{v}_2$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2$ for

$$A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

Problem Xc7.3-20. **(Eigenanalysis method** 3×3**)**
(a) Find λ_1, λ_2, λ_3, $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ in Fourier's model $A\,(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + c_3\lambda_3\mathbf{v}_3$ for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix}.$$

(b) Display the general solution of $\mathbf{x}' = A\mathbf{x}$.

Problem Xc7.3-30. **(Brine Tanks)**
Consider two brine tanks satisfying the equations

$$x_1'(t) = -k_1x_1 + k_2x_2, \quad x_2' = k_1x_1 - k_2x_2.$$

Assume $r = 10$ gallons per minute, $k_1 = r/V_1$, $k_2 = r/V_2$, $x_1(0) = 30$ and $x_2(0) = 0$. Let the tanks have volumes $V_1 = 50$ and $V_2 = 25$ gallons. Solve for $x_1(t)$ and $x_2(t)$.

Problem Xc7.3-40. **(Eigenanalysis method** 4×4**)**
Display (a) Fourier's model and (b) the general solution of $\mathbf{x}' = A\mathbf{x}$ for the 4×4 matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -16 & -4 \end{pmatrix}.$$

End of extra credit problems chapter 7.