

# Differential Equations and Linear Algebra

2250-2 10:10am 13 Dec 2007

**Instructions.** The time allowed is 120 minutes. The examination consists of six problems, one for each of chapters 3, 4, 5, 6, 7, 10, each problem with multiple parts. A chapter represents 20 minutes on the final exam.

The final exam counts as two midterm exams. For example, if exam scores earned were 90, 91, 92 and the final exam score is 89, then the exam average for the course is

$$\frac{90 + 91 + 92 + 89 + 89}{5}.$$

Each problem represents several textbook problems numbered (a), (b), (c),  $\dots$ . Choose the problems to be graded by check-mark ; the credits should add to 100. Each chapter (Ch3, Ch4, Ch5, Ch6, Ch7, Ch10) adds at most 100 towards the maximum final exam score of 600. The final exam score is reported as a percentage 0 to 100, which is the sum of the scores earned on six chapters divided by 600 to make a fraction, then converted to a percentage.

Calculators, books, notes and computers are not allowed.

Details count. Less than full credit is earned for an answer only, when details were expected. Generally, answers count only 25% towards the problem credit.

Answer checks are not expected or required. First drafts are expected, not complete presentations.

Please submit **exactly six** separately stapled packages of problems, one package per chapter.

**Please discard this page or keep it for your records.**

## Differential Equations and Linear Algebra 2250-2

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### Ch3. (Linear Systems and Matrices)

[40%] Ch3(a): Let  $B$  be the invertible matrix given below, where  $\square$  means the value of the entry does not affect the answer to this problem. The second matrix  $C$  is the adjugate (or adjoint) of  $B$ . Find the value of  $\det(2B^{-1}(C^T)^{-2})$ .

$$B = \begin{pmatrix} ? & ? & ? & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ ? & ? & ? & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 3 & 9 & 0 \\ -6 & -3 & 6 & 0 \\ -3 & 6 & 3 & 0 \\ 2 & 1 & 3 & -5 \end{pmatrix}$$

[40%] Ch3(b): State the three possibilities for a linear system  $A\mathbf{x} = \mathbf{b}$  [5%]. Determine which values of  $k$  correspond to these three possibilities, for the system  $A\mathbf{x} = \mathbf{b}$  given in the display below [20%].

$$A = \begin{pmatrix} 2 & 3 & -k \\ 2 & k & -2 \\ 0 & 0 & k-1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

[20%] Ch3(c): Assume  $A$  is an  $n \times m$  matrix. State a linear algebra theorem with the conclusion that  $A\mathbf{x} = \mathbf{0}$  has infinitely many solutions. No proofs, please!

If you solved (a), (b) and (c), then go on to Ch4. Otherwise, try (d), (e) and (f). Maximum credit is 100%.

[10%] Ch3(d): Give an example of a  $4 \times 3$  matrix  $A$  and a  $4 \times 1$  vector  $\mathbf{b}$  such that the system  $A\mathbf{x} = \mathbf{b}$  has a unique solution.

[20%] Ch3(e): Find the value of  $x_1$  by Cramer's Rule in the system  $C\mathbf{x} = \mathbf{b}$ , given  $C$  and  $\mathbf{b}$  below. Evaluate determinants by any hybrid method (triangular, swap, combo, multiply, cofactor). The use of  $2 \times 2$  Sarrus' rule is allowed. The  $3 \times 3$  Sarrus' rule is **disallowed**.

$$C = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 1 & 3 & -2 & 1 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

[10%] Ch3(f): Prove that if a  $4 \times 4$  triangular matrix  $C$  is invertible, then all diagonal entries of  $C$  are nonzero.

**Staple this page to the top of all Ch3 work. Submit one package per chapter.**

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### Ch4. (Vector Spaces)

[25%] Ch4(a): State (1) a rank test and (2) a determinant test to detect the independence or dependence of fixed vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  [5%]. Apply one of the tests and report for which values of  $x$  the three vectors are **independent** [15%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} -1 \\ 2 \\ 7 \\ x \end{pmatrix}.$$

[25%] Ch4(b): Let  $\mathbf{v}_1, \mathbf{v}_2$  be a basis for a 2-dimensional subspace  $S$  of vector space  $V = \mathcal{R}^n$ . Let  $\mathbf{w}_1, \mathbf{w}_2$  be any other independent set in  $S$ . Define  $E$  to be the augmented matrix of  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2$ . What condition on  $E$  implies the bases are equivalent? Apply the condition to the example

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

[50%] Ch4(c): Define the  $5 \times 5$  matrix  $A$  by the display below. Find a basis of fixed vectors in  $\mathcal{R}^5$  for (1) the column space of  $A$  [25%] and (2) the row space  $A$  [25%]. The two displayed bases **must** consist of columns of  $A$  and rows of  $A$ , respectively.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & -2 & 0 \\ 0 & 6 & 6 & 4 & 0 \\ 0 & -1 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 & 0 \end{pmatrix}$$

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If you did (a), (b) and (c), then 100% has been marked – go on to Ch5. Otherwise, unmark one of (a), (b) or (c), then complete (d). Only three problems will be graded.

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[25%] Ch4(d): Define  $S$  to be the set of all vectors  $\mathbf{x}$  in  $\mathcal{R}^4$  such that  $x_1 = x_3 + x_4$  and  $e^{x_1 - 2x_4} = 1$ . Prove or disprove that  $S$  is a subspace of  $\mathcal{R}^4$ .

