**Math 2250 Extra Credit Problems**

**Chapter 5**

**August 2008**

**Due date:** See the internet due date for 7.4, which is the due date for these problems. Records are locked on that date and only corrected, never appended.

**Submitted work.** Please submit one stapled package per problem. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., **XC5.2-18**. You may attach this printed sheet to simplify your work.

**Problem XCL5.2. (maple lab 5, row space)**

You may submit this problem only for score increases on maple lab 5.

Let \( A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 3 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 3 & 2 & 2 & 4 & 12 \end{pmatrix} \). Find two different bases for the row space of \( A \), using the following three methods.

1. The method of Example 2 in maple lab 5 (see the web site).
2. The **maple** command \([\text{rowspace}(A)]\).
3. The **ref**-method: select rows from \( \text{rref}(A) \).

Two of the methods produce exactly the same basis. Verify that the two bases \( \mathcal{B}_1 = \{ \mathbf{v}_1, \mathbf{v}_2 \} \) and \( \mathcal{B}_2 = \{ \mathbf{w}_1, \mathbf{w}_2 \} \) are equivalent. This means that each vector in \( \mathcal{B}_1 \) is a linear combination of the vectors in \( \mathcal{B}_2 \), and conversely, that each vector in \( \mathcal{B}_2 \) is a linear combination of the vectors in \( \mathcal{B}_1 \). See the examples in maple Lab 5, at the web site.

**Problem XCL5.3. (maple lab 5, Matrix Equations)**

You may submit this problem only for score increases on maple lab 5.

Let \( A = \begin{pmatrix} -6 & -4 & 11 \\ 3 & 1 & -3 \\ -4 & -4 & 9 \end{pmatrix} \), \( T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \). Let \( P \) denote a 3 \( \times \) 3 matrix. Assume the following result:

**Lemma 1.** The equality \( AP = PT \) holds if and only if the columns \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) of \( P \) satisfy \( Av_1 = v_1, \) \( Av_2 = -2v_2, \) \( Av_3 = 5v_3 \). [proved after Example 4, see maple lab 5, web site]

(a) Determine three specific columns for \( P \) such that \( \det(P) \neq 0 \) and \( AP = PT \). Infinitely many answers are possible. See Example 4 for the maple method that determines a column of \( P \).

(b) After reporting the three columns, check the answer by computing \( AP - PT \) (it should be zero) and \( \det(P) \) (it should be nonzero).

**Problem XC5.1-all. (Second order DE)**

This problem counts as 700 if 5.1 was not submitted and 100 otherwise. Solve the following seven parts.

(a) \( y'' + 4y' = 0 \)  
(b) \( 4y'' + 12y' + 9y = 0 \)  
(c) \( y'' + 2y' + 5y = 0 \)  
(d) \( 21y'' + 10y' + y = 0 \)  
(e) \( 16y'' + 8y' + y = 0 \)  
(f) \( y'' + 4y' + (4 + \pi)y = 0 \)  
(g) Find the differential equation \( ay'' + by' + cy = 0 \), if \( e^{-x} \) and \( e^x \) are solutions.

**Problem XC5.2-18. (Initial value problems)**

Given \( x^3y''' + 6x^2y'' + 4xy' - 4y = 0 \) has three solutions \( x, 1/x^2, \) \( \frac{\ln|x|}{x^2} \), prove by the Wronskian test that they are independent and then solve for the unique solution satisfying \( y(1) = 1, y'(1) = 5, y''(1) = -11 \).

**Problem XC5.2-22. (Initial value problem)**

Solve the problem \( y'' - 4y = 2x, y(0) = 2, y'(0) = -1/2 \), given a particular solution \( y_p(x) = -x/2 \).
Problem XC5.3-8. (Complex roots)
Solve \( y'' - 6y' + 25y = 0 \).

Problem XC5.3-10. (Higher order complex roots)
Solve \( y^{(v)} + \pi^2 y''' = 0 \).

