Introduction to Linear Algebra 2270-1  
Midterm Exam 3 Fall 2007  
Exam Date: Wednesday, 28 November 2007

Instructions. The exam is 50 minutes. Calculators are not allowed. Books and notes are not allowed.

1. (Kernel, Similarity) Complete two.

   (a) [50%] Use the identity $A_2 = E_1 E_2 \cdots E_k A_1$ to prove that two frames $A_1$ and $A_2$ of a frame sequence have exactly the same kernel.

   (b) [50%] Suppose that $\ker(A) = \{0\}$. Prove that $\ker(A^T A) = \{0\}$.

   (c) [50%] If you did both (a) and (b), then stop, otherwise proceed.
       Suppose $\ker(A)$ and $\ker(B)$ are isomorphic. Are $A$ and $B$ similar?

   (a) $A_2 = E A_1$ with $E$ invertible implies $\ker(A_2) = \ker(A_1)$.

   (b) Because $A x = 0 \Rightarrow A^T A x = 0$, then $\ker(A) \subseteq \ker(A^T A)$.
       If $x \in \ker(A^T A)$, then $A^T A x = 0$ and $A x \in \ker(A^T) = \text{Im}(A)^\perp$,
       implies $A x \in \text{Im}(A) \cap \text{Im}(A)^\perp = \{0\}$, then $A x = 0$, so $x \in \ker(A)$.

   (c) Choose $A, B$ not similar with $\ker(A) = \ker(B) = \{0\}$. Then they
       are not similar with isomorphic kernels. Example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Please start your solutions on this page. Additional pages may be stapled to this one.
2. (Abstract vector spaces, Linear transformations) Complete two.

Let $W$ be the set of all upper triangular $4 \times 4$ matrices (lower triangle all zeros).

(a) [50%] Define addition and scalar multiplication for $W$ and prove that $W$ is a vector space. You may use isomorphisms to shorten the proof.

(b) [50%] Let $V$ be the subset of $W$ all of whose diagonal elements are zero. Prove that $V$ is a subspace of $W$.

(c) [50%] If you did both (a) and (b), then stop, otherwise proceed.

Define $T(x) = y$ from $W$ to $V$ by the natural projection, in which $y$ equals matrix $x$ with all diagonal elements replaced by zero. Prove that $T$ is a linear transformation from $W$ to $V$ and determine $\ker(T)$.

(a) Define $S$ from all $4 \times 4$ matrices into $\mathbb{R}^{16}$ by $S(x) = \begin{pmatrix} a_{11} \\ \vdots \\ a_{44} \end{pmatrix}$.

Then $S$ is an isomorphism. Define $\phi$ on $W$ as usual matrix $t$, component-wise. Map $S$ sends $W$ to a subset $W_1$ of $\mathbb{R}^{16}$, defined by a matrix equation $Bx = 0$, $x \in \mathbb{R}^{16}$. Matrix $B$ is $6 \times 16$ and reflects the fact that the lower triangle of a matrix $A \in W$ is all zeros. Then $W_1 = \ker(B)$ is a subspace, hence $W = S^{-1}(W_1)$ is a subspace.

(b) Let $S$ be as in (a), define $V_1 = S(W)$. Then $V_1$ is given by a matrix equation $C\bar{x} = 0$, $\bar{x} \in \mathbb{R}^{16}$. The matrix $C$ has 4 ones and all other entries are zero. It is $4 \times 4$ and $C\bar{x} = 0$ just means the diagonal entries of $S^{-1}(\bar{x})$ are zeros. Because $V_1 = \ker(C)$ is a subspace, then so is $V$.

(c) Let $\bar{y} = T(c_1\bar{x}_1 + c_2\bar{x}_2)$. Then $\bar{y}$ equals matrix $c_1 \bar{x}_1 + c_2 \bar{x}_2$ with all 4 diagonal entries set to zero. Off-diagonal entries of $T(\bar{x}_1)$ and $T(\bar{x}_2)$ agree with those of $\bar{y}$. Hence $c_1 T(\bar{x}_1) + c_2 T(\bar{x}_2)$ has off-diagonal entries equal to those of $\bar{y}$.

On the diagonal, $T(\bar{x}_1)$ and $T(\bar{x}_2)$ have 4 zeros, therefore agree with $\bar{y}$, giving $T(c_1\bar{x}_1 + c_2\bar{x}_2) = \bar{y} = c_1 T(\bar{x}_1) + c_2 T(\bar{x}_2)$.

The kernel of $T$ is the set of all diagonal matrices.
3. (Orthogonality, Gram-Schmidt) Complete two.

(a) [50%] Find the orthogonal projection of \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) onto \( V = \text{span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\} \).

(b) [50%] Find the QR-factorization of \( A = \begin{pmatrix} 1 & 0 \\ 7 & 7 \\ 1 & 2 \end{pmatrix} \).

