2250-1 F2007 final

Ch3 (a) $BC = 15I$ because of $A\,\mathrm{adj}(A) = \det(A)\,I$. We get $15 = \mathrm{row}(B,4)\,\mathrm{col}(C,4) = \det(B)$. Then $\det(C) = \dfrac{15^4}{\det B} = 15^3$ and

$$\det(3\,B^{-2}C^T) = \frac{3^4 \det(C)}{\det(B)^2} = \frac{3^4\, 15^3}{15^2} = 3^4(15) = \boxed{5(3^5)}$$

Ch3 (b)
$$\left(\begin{array}{ccc|c} 2 & 3 & -k & 1 \\ 2 & k & -2 & 2 \\ 0 & 0 & k-2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 2 & 3 & -k & 1 \\ 0 & k-3 & k-2 & 1 \\ 0 & 0 & k-2 & 1 \end{array}\right)$$

If $k-2=0$, then the last eq is a signal eq $0=1$ and <u>no sol</u>.

If $k-3=0$, then $\left(\begin{array}{ccc|c} 2 & 3 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ and <u>∞-many sols</u>.

Otherwise $(k-2)(k-3) \neq 0$ and there is a <u>unique sol</u>.

Three possibilities: no sol, ∞-many sols, unique sol.

Ch3 (c) $\mathrm{rref}(A) = I \Rightarrow A^{-1}$ exists $\Rightarrow$ unique sol $x = A^{-1}b$

$\det(A) \neq 0 \Rightarrow A^{-1}$ exists $\Rightarrow$ unique sol $x = A^{-1}b$

rank $= n \Rightarrow$ " $\Rightarrow$ "

Ch3 (d)
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Ch3 (e) $x_2 = \dfrac{\Delta_2}{\Delta}$ $\quad \Delta = \begin{vmatrix} 1 & 2 & -1 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{vmatrix} = 4, \quad \Delta_2 = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 1 & -1 & -2 & 1 \\ 0 & -1 & 4 & 0 \\ 0 & 1 & 2 & 1 \end{vmatrix} = -11$

$x_2 = \dfrac{-11}{4}$

Ch3 (f) The determinant of a triangular matrix is the product of the diagonal entries, in this case nonzero, so $\det(C) \neq 0$. By standard theorems, C is invertible.

ch 4 (a) (1) $A = \operatorname{aug}(v_1, v_2, v_3)$ has rank $= 3 \iff v_1, v_2, v_3$ independent.

(2) If A is 3×3, then $\det A \neq 0 \iff v_1, v_2, v_3$ independent.

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ 1 & 4 & 7 \\ 0 & 1 & x \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \\ 0 & 1 & x \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & x \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & x-2 \\ 0 & 0 & 0 \end{pmatrix}$$

Cols of A are independent $\iff x-2 \neq 0$. ANS : $\boxed{\text{Dependent} \iff x=2}$

ch 4 (b) $$E = \begin{pmatrix} 2 & 1 & 2 & 1 \\ 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -2 & 1 & 2 \\ 0 & 5 & 0 & -9 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

E has rank 2. Further, v_1, v_2 are indep and w_1, w_2 are indep by inspection. Then $S_1 = S_2$. [See maple lab 5]

ch 4 (c)
$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & -3 & -3 & -2 & 0 \\ 0 & 6 & 6 & 4 & 0 \\ 0 & -1 & -2 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & -3 & -3 & -2 & 0 \\ 0 & 6 & 6 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -3 & 6 & -1 \\ 0 & -2 & -3 & 6 & -2 \\ 0 & -1 & -2 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 3 & -6 & 1 \\ 0 & 2 & 3 & -6 & 2 \\ 0 & 1 & 2 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 3 & -6 & 1 \\ 0 & 0 & -3 & 6 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 0 & 1 & 3 & -6 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots$(A^T) = 2, 3$
pivots$(A) = 2, 3$

ans: (1) colspace(A) = span of cols 2, 3 of A
(2) rowspace(A) = span of rows 2, 3 of A

ch 4 (d) Not a subspace, $\vec{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ fails to be in S due to $x_1 - 2x_4 = 1$.

