Introduction to Linear Algebra 2270-1
Midterm Exam 3 Fall 2007

Exam Date: Wednesday, 28 November 2007

Instructions. The exam is 50 minutes. Calculators are not allowed. Books and notes are not allowed.

1. (Kernel, Similarity) Complete two.

(a) [50%] Use the identity $A_2 = E_1E_2\cdots E_kA_1$ to prove that two frames A_1 and A_2 of a frame sequence have exactly the same kernel.

(b) [50%] Suppose that $\mathbf{ker}(A) = \{\mathbf{0}\}$. Prove that $\mathbf{ker}(A^TA) = \{\mathbf{0}\}$.

(c) [50%] If you did both (a) and (b), then stop, otherwise proceed.
Suppose $\mathbf{ker}(A)$ and $\mathbf{ker}(B)$ are isomorphic. Are A and B similar?

(a) $A_2 = EA_1$ with E invertible implies $\ker(A_2) = \ker(A_1)$.

(b) Because $Ax=0 \Rightarrow A^TAx=0$, then $\ker(A) \subseteq \ker(A^TA)$.
If $x \in \ker(A^TA)$, then $A^TAx=0$ and $Ax \in \ker(A^T) = \mathrm{Im}(A)^\perp$ implies $Ax \in \mathrm{Im}(A) \cap \mathrm{Im}(A)^\perp = \{0\}$. Then $Ax=0$, so $x \in \ker(A)$.

(c) Choose A, B not similar with $\ker(A) = \ker(B) = \{\vec{0}\}$. Then they are not similar with isomorphic kernels. Example: $A = \begin{pmatrix}1&0\\0&2\end{pmatrix}$, $B = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

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2. (Abstract vector spaces, Linear transformations) Complete two.

Let W be the set of all upper triangular 4×4 matrices (lower triangle all zeros).

(a) [50%] Define addition and scalar multiplication for W and prove that W is a vector space. You may use isomorphisms to shorten the proof.

(b) [50%] Let V be the subset of W all of whose diagonal elements are zero. Prove that V is a subspace of W.

(c) [50%] If you did both (a) and (b), then stop, otherwise proceed.
Define $T(\mathbf{x}) = \mathbf{y}$ from W to V by the natural projection, in which $\mathbf{y}$ equals matrix $\mathbf{x}$ with all diagonal elements replaced by zero. Prove that T is a linear transformation from W to V and determine $\mathbf{ker}(T)$.

(a) Define S from all 4×4 matrices into $\mathbb{R}^{16}$ by $S(A) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{44} \end{pmatrix}$. Then S is an isomorphism. Define $+, \cdot$ on W as usual matrix $+, \cdot$, componentwise. Map S sends W to a subset W_1 of $\mathbb{R}^{16}$, defined by a matrix equation $B\vec{x} = \vec{0}$, $\vec{x} \in \mathbb{R}^{16}$. Matrix B is 6×16 and reflects the info that the lower triangle of a matrix $A \in W$ is all zeros. Then $W_1 = \ker(B)$ is a subspace, hence $W = S^{-1}(W_1)$ is a subspace.

(b) Let S be as in (a). Define $V_1 = S(V)$. Then V_1 is given by a matrix equation $C\vec{x} = \vec{0}$, $\vec{x} \in \mathbb{R}^{16}$. The matrix C has 4 ones and all other entries are zero. It is 4×4 and $C\vec{x} = \vec{0}$ just means the diagonal entries of $S^{-1}(\vec{x})$ are zeros. Because $V_1 = \ker(C)$ is a subspace, then so is V.

(c) Let $\vec{y} = T(c_1\vec{x}_1 + c_2\vec{x}_2)$. Then $\vec{y}$ equals matrix $c_1\vec{x}_1 + c_2\vec{x}_2$ with all 4 diagonal entries set to zero. Off-diagonal entries of $T(\vec{x}_1)$ and $T(\vec{x}_2)$ agree with those of $\vec{x}_1$ and $\vec{x}_2$, therefore $c_1 T(\vec{x}_1) + c_2 T(\vec{x}_2)$ has off-diagonal entries equal to those of $\vec{y}$. On the diagonal, $T(\vec{x}_1)$ and $T(\vec{x}_2)$ have 4 zeros, therefore agreeing with $\vec{y}$, giving $T(c_1\vec{x}_1 + c_2\vec{x}_2) = \vec{y} = c_1 T(\vec{x}_1) + c_2 T(\vec{x}_2)$.

The kernel of T is the set of all diagonal matrices.

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3. (Orthogonality, Gram-Schmidt) Complete two.

(a) [50%] Find the orthogonal projection of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto $V = \textbf{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$.

(b) [50%] Find the QR-factorization of $A = \begin{pmatrix} 1 & 0 \\ 7 & 7 \\ 1 & 2 \end{pmatrix}$.

