Name. Marissa Saunders

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Applied Linear Algebra 2270-1
Midterm Exam 1
Wednesday, 26 September 2007

Instructions: This in-class exam is designed for 50 minutes. No tables, notes, books or calculators allowed.

1. **(Inverse of a matrix)** Supply details for two of these:

   a. If A and B are $n \times n$ invertible, then $(A+B)^{-1} = A^{-1} + B^{-1}$ can be false.

   b. If square matrices A and B satisfy $AB = I$, then $A\mathbf{x} = \mathbf{0}$ cannot have infinitely many solutions.

   c. Give an example of a 3×3 matrix A and a frame sequence with three or more frames, starting at A, which proves that $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

   d. Give an example of a 4×3 system having a unique solution.

c. $A =$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \text{combo}(1,3,-1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{combo}(2,3,1)$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{mult}(2, 1/2)$$

rank = 2 nullity = 1 $\therefore$ ∞ many solns.

✓

d. $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

3 variables, 3 leading ones $\Rightarrow$ unique soln

✓

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2. (Elementary Matrices) Let A be a 3×3 matrix. Let

$$F = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

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Assume F is obtained from A by the following sequential row operations: (1) Swap rows 1 and 3; (2) Add -2 times row 2 to row 3; (3) Add 3 times row 1 to row 2; (4) Multiply row 2 by 5.

a. Write a matrix multiplication formula for F in terms of explicit elementary matrices and the matrix A. (80%)

b. Find A. (20%)

a. $F = E_4 E_3 E_2 E_1 A$

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A$$

b. $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} F.$

$$E_1^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -3 & -3 & -29/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -3 & -3 & -29/5 \\ -6 & -6 & -58/5 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & -6 & -58/5 \\ -3 & -3 & -29/5 \\ 1 & 1 & 2 \end{bmatrix}$$

3. (RREF method) | 80

Part I. State a theorem about homogeneous systems which concludes that the $m \times n$ system has at least one nonzero solution. Then supply a proof of the theorem. [20%]

Part II. Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & b-c & a \\ 1 & c & -a \\ 2 & b & a \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -a \\ a \\ a \end{pmatrix}$$

a. Determine those values of a, b and c such that the system has a unique solution. (40%) $2c-b \neq 0$ $\quad a \neq 0$

b. Determine those values of a, b and c such that the system has no solution. (20%)

$2c-b=0$ $\quad a \neq 0$

c. Determine those values of a, b and c such that the system has infinitely many solutions. (20%) $a=0$

To save time, don't attempt to solve the equations or to produce a complete frame sequence.

Part I $A\vec{x}=0$ has at least one non-zero solution $\vec{x}$ for all A.
This gives augmented matrix $\left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & 0 \\ \vdots & & & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 \end{array}\right)$

When reduced to rref, either ∞ many solns or no solution
- cannot get a signal equation for the homogeneous system since will always get either $0=0$ or an equation $=0$.

If the rref has 1^+ rows of zeros $\Rightarrow \infty$ many solns

If rref has no no rows of zeros $\Rightarrow$ unique sol

Part II

$$\left(\begin{array}{ccc|c} 1 & b-c & a & -a \\ 1 & c & -a & a \\ 2 & b & a & a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & b-c & a & -a \\ 1 & c & -a & a \\ 1 & b-c & 2a & 0 \end{array}\right) \text{combo}(2,3,-1)$$

$$\left(\begin{array}{ccc|c} 1 & b-c & a & -a \\ 1 & c & -a & a \\ 0 & 0 & a & a \end{array}\right) \text{combo}(3,1,-1)$$

$$\left(\begin{array}{ccc|c} 1 & b-c & a & -a \\ 0 & 2c-b & -2a & 2a \\ 0 & 0 & a & a \end{array}\right) \text{combo}(1,3,-1)$$

If $2c-b=0$ $\quad b=2c$

$$\left(\begin{array}{ccc|c} 1 & c & a & -a \\ 0 & 0 & -2a & 2a \\ 0 & 0 & a & a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & c & a & -a \\ 0 & 0 & 0 & 4a \\ 0 & 0 & a & a \end{array}\right) \text{combo}(3,2,2)$$

no soln if $a \neq 0$

∞ soln if $a=0$

If $2c-b \neq 0$

let $f = -2a/(2c-b)$

$$\left(\begin{array}{ccc|c} 1 & b-c & a & a \\ 0 & 1 & -f & f \\ 0 & 0 & a & a \end{array}\right)$$

$b-2c \neq 0$ and if $a \neq 0$ unique soln

If $a=0$ ∞ many soln

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4. **(Matrix algebra)**
Do two of these:

a. Find all 2×2 matrices A such that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A = A \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + A$.

b. Let A be a 4×3 matrix and B a 3×4 matrix. Explain using matrix algebra and the three possibilities why the 4×4 matrix $C = AB$ cannot start a frame sequence that ends in the identity matrix.

c. For 2×2 matrices A, B, prove that $(A+B)(A-B) = A^2 - B^2$ implies that A and B commute.

