Introduction to Linear Algebra 2270-1
Final Exam Fall 2007

Instructions. The time allowed is 120 minutes. The examination consists of five problems, one for each of chapters 3, 4, 5, 6, 7, each problem with multiple parts. A chapter represents 25 minutes on the final exam. Each problem represents several textbook problems numbered (a), (b), (c), $\cdots$. Please solve enough parts to make 100% on each chapter. Choose the problems to be graded by check-mark [X]; the credits should add to 100.

Calculators, books, notes and computers are not allowed.

Answer checks are not expected or required. First drafts are expected, not complete presentations.

Please submit **exactly five** separately stapled packages of problems.

Keep this page for your records.

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Ch3. (Subspaces of $\mathcal{R}^n$ and Their Dimensions)

☐ [30%] Ch3(a): Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 4 & 2 & 2 & 0 & 0 \end{bmatrix}$. Find bases for the image and kernel of A.

☐ [40%] Ch3(b): Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ the columns of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Define $T(\mathbf{x}) = A\mathbf{x}$. Find the matrix of T relative to the basis $\mathbf{v}_1 + \mathbf{v}_3$, $\mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_1 + \mathbf{v}_2$.

☐ [30%] Ch3(c): Let V be the vector space of all continuously differentiable functions $f(x)$ defined on $0 \le x \le 1$. Let S be the subset of V defined by $f(1) = f'(0) + \int_0^1 f'(x)x^2 dx$, $f'(1/3) = f(1/3)$. Prove that S is a subspace of V.

☐ [30%] Ch3(d):

(1) [10%] Prove that the kernel of a matrix defines a subspace S of $\mathcal{R}^n$.

(2) [10%] Find a basis for the subspace $S = \textbf{span}\{e^x, \sin x, 1 - \sin x, 2 + x, 3 + x\}$, in the linear space V of all functions on the real line.

(3) [10%] Prove that the intersection of two subspaces S_1 and S_2 is also a subspace.

☐ [40%] Ch3(e): Let V be the vector space of all data packages $\mathbf{v} = \begin{pmatrix} f \\ x_0 \\ y_0 \end{pmatrix}$, where f is a continuous function defined on $0 \le x \le 1$ and x_0, y_0 are real values. Define $\boxed{+}$ and $\boxed{\cdot}$ componentwise. Let S be the subset of V defined by $f(0) = f(1)$, $f(1/2) + y_0 = 0$. Prove or disprove: S is a subspace of V.

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Ch4. (Linear Spaces)

☐ [30%] Ch4(a): Let $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Let V be the linear space of all 3×3 matrices. Let S be the set of all 3×3 matrices A such that $\mathbf{x}$ belongs to the kernel of A. Prove or disprove: S is a subspace of V.

☐ [40%] Ch4(b): Let V be the linear space of all functions $f(x) = c_0 + c_1 x + c_2 x^2$. Define $T(f) = c_2(1-x)^2$ from V to V. Find bases for the image and kernel of T and report the rank and nullity of T.

☐ [30%] Ch4(c): Let V be the linear space of all real 3×3 matrices M. Let T be defined on V by $T(\mathbf{M}) = \mathbf{N}$ where $\mathbf{N} = \mathbf{M}$ except for the lower triangle, which is all zeros. Find bases for the image and kernel of T.

☐ [40%] Ch4(d): Let $A = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$. Find the set X of all matrices B not similar to A. For example, $B = 0$ is in X, because $AS = SB$ implies $AS = 0$ and then $A = 0$, a false statement.

Please start your solutions on this page. Staple on additional pages.

Ch5. (Orthogonality and Least Squares)

☐ [20%] Ch5(a): Find the orthogonal projection of $\mathbf{v}$ onto $V = \mathbf{span}(\mathbf{v}_1, \mathbf{v}_2)$, given

$$\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} 0 \\ 4 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}.$$

☐ [40%] Ch5(b): Derive the equations for m and b in the least squares fit of $y = mx + b$ to data points (x_i, y_i), $i = 1, \ldots, n$. State what you assume and try to prove the result from the normal equations in the theory of least squares.

