Math 2250 Extra Credit Problems
Chapter 7
August 2008

**Due date**: Submit these problems before the last day of classes. Records are locked on that date and only corrected, never appended.

**Submitted work**. Please submit one stapled package per problem. Kindly label problems [Extra Credit]. Label each problem with its corresponding problem number, e.g., [Xc7.1-8]. You may attach this printed sheet to simplify your work.

**Problem Xc7.1-8. (Transform to a first order system)**
Use the position-velocity substitution $u_1 = x(t), u_2 = x'(t), u_3 = y(t), u_4 = y'(t)$ to transform the system below into vector-matrix form $u'(t) = Au(t)$. Do not attempt to solve the system.

$$
x'' - 2x' + 5y = 0, \quad y'' + 2y' - 5x = 0.
$$

**Problem Xc7.1-20a. (Dynamical systems)**
Prove this result for system

\[
\begin{align*}
x' &= ax + by, \\
y' &= cx + dy.
\end{align*}
\]

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**Theorem.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $\text{trace}(A) = a + d$. Then $p_1 = -\text{trace}(A), p_2 = \text{det}(A)$ are the coefficients in the determinant expansion

\[
\text{det}(A - rI) = r^2 + p_1 r + p_2
\]

and $x(t)$ and $y(t)$ in equation (1) are both solutions of the differential equation $u'' + p_1 u' + p_2 u = 0$.

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**Problem xC7.1-20b. (Solve dynamical systems)**
(a) Apply the previous problem to solve

\[
\begin{align*}
x' &= 2x - y, \\
y' &= x + 2y.
\end{align*}
\]

(b) Use first order methods to solve the system

\[
\begin{align*}
x' &= 2x - y, \\
y' &= 2y.
\end{align*}
\]

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**Problem Xc7.2-12. (General solution answer check)**
(a) Verify that $x_1(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $x_2(t) = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are solutions of $x' = Ax$, where

\[
A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}.
\]

(b) Apply the Wronskian test $\text{det}(\text{aug}(x_1, x_2)) \neq 0$ to verify that the two solutions are independent.

(c) Display the general solution of $x' = Ax$.

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Problem Xc7.2-14. (Particular solution)

(a) Find the constants $c_1, c_2$ in the general solution

$$x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

satisfying the initial conditions $x_1(0) = 4, x_2(0) = -1$.

(b) Find the matrix $A$ in the equation $x' = Ax$. Use the formula $AP = PD$ and Fourier’s model for $A$, which is given implicitly in (a) above.

Problem Xc7.3-8. (Eigenanalysis method $2 \times 2$)

(a) Find $\lambda_1, \lambda_2, v_1, v_2$ in Fourier’s model $A (c_1 v_1 + c_2 v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$ for

$$A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.$$ 

(b) Display the general solution of $x' = Ax$.

Problem Xc7.3-20. (Eigenanalysis method $3 \times 3$)

(a) Find $\lambda_1, \lambda_2, \lambda_3, v_1, v_2, v_3$ in Fourier’s model $A (c_1 v_1 + c_2 v_2 + c_3 v_3) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3$ for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix}.$$ 

(b) Display the general solution of $x' = Ax$.

Problem Xc7.3-30. (Brine Tanks)

Consider two brine tanks satisfying the equations

$$x_1'(t) = -k_1 x_1 + k_2 x_2, \quad x_2'(t) = k_1 x_1 - k_2 x_2.$$ 

Assume $r = 10$ gallons per minute, $k_1 = r/V_1, \ k_2 = r/V_2, \ x_1(0) = 30$ and $x_2(0) = 0$. Let the tanks have volumes $V_1 = 50$ and $V_2 = 25$ gallons. Solve for $x_1(t)$ and $x_2(t)$.

Problem Xc7.3-40. (Eigenanalysis method $4 \times 4$)

Display (a) Fourier’s model and (b) the general solution of $x' = Ax$ for the $4 \times 4$ matrix

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -16 & -4 \end{pmatrix}.$$ 

End of extra credit problems chapter 7.