**Staple this page to the top of all Ch4 work. Submit one package per chapter.**

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**Ch5. (Linear Equations of Higher Order)** [10%] Ch5(a): Find the general solution of the differential equation

$$y'' + \frac{5}{2}y' + y = 0.$$

 [20%] Ch5(b): Find the homogeneous differential equation general solution, given characteristic equation

$$(r^2 + 3r)^3(r^4 - 9r^2)(r^2 + 1) = 0.$$

 [10%] Ch5(c): Given a damped spring-mass system  $mx''(t) + cx'(t) + kx(t) = 0$  with  $m = 15$ ,  $c = 17$  and  $k = 4$ , classify the answer as over-damped, critically damped or under-damped. Please, **do not solve** the differential equation! [40%] Ch5(d): Assume a ninth order constant-coefficient differential equation has characteristic equation  $r(r^2 + 1)^2(r^2 - 4)^2 = 0$ . Suppose the right side of the differential equation is

$$f(x) = x(x + 2e^{2x}) + x \sin x + 3 \cos x$$

Determine the **corrected** trial solution for  $y_p$  according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients! [20%] Ch5(e): Find the steady-state periodic solution for the equation

$$x'' + 4x' + 15x = \cos(5t).$$

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### Ch6. (Eigenvalues and Eigenvectors)

[30%] Ch6(a): Find the eigenvalues of the matrix  $A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -3 & 0 \\ 1 & 2 & 3 & 4 & 0 \end{pmatrix}$ . To save time, **do**

**not** find eigenvectors!

[20%] Ch6(b): Consider the  $3 \times 3$  matrix

$$A = \begin{pmatrix} 4 & 2 & -2 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Assume the eigenpairs are

$$\left( 2, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right), \quad \left( 4, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right), \quad \left( 4, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

(1) [10%] Display an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $AP = PD$ .

(2) [10%] Display explicitly Fourier's model for  $A$ .

[50%] Ch6(c): The matrix  $A$  below has eigenvalue package

$$D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{pmatrix}$$

Test  $A$  to see it is diagonalizable, and if it is, then display the matrix package  $P$  of eigenvectors such that  $AP = PD$ .

$$A = \begin{pmatrix} 13 & -4 & 2 \\ -2 & 11 & 2 \\ 0 & 0 & 15 \end{pmatrix}$$

**Staple this page to the top of all Ch6 work. Submit one package per chapter.**

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**Ch7. (Linear Systems of Differential Equations)** [50%] Ch7(a): Apply the eigenanalysis method to solve the system  $\mathbf{u}' = A\mathbf{u}$ , given

$$A = \begin{pmatrix} -3 & 5 & -10 \\ 0 & 2 & 0 \\ 5 & -5 & 12 \end{pmatrix}$$

 [25%] Ch7(c): Solve for  $y(t)$  in the system below.

$$\begin{aligned} x' &= x + 3y, \\ y' &= -3x + y. \end{aligned}$$

 [15%] Ch7(c): Solve the system  $\mathbf{u}' = A\mathbf{u}$  for matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

 [10%] Ch7(d): Assume  $A$  is  $2 \times 2$  and the general solution of  $\mathbf{u}' = A\mathbf{u}$  is given by

$$\mathbf{u}(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Find  $A$ .**Staple this page to the top of all Ch7 work. Submit one package per chapter.**

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### Ch10. (Laplace Transform Methods)

It is assumed that you have memorized the basic 4-item Laplace integral table and know the 6 basic rules for Laplace integrals. No other tables or theory are required to solve the problems below. If you don't know a table entry, then leave the expression unevaluated for partial credit.

[35%] Ch10(a): Apply Laplace's method to the system. Find a  $2 \times 2$  system for  $\mathcal{L}(x)$ ,  $\mathcal{L}(y)$  [20%]. Solve it **only** for  $\mathcal{L}(x)$  [15%]. Do not solve for  $x(t)$  or  $y(t)$ !

$$\begin{aligned}x'' &= x - 3y, \\y'' &= 3x, \\x(0) &= 0, \quad x'(0) = 0, \\y(0) &= 0, \quad y'(0) = 1.\end{aligned}$$

[35%] Ch10(b): Solve for  $f(t)$ , given

$$\mathcal{L}(f(t)) = \left( \left( \frac{d}{ds} \right)^2 \left( \mathcal{L}(t^3 e^t \sin 3t) \right) \right) \Big|_{s \rightarrow (s+2)} + \frac{s^2 + s + 2}{(s-1)^3} + \mathcal{L}(e^t \sin 2t \cos 2t)$$

[30%] Ch10(c): Find  $f(t)$  by partial fraction methods, given

$$\mathcal{L}(f(t)) = \frac{2-2s}{s^2+2s} + \frac{5s-15}{(s+2)^2(s^2+1)}.$$

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If you solved (a), (b) and (c), then you have 100%. To select (d), unmark one of the previous three. Only three parts will be graded. If you did not solve (a) or (b), then the maximum score is 95.

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[30%] Ch10(d): Find  $\mathcal{L}(f(t))$ , given  $f(t) = e^{2t}(\cos(t) - 1)/t$ .