C.

$(A+B)(A-B) = A^2 - B^2$

(distribute) $A^2 + BA - AB - B^2 = A^2 - B^2$

(add $-A^2 + B^2$ to both sides) $BA - AB = 0$.

$BA = AB$

Thus A and B commute

A. let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} 2a & a \\ 2c & c \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\begin{pmatrix} c & d \\ a & b \end{pmatrix} = \begin{pmatrix} 3a & a+b \\ 3c & c+d \end{pmatrix}$

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{pmatrix} 3 & 0 & -1 & 0 \\ -1 & 0 & 3 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 5 & 0 \\ -1 & 0 & 3 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \text{ combo}(2,1,2)$$

$$\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \text{ combo}(1,2,1)$$

$$\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 1 & -5 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \text{ combo}(1,3,-1)$$

$$\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 1 & -5 & -1 \\ 0 & 0 & -4 & 0 \end{pmatrix} \text{ combo}(3,4,1)$$

$$\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & -4 & 0 \end{pmatrix} \text{ mult}(2, 1/8)$$

$$\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{pmatrix} \text{ swap}(2,3)$$

$$\begin{pmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{pmatrix} \text{ combo}(3,2,5)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{pmatrix} \text{ combo}(3,1,-5)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ combo}(3,4,4)$$

general soln: $a = 0$, $b = -t_1$, $c = 0$, $d = t_1$

$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} t_1$ for all t_1

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scale kI rotation $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

reflection $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ h shear $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ v shear $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$

5. (Geometry and linear transformations)

Part I. Classify $T(\mathbf{x}) = A\mathbf{x}$ geometrically as scaling, projection onto line L, reflection in line L, pure rotation by angle θ, rotation composed with scaling, horizontal shear, vertical shear. Find an equation for L and define θ where applicable. [60%]

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a. $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$ rotation + scaling $k=2$ $A = 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ cc rotation by 90°.

b. $A = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$ horizontal shear (form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$) $k=5$.

? c. $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ projection $\begin{pmatrix} u_1^2 & u_2u_1 \\ u_1u_2 & u_2^2 \end{pmatrix}$ $u_1 = 0$, $u_2 = 1$ projection onto L: $x=0$.

d. $A = \frac{1}{4}\begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$ projection $u_1 = \sqrt{3}/2$, $u_2 = 1/2$ $L: y = \sqrt{3}x$.

e. $A = \frac{1}{2}\begin{pmatrix} -\sqrt{3} & 1 \\ 1 & \sqrt{3} \end{pmatrix}$ reflection in line L.
$a = -\frac{\sqrt{3}}{2}$
$b = 1/2$
$a^2+b^2=1$.

$2u_1^2 - 1 = -\sqrt{3}/2$ $L: y = (2-\sqrt{3})x$

$u_1^2 = \frac{1}{2} - \frac{\sqrt{3}}{4}$

$u_1 = \frac{\sqrt{2-\sqrt{3}}}{2}$

$u_2 = \frac{\sqrt{2+\sqrt{3}}}{2}$

Part II. Give details. [40%]

f. Define reflection in a line L in $\mathcal{R}^3$.

reflection $A = 2\,\mathrm{aug}(u_1\vec{u} \ldots u_n\vec{u}) - I$ In $\mathbb{R}^3$ $A = \begin{pmatrix} 2u_1^2-1 & u_2u_1 & u_3u_1 \\ u_1u_2 & 2u_2^2-1 & \cancel{u_3u_2} \\ u_1u_3 & u_2u_3 & 2u_3^2-1 \end{pmatrix}$ ✓

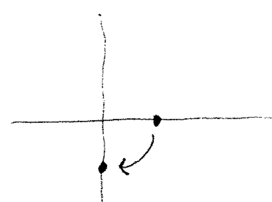
g. Display the matrix A of a projection onto a line L in $\mathcal{R}^3$.

projection $A = \mathrm{aug}(u_1\vec{u} \ldots u_n\vec{u})$ in $\mathbb{R}^3$ - $A = \begin{pmatrix} u_1^2 & u_2u_1 & u_3u_1 \\ u_1u_2 & u_2^2 & u_3u_2 \\ u_1u_3 & u_2u_3 & u_3^2 \end{pmatrix}$ ✓

h. Define rotation clockwise by angle θ in $\mathcal{R}^2$.

rotation : $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ ✓

Test sign for 90° $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$\Rightarrow (0,-1)$

$\therefore$ sign is correct .

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