Problem XC5.3-16. (Fourth order DE)
Solve the fourth order homogeneous equation whose characteristic equation is \((r - 1)(r^3 - 1) = 0\).

Problem XC5.3-32. (Theory of equations)
Solve \( y^{(v)} - y''' + y'' - 3y' - 6y = 0 \). Use the theory of equations [factor theorem, root theorem, rational root theorem, division algorithm] to completely factor the characteristic equation. You may check answers by computer, but please display the complete details of factorization.

Problem XC5.4-20. (Overdamped, critically damped, underdamped)
(a) Consider \( 2x''(t) + 12x'(t) + 50x(t) = 0 \). Classify as overdamped, critically damped or underdamped.
(b) Solve \( 2x''(t) + 12x'(t) + 50x(t) = 0 \), \( x(0) = 0 \), \( x'(0) = -8 \). Express the answer in the form \( x(t) = C_1 e^{at} \cos(b t - \theta) \).
(c) Solve the zero damping problem \( 2u''(t) + 50u(t) = 0 \), \( u(0) = 0 \), \( u'(0) = -8 \). Express the answer in phase-amplitude form \( u(t) = C_2 \cos(b t - \theta_2) \).
(d) Using computer assist, display on one graphic plots of \( x(t) \) and \( u(t) \). The plot should showcase the damping effects. A hand-made replica of a computer or calculator graphic is sufficient.

Problem XC5.4-34. (Inverse problem)
A body weighing 100 pounds undergoes damped oscillation in a spring-mass system. Assume the differential equation is \( m x'' + c x' + kx = 0 \), with \( t \) in seconds and \( x(t) \) in feet. Observations give \( x(0.4) = 6.1/12 \), \( x'(0.4) = 0 \) and \( x(1.2) = 14/12 \), \( x'(1.2) = 0 \) as successive maxima of \( x(t) \). Then \( m = 3.125 \) slugs. Find \( c \) and \( k \).

Atoms. An atom is a term of the form \( x^k e^{\alpha x} \), \( x^k e^{\alpha x} \cos bx \) or \( x^k e^{\alpha x} \sin bx \). The symbol \( k \) is a non-negative integer. Symbols \( a \) and \( b \) are real numbers with \( b > 0 \). In particular, 1, \( x \), \( x^2 \), \( e^x \), \( \cos x \), \( \sin x \) are atoms. Any distinct list of atoms is linearly independent.

Roots and Atoms. Define atomRoot\((A)\) as follows. Symbols \( \alpha \), \( \beta \), \( r \) are real numbers, \( \beta > 0 \) and \( k \) is a non-negative integer.

<table>
<thead>
<tr>
<th>atom ( A )</th>
<th>atomRoot((A))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^k e^{\alpha x} )</td>
<td>( r )</td>
</tr>
<tr>
<td>( x^k e^{\alpha x} \cos \beta x )</td>
<td>( \alpha + i \beta )</td>
</tr>
<tr>
<td>( x^k e^{\alpha x} \sin \beta x )</td>
<td>( \alpha + i \beta )</td>
</tr>
</tbody>
</table>

The fix-up rule for undetermined coefficients can be stated as follows:

Compute atomRoot\((A)\) for all atoms \( A \) in the trial solution. Assume \( r \) is a root of the characteristic equation of multiplicity \( k \). Search the trial solution for atoms \( B \) with atomRoot\((B)\) = \( r \), and multiply each such \( B \) by \( x^k \). Repeat for all roots of the characteristic equation.

Problem Xc5.5-1A. (AtomRoot Part 1)

1. Evaluate atomRoot\((A)\) for \( A = 1, x, x^2, e^{-x}, \cos 2x, \sin 3x, x \cos \pi x, e^{-x} \sin 3x, x^3, e^{2x}, \cos x/2, \sin 4x, x^2 \cos x, e^{3x} \sin 2x \).

