Atoms

An atom is a term with coefficient 1 obtained by taking the real and imaginary parts of

\[ x^j e^{ax + ix}, \quad j = 0, 1, 2, \ldots, \]

where \( a \) and \( c \) represent real numbers and \( c \geq 0 \).

**Theorem 1 ((Independence of Atoms))**

Any finite list of distinct atoms is linearly independent.

**Details and Remarks**

- The definition plus Euler’s formula \( e^{i\theta} = \cos \theta + i \sin \theta \) implies that an atom is a term of one of the following types:

  \[ x^n, \; x^n e^{ax}, \; x^n e^{ax} \cos bx, \; x^n e^{ax} \sin bx. \]

  The symbol \( n \) is an integer 0, 1, 2, \ldots and \( a, b \) are real numbers with \( b > 0 \).

- In particular, 1, \( x, \; x^2, \ldots, \; x^k \) are atoms and this list is independent.

- The term that makes up an atom has coefficient 1, therefore \( 2e^x \) is not an atom, but the 2 can be stripped off to create the atom \( e^x \). Linear combinations like \( 2x + 3x^2 \) are not atoms, but the individual terms \( x \) and \( x^2 \) are indeed atoms. Terms like \( e^{x^2}, \ln |x| \) and \( x/(1 + x^2) \) are not atoms, nor are they constructed from atoms.
Construction of the general solution from a list of distinct atoms

- The general solution \( y \) of a homogeneous constant-coefficient linear differential equation
  \[ y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1y' + p_0y = 0 \]

is known to be a formal linear combination of the atoms of this equation, using symbols \( c_1, \ldots, c_n \) for the coefficients:
\[
y = c_1(\text{atom } 1) + \cdots + c_n(\text{atom } n).\]

In particular, each atom listed is itself a solution of the differential equation.

- **Euler’s theorem infra** explains how to construct a list of distinct atoms, each of which is a solution of the differential equation, from the roots of the characteristic equation
  \[ r^n + p_{n-1}r^{n-1} + \cdots + p_1r + p_0 = 0. \]

- The **Fundamental Theorem of Algebra** states that there are exactly \( n \) roots \( r \), real or complex, for an \( n \)th order polynomial equation. The result explains how we know that the characteristic equation has exactly \( n \) roots.

- **Picard’s theorem** says that the constructed atom list is a basis for the solution space of the differential equation, provided it contains \( n \) independent elements.

  Because the list of atoms constructed by Euler’s theorem has \( n \) distinct elements, which are independent, then these atoms form a basis for the general solution of the differential equation.
Euler’s Theorem

Theorem 2 (L. Euler)
The function \( y = x^j e^{r_1 x} \) is a solution of a constant-coefficient linear homogeneous differential of the \( n \)th order if and only if \((r - r_1)^{j+1}\) divides the characteristic polynomial.

The Atom List

1. If \( r_1 \) is a real root, then the atom list for \( r_1 \) begins with \( e^{r_1 x} \). The revised atom list is

\[ e^{r_1 x}, xe^{r_1 x}, \ldots, x^{k-1} e^{r_1 x} \]

provided \( r_1 \) is a root of multiplicity \( k \), that is, \((r - r_1)^k\) divides the characteristic polynomial, but \((r - r_1)^{k+1}\) does not.

2. If \( r_1 = \alpha + i\beta \), with \( \beta > 0 \), is a complex root along with its conjugate root \( r_2 = \alpha - i\beta \), then the atom list for this pair of roots (both \( r_1 \) and \( r_2 \) counted) begins with

\[ e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x. \]

If the roots have multiplicity \( k \), then the list of \( 2k \) atoms are

\[ e^{\alpha x} \cos \beta x, \quad xe^{\alpha x} \cos \beta x, \quad \ldots, \quad x^{k-1} e^{\alpha x} \cos \beta x, \]
\[ e^{\alpha x} \sin \beta x, \quad xe^{\alpha x} \sin \beta x, \quad \ldots, \quad x^{k-1} e^{\alpha x} \sin \beta x. \]
Explanation of steps 1 and 2

1. Root \( r_1 \) always produces atom \( e^{r_1x} \), but if the multiplicity is \( k > 1 \), then \( e^{r_1x} \) is multiplied by \( 1, x, \ldots, x^{k-1} \).

2. The expected first terms \( e^{r_1x} \) and \( e^{r_2x} \) \([e^{\alpha x + i\beta x} \text{ and } e^{\alpha x - i\beta x}] \) are not atoms, but they are linear combinations of atoms:

\[
e^{\alpha x \pm i\beta x} = e^{\alpha x} \cos \beta x \pm ie^{\alpha x} \sin \beta x.
\]

The atom list for a complex conjugate pair of roots \( r_1 = \alpha + i\beta, r_2 = \alpha - i\beta \) is obtained by multiplying the two real atoms

\[
e^{\alpha x} \cos \beta x, \quad e^{\alpha x} \sin \beta x
\]

by the powers

\[
1, x, \ldots, x^{k-1}
\]

to obtain the \( 2k \) distinct real atoms in item 2 above.