(c) [50%] If you did both (a) and (b), then stop, else proceed. Prove that the product \( AB \) of two orthogonal matrices \( A \) and \( B \) is again orthogonal.

\[ \begin{align*}
\vec{u}_1 &= \frac{1}{\sqrt{3}} \vec{v}_1, \quad \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \text{proj}_V(x) = (x^T \vec{u}_1) \vec{u}_1 + (x^T \vec{u}_2) \vec{u}_2 = \begin{pmatrix} V_L \\ 0 \end{pmatrix} \\
\vec{u}_1 &= \frac{1}{\sqrt{51}} \vec{v}_1, \quad \vec{u}_2 = \frac{-1}{11 \sqrt{2}} \vec{v}_2, \quad \vec{v}_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} \vec{u}_1^T & \vec{u}_2^T \end{pmatrix}, \quad R = \begin{pmatrix} V_1 \sqrt{51} \\ 0 \sqrt{2} \end{pmatrix}
\end{align*} \]

\[ \begin{align*}
\|AB\vec{x}\| &= \|B\vec{x}\| \quad \text{because} \quad \|A\vec{u}\| = \|\vec{u}\|
\end{align*} \]

Therefore, \( AB \perp \).

Please start your solutions on this page. Additional pages may be stapled to this one.
4. **Orthogonality and least squares** Complete all three.

(a) [20%] For an inconsistent system $Ax = b$, the least squares solutions $x$ are the exact solutions of the normal equation. Define the normal equation and display the unique solution $x = \hat{x}$ when $\ker(A) = \{0\}$.

(b) [20%] State the near point theorem. Then explain how the near point to $x$ can equal $x$ itself.

(c) [60%] Fit $c_0 + c_1x$ to the data points $(0, 2), (1, 0), (2, 1), (3, 1)$ using least squares. Sketch the solution and the data points as an answer check. This is a $2 \times 2$ system problem, and should take only 1-3 minutes to complete.

\( A^T A \hat{x} = A^T b \), \( \hat{x} = (A^T A)^{-1} A^T b \)

(b) The near point equals itself when $\overline{x} \in V$.

**Theorem.** $\| \overline{x} - \overline{v} \|$ is a minimum when $\overline{v} = \text{proj}_V(\overline{x})$. This means $\| \overline{x} - \text{proj}_V(\overline{x}) \| \leq \| \overline{x} - \overline{v} \|$ for all $\overline{v} \in V$.

(c) The normal equation $A^T A \hat{x} = A^T b$ is $(4 \ 6) \hat{x} = (5)$, when $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$, \( b = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \). Then $\hat{x} = \begin{pmatrix} 13/10 \\ -1/5 \end{pmatrix}$. The ans check graphic shows a line of negative slope with 3 data points above, one below.
5. (Miscellany) Complete two.

(a) [50%] Let $3 \times 3$ matrix $A$ be invertible and assume $\text{rref}(A) = E_3 E_2 E_1 A$. The elementary matrices $E_1, E_2, E_3$ represent $\text{comb}(1,3,-5)$, $\text{swap}(1,3)$, $\text{mult}(2,-2)$, respectively. Find $A^2$.

(b) [50%] Prove that $\text{im}(A^T B^T) = \ker(BA)^\perp$, when matrix product $BA$ is defined.

(c) [50%] If you did (a) and (b), then stop, else proceed.

Prove that the span of the Gram-Schmidt vectors $u_1, \ldots, u_k$ equals exactly the span of the independent vectors $v_1, \ldots, v_k$ used to construct them.

\[ A = E_1^{-1} E_2^{-1} E_3^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -5 & 0 \\ 1 & 0 & -5 \end{pmatrix}, \quad A^2 = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 25 & 0 \\ 5 & 0 & 26 \end{pmatrix} \]

\[ \text{im}(A^T B^T) = \left( \left( \text{im}(A^T B^T) \right)^\perp \right)^\perp = \left( \text{ker}(A^T B^T)^T \right)^\perp = \text{ker}(BA)^\perp \]

\[ \text{Each } \tilde{u}_j \text{ belongs to the span of } \tilde{v}_i, \ldots, \tilde{v}_j \text{ because of the GS formula.} \]

\[ \tilde{u}_j = \frac{\tilde{u}_j}{\| \tilde{u}_j \|}, \quad \tilde{v}_j = \tilde{v}_j - \text{proj}_{\text{span}(\tilde{v}_1, \ldots, \tilde{v}_{j-1})}(\tilde{v}_j) \]

Therefore, $\tilde{u}_1, \ldots, \tilde{u}_k$ span the same set as $\tilde{v}_1, \ldots, \tilde{v}_k$. Because they are independent and so are $\tilde{v}_1, \ldots, \tilde{v}_k$, then they span the same set.