Name Andy Thaler **2270-1 Midterm 3 F2007**

2. (Abstract vector spaces, Linear transformations) Complete two.

Let W be the set of all upper triangular 4×4 matrices (lower triangle all zeros).
(a) [50%] Define addition and scalar multiplication for W and prove that W is a vector space. You may use isomorphisms to shorten the proof.

(b) [50%] Let V be the subset of W all of whose diagonal elements are zero. Prove that V is a subspace of W.

(c) [50%] If you did both (a) and (b), then stop, otherwise proceed.
Define $T(\mathbf{x}) = \mathbf{y}$ from W to V by the natural projection, in which $\mathbf{y}$ equals matrix $\mathbf{x}$ with all diagonal elements replaced by zero. Prove that T is a linear transformation from W to V and determine $\mathbf{ker}(T)$.

b) Use the subspace criterion:

$V = \{\text{UT } 4\times 4 \text{ matrices}: \text{diagonal elements} = 0\}$.

• Is the zero element in V? Yes

The 0 matrix is an upper triangular matrix w/ diagonal elements equal to 0. ✓

• Is V closed under scalar multiplication? Yes ✓

Let $A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Show that kA is in V, where k is an arbitrary constant.

$kA = \begin{bmatrix} 0 & ka & kb & kc \\ 0 & 0 & kd & ke \\ 0 & 0 & 0 & kf \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is in V.

Is V closed under addition? Yes ✓

Let $A = \begin{bmatrix} 0 & a & b & c \\ 0 & 0 & d & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & u & v & w \\ 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Show that $A+B$ is in V.

$A+B = \begin{bmatrix} 0 & a+u & b+v & w+c \\ 0 & 0 & d+x & e+y \\ 0 & 0 & 0 & f+z \\ 0 & 0 & 0 & 0 \end{bmatrix}$, which is in V

Please start your solutions on this page. Additional pages may be stapled to this one.

$\therefore$ V is a subspace of W, by the subspace criterion. ✓

(c) $T(\vec{x}) = \vec{y}$

Show: $T(k_1\vec{x}_1 + k_2\vec{x}_2) = k_1 T(\vec{x}_1) + k_2 T(\vec{x}_2)$

$LHS = T(k_1\vec{x}_1 + k_2\vec{x}_2)$

Let $\vec{x}_1 = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$ $\vec{x}_2 = \begin{bmatrix} l & m & n & o \\ 0 & p & q & r \\ 0 & 0 & s & t \\ 0 & 0 & 0 & u \end{bmatrix}$

then $k_1\vec{x}_1 + k_2\vec{x}_2 = \begin{bmatrix} k_1a+k_2l & k_1b+k_2m & k_1c+k_2n & k_1d+k_2o \\ 0 & k_1e+k_2p & k_1f+k_2q & k_1g+k_2r \\ 0 & 0 & k_1h+k_2s & k_1i+k_2t \\ 0 & 0 & 0 & k_1j+k_2u \end{bmatrix}$

and $T(k_1\vec{x}_1 + k_2\vec{x}_2) = \begin{bmatrix} 0 & k_1b+k_2m & k_1c+k_2n & k_1d+k_2o \\ 0 & 0 & k_1f+k_2q & k_1g+k_2r \\ 0 & 0 & 0 & k_1i+k_2t \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= \begin{bmatrix} 0 & k_1b & k_1c & k_1d \\ 0 & 0 & k_1f & k_1g \\ 0 & 0 & 0 & k_1i \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & k_2m & k_2n & k_2o \\ 0 & 0 & k_2q & k_2r \\ 0 & 0 & 0 & k_2t \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$= k_1 T(\vec{x}_1) + k_2 T(\vec{x}_2)$

$\ker(T) = \{\vec{x} : T(\vec{x}) = \vec{0}\}$.

For a matrix $\vec{x} = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & j \end{bmatrix}$ in W, $T(\vec{x})$ will equal $\vec{0}$ if entries b, c, d, f, g, and $i = 0$. ✓

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