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Introduction to Linear Algebra 2270-1
Sample Midterm Exam 2 Fall 2007
Exam Date: 31 October

Instructions. This exam is designed for 50 minutes. Calculators, books, notes and computers are not allowed.

1. (Matrices, determinants and independence) Do two parts.

(a) Prove that the pivot columns of A form a basis for $\mathbf{im}(A)$.

(b) Suppose A and B are both $n \times m$ of rank m and $\mathbf{rref}(A) = \mathbf{rref}(B)$. Prove or give a counterexample: the column spaces of A and B are identical.

Start your solution on this page. Please staple together any additional pages for this problem.

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2. (Kernel and similarity) Do two parts.

(a) Illustrate the relation $\mathbf{rref}(A) = E_k \cdots E_2 E_1 A$ by a frame sequence and explicit elementary matrices for the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}.$$

(b) Prove or disprove: $\mathbf{ker}(\mathbf{rref}(BA)) = \mathbf{ker}(A)$, for all invertible matrices B.

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3. **(Independence and bases)** Do two parts.

(a) Let A be a 12×15 matrix. Suppose that, for any possible independent set $\mathbf{v}_1, \ldots, \mathbf{v}_k$, the set $A\mathbf{v}_1, \ldots, A\mathbf{v}_k$ is independent. Prove or give a counterexample: $\mathbf{ker}(A) = \{\mathbf{0}\}$.

(b) Let V be the vector space of all polynomials $c_0 + c_1 x + c_2 x^2$ under function addition and scalar multiplication. Prove that $1 - x$, $2x$, $(x-1)^2$ form a basis of V.

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4. (Linear transformations) Do two parts.

(a) Let L be a line through the origin in $\mathcal{R}^3$ with unit direction $\mathbf{u}$. Let T be a reflection through L. Define T precisely. Display its representation matrix A, i.e., $T(\mathbf{x}) = A\mathbf{x}$.

(b) Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^m$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be the columns of I and let A be the matrix whose columns are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$. Prove that $T(\mathbf{x}) = A\mathbf{x}$.

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5. (Vector spaces)

(a) Show that the set of all 4×3 matrices A which have exactly one element equal to 1, and all other elements zero, form a basis for the vector space of all 4×3 matrices.

(b) Let $S = \{ \begin{pmatrix} a & b \\ -a & 2b \end{pmatrix} : a, b \text{ real} \}$. Find a basis for S.

(c) Let V be the vector space of all functions defined on the real line, using the usual definitions of function addition and scalar multiplication. Let S be the set of all polynomials of degree less than 5 (e.g., $x^4 \in V$ but $x^5 \notin V$) that have zero constant term. Prove that S is a subspace of V.

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1. (Matrix facts)

(a) Let A be a given matrix. Assume $\mathbf{rref}(A) = E_1 E_2 \cdots E_k A$ for some elementary matrices E_1, E_2, $\ldots$, E_k. Prove that if A is invertible, then A^{-1} is the product of elementary matrices.

(b) Suppose $A^2 = \mathbf{0}$ for some square matrix A. Prove that $I + 2A$ is invertible.

(c) Prove using non-determinant results that an invertible matrix cannot have two equal rows.

(d) Prove using non-determinant results that an invertible matrix cannot have a row of zeros.

(e) Prove that the column positions of leading ones in $\mathbf{rref}(A)$ identify independent columns of A. Use $\mathbf{rref}(A) = E_1 E_2 \cdots E_k A$ from (a) above in your proof details.

3. (Kernel properties)

(a) Prove or disprove: $\mathbf{ker}(\mathbf{rref}(A)) = \mathbf{ker}(A)$.

(b) Prove or disprove: $AB = I$ with A, B possibly non-square implies $\mathbf{ker}(B) = \{\mathbf{0}\}$.

(c) Find the best general values of c and d in the inequality $c \le \dim(\mathbf{im}(A)) \le d$. The constants depend on the row and column dimensions of A.

(d) Prove that similar matrices A and $B = S^{-1}AS$ satisfy $\mathbf{nullity}(A) = \mathbf{nullity}(B)$.

(e) Find a matrix A of size 3×3 that is not similar to a diagonal matrix.

4. (Independence and bases)

(a) Show that the set of all $m \times n$ matrices A which have exactly one element equal to 1, and all other elements zero, forms a basis for the vector space of all $m \times n$ matrices.

(b) Let V be the vector space of all polynomials under function addition and scalar multiplication. Prove that 1, x, $\ldots$, x^n are independent in V.

(c) Let A be an $n \times m$ matrix. Find a condition on A such that each possible set of independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is mapped by A into independent vectors $A\mathbf{v}_1, \ldots, A\mathbf{v}_k$. Prove assertions.

(d) Prove that vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ orthonormal in $\mathcal{R}^n$ are linearly independent.

(e) Let V be the vector space of all polynomials $c_0 + c_1 x + c_2 x^2$ under function addition and scalar multiplication. Prove that $1 - x$, $2x$, $(x-1)^2$ form a basis of V.

5. (Linear transformations)

(a) Let L be a line through the origin in $\mathcal{R}^2$ with unit direction $\mathbf{u}$. Let T be a reflection through L. Define T precisely. Display its representation matrix A, i.e., $T(\mathbf{x}) = A\mathbf{x}$.

(b) Let T be a linear transformation from $\mathcal{R}^n$ into $\mathcal{R}^m$. Given a basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of $\mathcal{R}^n$, let A be the matrix whose columns are $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)$. Prove that $T(\mathbf{x}) = A\mathbf{x}$.

(c) State and prove a theorem about the matrix of representation for a composition of two linear transformations T_1, T_2.

(d) Define linear isomorphism. Give an example of how an isomorphism can be used to find a basis for a subspace S of a vector space V of functions.