ch 4 (e) $a =$ $b = 1$ $c =$ $d = 4$

ch5 (a) $2r^2+15r+7 = (2r+1)(r+7)$

$y = c_1 e^{-x/2} + c_2 e^{-7x}$

ch5 (b) $r^2(r+2)^2 r^2 (r-2)(r+2)\left((r+2)^2+4\right) = 0$

atoms = $1, x, x^2, x^3, e^{2x}, e^{-2x}, xe^{-2x}, x^2e^{-2x},$

$e^{-2x}\cos 2x, e^{-2x}\sin 2x$

y = linear combination of the 10 atoms

ch5 (c) $r^2+6r+10 = (r+3)^2+1$ complex roots $\Rightarrow$ $\boxed{\text{under damped}}$

ch5 (d) $y = x^{s_1}(d_1 + d_2 x + d_3 x^2)$

$+ x^{s_2}(d_4 + d_5 x)e^{3x}$

$+ x^{s_3}(d_6 \cos x + d_7 \sin x + d_8 x\cos x + d_9 x \sin x)$

atom Root$(x^2) = 0$ $\Rightarrow$ $\boxed{s_1 = 3}$ = multiplicity of root 0 in char eq $r^3(r^2+1)^2(r^2-9) = 0$

atom Root$(xe^{3x}) = 3$ $\Rightarrow$ $\boxed{s_2 = 1}$ = mult. of root 3 in char eq

atom Root$(\cos x) = i$ $\Rightarrow$ $\boxed{s_3 = 2}$ = mult. of root i in char eq.

ch5 (e) $x = d_1\cos(5t) + d_2 \sin(5t)$

$d_1 = -3/65$, $d_2 = 2/65$ from $\begin{pmatrix} -15 & 10 \\ -10 & -15 \end{pmatrix}\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

ch6 (a) $\det(A-\lambda I) = (2-\lambda)(3-\lambda)\left[1+\lambda-\lambda^3-\lambda^2\right]$

$= (2-\lambda)(3-\lambda)(1+\lambda)(1-\lambda^2)$

roots $= -1, -1, 1, 2, 3$

ch6 (b) $D = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4i & 0 \\ 0 & 0 & 0 & -4i \end{pmatrix}$ $\quad P = \begin{pmatrix} 2 & -11 & 1 & 1 \\ 1 & -3 & i & -i \\ -4 & 10 & 0 & 0 \\ 4 & -30 & 0 & 0 \end{pmatrix}$

$$A\left(c_1\begin{pmatrix}2\\1\\-4\\4\end{pmatrix}+c_2\begin{pmatrix}-11\\-3\\10\\-30\end{pmatrix}+c_3\begin{pmatrix}1\\i\\0\\0\end{pmatrix}+c_4\begin{pmatrix}1\\-i\\0\\0\end{pmatrix}\right) = 4c_1\begin{pmatrix}2\\1\\-4\\4\end{pmatrix}+2c_2\begin{pmatrix}-11\\-3\\10\\-30\end{pmatrix}+4ic_3\begin{pmatrix}1\\i\\0\\0\end{pmatrix}$$
$$-4ic_4\begin{pmatrix}1\\-i\\0\\0\end{pmatrix}$$

ch6 (c)

$$A-2I = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & -2 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 - x_2 = 0 \\ x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \vec{v}_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$A-5I = \begin{pmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 - x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -2t_1 + t_2 \\ x_2 = t_1 \\ x_3 = t_2 \end{cases} \Rightarrow \vec{v}_2 = \begin{pmatrix}-2\\1\\0\end{pmatrix}, \vec{v}_3 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \quad P = \begin{pmatrix} 1 & -2 & 1 \\ 1 & 1 & 0 \\ 0 & \underbrace{0 \quad 1}_{\text{can be swapped}} \end{pmatrix}$$