(c) [50%] If you did both (a) and (b), then stop, else proceed.
Prove that the product AB of two orthogonal matrices A and B is again orthogonal.

ⓐ $\vec{u}_1 = \frac{1}{\sqrt{3}}\vec{v}_1$, $\vec{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, $\text{proj}_V(\vec{x}) = (\vec{x}\cdot\vec{u}_1)\vec{u}_1 + (\vec{x}\cdot\vec{u}_2)\vec{u}_2 = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$

ⓑ $\vec{u}_1 = \frac{1}{\sqrt{51}}\vec{v}_1$, $\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\|\vec{v}_2^{\perp}\|}$, $\vec{v}_2^{\perp} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$Q = \text{aug}(\vec{u}_1, \vec{u}_2)$, $R = \begin{pmatrix} \|\vec{v}_1\| & \vec{u}_1\cdot\vec{v}_2 \\ 0 & \|\vec{v}_2^{\perp}\| \end{pmatrix} = \begin{pmatrix} \sqrt{51} & \sqrt{51} \\ 0 & \sqrt{2} \end{pmatrix}$

ⓒ $\|AB\vec{x}\| = \|B\vec{x}\|$ because $\|A\vec{u}\| = \|\vec{u}\|$

$= \|\vec{x}\|$ because $\|B\vec{x}\| = \|\vec{x}\|$

Therefore, AB is $\perp$.

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4. (Orthogonality and least squares) Complete all three.

(a) [20%] For an inconsistent system $A\mathbf{x} = \mathbf{b}$, the least squares solutions $\mathbf{x}$ are the exact solutions of the normal equation. Define the normal equation and display the unique solution $\mathbf{x} = \mathbf{x}^*$ when $\mathbf{ker}(A) = \{\mathbf{0}\}$.

(b) [20%] State the *near point theorem*. Then explain how the near point to $\mathbf{x}$ can equal $\mathbf{x}$ itself.

(c) [60%] Fit $c_0 + c_1 x$ to the data points $(0,2)$, $(1,0)$, $(2,1)$, $(3,1)$ using least squares. Sketch the solution and the data points as an answer check. This is a 2×2 system problem, and should take only 1-3 minutes to complete.

(a) $A^T A \vec{x}^* = A^T \vec{b}$, $\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$

(b) The near point equals itself when $\vec{x} \in V$.

Theorem. $\|\vec{x} - \vec{v}\|$ is a minimum when $\vec{v} = \mathrm{proj}_V(\vec{x})$. This means $\|\vec{x} - \mathrm{proj}_V(\vec{x})\| \le \|\vec{x} - \vec{v}\|$ for all $\vec{v} \in V$,

(c) The normal equation $A^T A \vec{x} = A^T \vec{b}$ is $\begin{pmatrix} 4 & 6 \\ 6 & 14 \end{pmatrix} \vec{x} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ 1 \end{pmatrix}$. Then $\vec{x} = \begin{pmatrix} 13/10 \\ -1/5 \end{pmatrix}$. The ans check graphic shows a line of negative slope with 3 data points above, one below.

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5. (Miscellany) Complete two.

(a) [50%] Let 3×3 matrix A be invertible and assume $\mathbf{rref}(A) = E_3 E_2 E_2 A$. The elementary matrices E_1, E_2, E_3 represent `combo(1,3,-5)`, `swap(1,3)`, `mult(2,-2)`, respectively. Find A^2.

(b) [50%] Prove that $\mathbf{im}(A^T B^T) = \mathbf{ker}(BA)^{\perp}$, when matrix product BA is defined.

(c) [50%] If you did (a) and (b), then stop, else proceed.
Prove that the span of the Gram-Schmidt vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ equals exactly the span of the independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ used to construct them.

(a) $A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -.5 & 0 \\ 1 & 0 & 5 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 0 & 5 \\ 0 & .25 & 0 \\ 5 & 0 & 26 \end{pmatrix}$

(b)
$$\begin{aligned} \mathrm{im}(A^T B^T) &= \left(\left(\mathrm{im}(A^T B^T)\right)^{\perp}\right)^{\perp} \\ &= \left(\ker (A^T B^T)^T\right)^{\perp} \\ &= (\ker(BA))^{\perp} \end{aligned}$$

(c) Each $\vec{u}_j$ belongs to the span of $\vec{v}_1, \ldots, \vec{v}_j$ because of the GS-formulas.

$$\vec{u}_j = \frac{\vec{v}_j^{\perp}}{\|\vec{v}_j^{\perp}\|}, \quad \vec{v}_j^{\perp} = \vec{v}_j - \mathrm{proj}_{\mathrm{span}(v_1,\ldots,v_{j-1})}(\vec{v}_j)$$

Therefore, $\vec{u}_1, \ldots, \vec{u}_k$ span the same set as $\vec{v}_1, \ldots, \vec{v}_k$. Because they are independent and so are $\vec{v}_1, \ldots, \vec{v}_k$, then the spans are equal.