2. Let \( A = xe^{-2x} \) and \( B = x^2 e^{-2x} \). Verify that atomRoot\((A)\) = atomRoot\((B)\).

Problem Xc5.5-1B. (AtomRoot Part 2)

3. Let \( A = xe^{-2x} \) and \( B = x^2 e^{2x} \). Verify that atomRoot\((A)\) \( \neq \) atomRoot\((B)\).
4. Atoms $A$ and $B$ are said to be related if and only if the derivative lists $A$, $A'$, ... and $B$, $B'$, ... share a common atom. Prove: atoms $A$ and $B$ are related if and only if \( \text{atomRoot}(A) = \text{atomRoot}(B) \).

**Problem XC5.5-6. (Undetermined coefficients, fixup rule)**

Find a particular solution \( y_p(x) \) for the equation \( y^{iv} - 4y'' + 4y = xe^{2x} + x^2e^{-2x} \). Check your answer in maple.

**Problem XC5.5-12. ()**

Find a particular solution \( y_p(x) \) for the equation \( y^{iv} - 5y'' + 4y = xe^x + x^2e^{2x} + \cos x \). Check your answer in maple.

**Problem XC5.5-22. (Fixup rule, trial solution)**

Report a trial solution \( y \) for the calculation of \( y_p \) by the method of undetermined coefficients, after the fixup rule has been applied. To save time, do not do any further undetermined coefficients steps.

\[
y^{iv} + 2y''' + 2y'' = 5x^3 + e^{-x} + 4\cos x.
\]

Hint: Test \( r^2(r^2 + 2r + 2) = 0 \) when \( r = \text{atomRoot}(B) \) and \( B \) is an atom in the initial trial solution. This means a test only for \( r = 0, -1, i \).

**Problem XC5.5-54. (Variation of parameters)**

Solve by variation of parameters for \( y_p(x) \) in the equation \( y^{iv} - 16y = xe^{4x} \). Check your answer in maple.

**Problem XC5.5-58. (Variation of parameters)**

Solve by the method of variation of parameters for \( y_p(x) \) in the equation \((x^2 - 1)y^{iv} - 2xy' + 2y = x^2 - 1\). Use the fact that \( \{x, 1 + x^2\} \) is a basis of the solution space of the homogeneous equation. Apply (33) in the textbook, after division of the leading coefficient \((x^2 - 1)\). Check your answer in maple.

**Problem XC5.6-4. (Harmonic superposition)**

Write the general solution \( x(t) \) as the superposition of two harmonic oscillations of frequencies 2 and 3, for the initial value problem \( x''(t) + 4x(t) = 16\sin 3t, \; x(0) = 0, \; x'(0) = 0 \).

**Problem XC5.6-8. (Steady-state periodic solution)**

The equation \( x''(t) + 3x'(t) + 3x(t) = 8\cos 10t + 6\sin 10t \) has a unique steady-state periodic solution of period \( 2\pi/10 \). Find it.

**Problem XC5.6-18. (Practical resonance)**

Use the equation \( \omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} \) to decide upon practical resonance for the equation \( mx'' + cx' + kx = F_0\cos \omega t \) when \( F_0 = 10, \; m = 1, \; c = 4, \; k = 5 \). Sketch the graph of \( C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \) and mark on the graph the location of the resonant frequency (if any). See Figure 5.6.9 in Edwards-Penney.

**Problem XC-EPbvp-3.7-4. (LR-circuit)**

An LR-circuit with emf \( E(t) = 100e^{-12t} \), inductor \( L = 2 \), resistor \( R = 40 \) is initialized with \( i(0) = 0 \). Find the current \( i(t) \) for \( t \geq 0 \) and argue physically and mathematically why the observed current at \( t = \infty \) should be zero.

**Problem XC-EPbvp-3.7-12. (Steady-state of an RLC-circuit)**

Compute the steady-state current in an RLC-circuit with parameters \( L = 5, \; R = 50, \; C = 1/200 \) and emf \( E(t) = 30\cos 100t + 40\sin 100t \). Report the amplitude, phase-lag and period of this solution.

End of extra credit problems chapter 5.