Ch 7 (a) $\det(A-\lambda I) = (3-\lambda)(3-\lambda)(3+\lambda)$

Eigenpairs $\left(3, \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right), \left(3, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right), \left(-3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right)$

$$\vec{u}(t) = e^{3t} c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + e^{3t} c_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + e^{-3t} c_3 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Ch 7 (b) $r^2 - 2r + 5 = 0 \Rightarrow (r-1)^2 + 4 = 0 \Rightarrow r = 1 \pm 2i$

$$\boxed{x(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t}$$

$$\boxed{y(t) = c_2 e^t \cos 2t - c_1 e^t \sin 2t}$$

Ch 7 (c) $\begin{cases} x' = 0 \\ y' = x \end{cases}$ $\boxed{\begin{array}{l} x = c_1 \\ y = c_1 t + c_2 \end{array}}$

Ch 7 (d) $AP = PD$ $D = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$A = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

ch 10 (a)

$$-1 + s^2 x_1 = x_1 - 2x_2 \qquad x_1 = \mathcal{L}(x(t))$$
$$s^2 x_2 = 3x_1 \qquad x_2 = \mathcal{L}(y(t))$$

$$\begin{cases} (s^2-1)x_1 + 2x_2 = 1 \\ (-3)x_1 + s^2 x_2 = 0 \end{cases}$$

$$\mathcal{L}(x) = x_1 = \frac{\Delta_1}{\Delta} \qquad \Delta = \begin{vmatrix} s^2-1 & 2 \\ -3 & s^2 \end{vmatrix} = s^4 - s^2 + 6$$

$$= \boxed{\frac{s^2}{s^4-s^2+6}} \qquad \Delta_1 = \begin{vmatrix} 1 & 2 \\ 0 & s^2 \end{vmatrix} = s^2$$

ch 10 (b)

$$\mathcal{L}(f_1) = \left(\frac{d}{ds}\right)^2 \mathcal{L}(t^3 \cosh t)\Big|_{s \to s+2}$$
$$= \mathcal{L}((-t)^2 t^3 \cosh t)\Big|_{s \to s+2}$$
$$= \mathcal{L}(e^{-2t} t^5 \cosh t)$$

$$\mathcal{L}(f_2) = \frac{(s-1)^2}{(s+1)^3} = \frac{(s-2)^2}{s^3}\Big|_{s \to s+1}$$
$$= \frac{s^2 - 4s + 4}{s^3}\Big|_{s \to s+1}$$
$$= \left(\frac{1}{s} - \frac{4}{s^2} + \frac{4}{s^3}\right)\Big|_{s \to s+1}$$
$$= \mathcal{L}(e^{-t}(1 - 4t + 2t^2))$$

$$\mathcal{L}(f_3) = \mathcal{L}(e^{2t} \cos t \sinh t)$$

$$f = f_1 + f_2 + f_3 = \boxed{t^5 e^{-2t} \cosh(t) + e^{-t}(1 - 4t + 2t^2) + e^{2t} \cos(t) \sinh(t)}$$

ch 10 (c)

$$\frac{2+2s}{s^2-2s} = \frac{3}{s-2} - \frac{1}{s}$$

$$\frac{5s+15}{(s-2)^2(s^2+1)} = \frac{-3}{s-2} + \frac{5}{(s-2)^2} + \frac{3s+1}{s^2+1}$$

$$\boxed{f(t) = -1 + 5te^{2t} + 3\cos t + \sin t}$$

ch 10 (d) $\mathcal{L}(tf(t)) = \mathcal{L}(e^{-t} \sin t) = \dfrac{1}{(s+1)^2+1}$ but $-\dfrac{d}{ds}F(s) = \mathcal{L}(tf(t))$

Then solve the DE by quadrature to get $F(s) = -\tan^{-1}(s+1) + c$

At $s = \infty$, $F = 0$ and $\tan^{-1}(\infty) = \frac{\pi}{2}$. $\boxed{\mathcal{L}(f(t)) = \frac{\pi}{2} - \tan^{-1}(s+1)}$