☐ [20%] Ch5(c): Let A be 3×4 with kernel zero. Prove or give a counterexample: $\dim \left(\mathbf{im}(A^T A)\right) + \dim\left(\mathbf{ker}\,(A)\right) = 4$.

☐ [30%] Ch5(d): Consider the linear space V of polynomials $f(t) = c_0 + c_1 t + c_2 t^2$ on $0 \le t \le 1$ with inner product

$$< f, g >= \int_0^1 f(t) g(t) dt.$$

Find a basis for the subspace S of all f in V orthogonal to $1 + t$ satisfying the additional restriction equation $f(1/2) = 0$.

☐ [20%] Ch5(e): Find the Gram-Schmidt orthonormal vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ for the following independent set:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \end{pmatrix}.$$

☐ [40%] Ch5(f): Find the QR-factorization of $A = \begin{pmatrix} 2 & -2 & 18 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$.

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Ch6. (Determinants)

☐ [50%] Ch6(a): Let B be the invertible matrix given below, where $\boxed{?}$ means the value of the entry does not affect the answer to this problem. The second matrix C is the adjugate (or adjoint) of B. Find the value of $\det(2B^{-1}(C^T)^{-2})$.

$$B = \begin{pmatrix} ? & ? & ? & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ ? & ? & ? & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 3 & 9 & 0 \\ -6 & -3 & 6 & 0 \\ -3 & 6 & 3 & 0 \\ 2 & 1 & 3 & -5 \end{pmatrix}$$

☐ [25%] Ch6(b): Assume $A = \mathbf{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is 3×3 and $B = \mathbf{aug}(\mathbf{v}_1 + \mathbf{v}_3, \mathbf{v}_3 - \mathbf{v}_2, 3\mathbf{v}_2 - \mathbf{v}_3)$. Suppose $\det(A+B) + 2\det(A^2) = 0$. Find all possible values of $\det(A)$.

☐ [25%] Ch6(c): Prove from the Four Rules that $\det(A) = 0$ if two columns of A are linearly dependent.

☐ [25%] Ch6(d): Assume given 3×3 matrices A, B. Suppose $E_5E_4B = E_3E_2E_1A$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a a combination, a multiply by -2, a combination, a swap and a multiply by 5. Assume $\det(A) = -1$. Find $\det(3AB^2)$.

☐ [25%] Ch6(e): Evaluate $\det(A)$ by any hybrid method. Symbol x is a variable.

$$A = \begin{pmatrix} x & 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & -3 & 0 \\ 1 & 2 & 3 & 4 & 1 \end{pmatrix}$$

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Ch7. (Eigenvalues and Eigenvectors)

☐ [40%] Ch7(a): Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$. To save time, **do not** find eigenvectors!

☐ [40%] Ch7(b): Given $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, find an invertible matrix P and a diagonal matrix D such that $AP = PD$.

☐ [20%] Ch7(c): Consider the 3×3 matrix

$$A = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Assume the eigenpairs are

$$\left(2, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

(1) [10%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(2) [10%] Display explicitly Fourier's model for A.

☐ [40%] Ch7(d): Consider a discrete dynamical system $\mathbf{x}(n+1) = A\mathbf{x}(n)$. Given A and $\mathbf{x}(0)$ below, find exact formulas for the vectors $\mathbf{x}(n)$ and $\lim_{n\to\infty} \mathbf{x}(n)$.

$$A = \frac{1}{9}\begin{pmatrix} 7 & 1 \\ -2 & 10 \end{pmatrix}, \quad \mathbf{x}(0) = \begin{pmatrix} 36 \\ 45 \end{pmatrix}.$$

Please start your solutions on this page. Staple on additional pages.