2250-2 F2007 final

Ch3(a) $BC = 15I$ because $\det(B) = 15 = \text{row}(B,4)\,\text{col}(C,4)$ by the adjugate formula $A\,\text{adj}(A) = \det(A) I$. Because $\det(A^{-1}) = \frac{1}{\det A}$ and $\det(A^T) = \det(A)$, then $\det(2B^{-1}(C^T)^{-2}) = \dfrac{\det(2I)}{\det B\,\det(C)^2} =$ $\boxed{\dfrac{2^4}{15\,(15^3)^2}}$, where $BC = 15I$ implies $\det C = 15^3$.

Ch3(b) $\left(\begin{array}{ccc|c} 2 & 3 & -k & 1 \\ 2 & k & -2 & 2 \\ 0 & 0 & k-1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & -k & 1 \\ 0 & k-3 & k-2 & 1 \\ 0 & 0 & k-1 & 3 \end{array}\right)$

No sol. if $k-1=0$. Unique sol for $k-1 \neq 0, k-3 \neq 0$. If $k-3=0$, then $\left(\begin{array}{ccc|c} 2 & 3 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 3 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right) \leftarrow$ signal eq and there is no sol.

Ch3(c) If a system $A\vec{x} = \vec{0}$ has more variables than equations then there is at least one free variable and ∞-many solutions

Ch3(d) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

ch3(e) $x_1 = \dfrac{\Delta_1}{\Delta}$ $\quad \Delta_1 = \begin{vmatrix} 0 & 2 & -1 & 0 \\ -1 & 3 & -2 & 1 \\ -1 & 3 & 4 & 0 \\ 1 & 0 & 2 & 1 \end{vmatrix} = 21$, $\Delta = \det(A) = 4$ $\quad \boxed{x_1 = \dfrac{21}{4}}$

ch3(f) If C is invertible, and triangular, then $\det(C) \neq 0$ and also $\det(C) =$ product of diagonal elements by the triangular rule. Therefore, no diagonal element of C can be zero.

ch4 (a) (1) v_1, v_2, v_3 independent $\iff$ rank(aug(v_1, v_2, v_3)) = 3

(2) If v_1, v_2, v_3 are in $\mathbb{R}^3$, then v_1, v_2, v_3 are independent $\iff$ det(aug(v_1, v_2, v_3)) $\neq 0$.

Apply: $A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 0 & 2 \\ 1 & 3 & 7 \\ 0 & 1 & x \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \\ 0 & 1 & x \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & x \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & x-2 \\ 0 & 0 & 0 \end{pmatrix}$

the rank is 3 $\iff x-2 \neq 0$. $\boxed{\text{Independent} \iff x \neq 2}$

ch4 (b) The condition is rank(E) = 2.

v_1, v_2 span the xy-plane and so do w_1, w_2. So they are different bases for $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : z = 0 \right\}$ = xy-plane.

$E = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, rank(E) = 2.

ch4 (c) $\text{rref}(A^T) = \begin{pmatrix} 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $\text{rref}(A) = \begin{pmatrix} 0 & 1 & 0 & 1/3 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

rowspace(A) = span{row(A, 2), row(A, 4)}

colspace(A) = span{col(A, 2), col(A, 3)}

ch4 (d) Because $e^{x_1 - 2x_4} = 1$ is equivalent to $x_1 - 2x_4 = 0$, then $S = \{\vec{x} : A\vec{x} = \vec{0}\}$ where $A = \begin{pmatrix} 1 & 0 & -1 & -1 \\ 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. By the Kernel Theorem, S is a subspace of $\mathbb{R}^4$.

ch 5 (a) $(2r+1)(5r+10)=0$, $r=-1/2,-2$, $y(x)=c_1e^{-x/2}+c_2e^{-2x}$

ch 5 (b) $(r(r+3))^3(r^2(r-3)(r+3))(r^2+1)=0$

$r^5(r+3)^4(r-3)(r^2+1)=0$

$$
\begin{aligned}
y(x) = {} & c_1+c_2x+c_3x^2+c_4x^3+c_5x^4 \\
& + (c_6+c_7x+c_8x^2+c_9x^3)e^{-3x} \\
& + c_{10}e^{3x} \\
& + c_{11}\cos x + c_{12}\sin x
\end{aligned}
$$

ch 5 (c) $15r^2+17r+4 = (3r+1)(5r+4)$ real roots <u>overdamped</u>

ch 5 (d) $r(r^2+1)^2(r-2)^2(r+2)^2=0$

$$
\begin{aligned}
y = {} & x^{s_1}(d_1+d_2x+d_3x^2) \\
& + x^{s_2}(d_4+d_5x)e^{2x} \\
& + x^{s_3}(d_6\cos x + d_7\sin x + d_8x\cos x + d_9 x\sin x)
\end{aligned}
$$

atomRoot$(x^2)=0$, implies $\boxed{s_1=1}$

atomRoot$(e^{2x})=2$, implies $\boxed{s_2=2}$

atomRoot$(\cos x)=i$, implies $\boxed{s_3=2}$

ch 5 (e) $x(t)=\frac{1}{25}\sin(5t)-\frac{1}{50}\cos(5t)$

ch6 (a) $-3, -1, -1, 1, 1$

ch6 (b) $P = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

$$A\left(c_1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = 2c_1\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + 4c_2\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 4c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

ch6 (c) $P = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ for $D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{pmatrix}$

Last two columns of P can be swapped, or other independent eigenvectors.

ch 7 (a) $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$, $P = \begin{pmatrix} 1 & -2 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$

$$\vec{u}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{7t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

ch 7 (b) $r^2 - 2r + 10 = 0$

$(r-1)^2 + 9 = 0$

$r = 1 \pm 3i$

$x(t) = c_1 e^t \cos 3t + c_2 e^t \sin 3t$

$y(t) = \frac{1}{3}(x' - x)$

$= \frac{1}{3}(-3c_1 e^t \sin 3t + 3c_2 e^t \cos 3t)$

ch 7 (c) $\begin{cases} x' = y \\ y' = 0 \end{cases}$

$\boxed{\begin{array}{l} y = c_2 \\ x = c_2 t + c_1 \end{array}}$ or $\vec{u}(t) = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}$

$= c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} t \\ 1 \end{pmatrix}$

ch 7 (d) $\begin{cases} x = c_1 e^t - c_2 \\ y = c_2 \end{cases}$

$\begin{cases} x' = c_1 e^t = x + c_2 = x + y \\ y' = 0 \end{cases}$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}$$

ch10 ⓐ Let $X_1 = \mathcal{L}(x)$, $X_2 = \mathcal{L}(y)$. Then

$$s^2 X_1 = X_1 - 3X_2 \quad \text{and} \quad -1 + s^2 X_2 = 3X_1$$

$$\begin{cases} (s^2-1)X_1 + 3X_2 = 0 \\ (-3)X_1 + s^2 X_2 = 1 \end{cases}$$

System found

$$X_1 = \frac{\begin{vmatrix} 0 & 3 \\ 1 & s^2 \end{vmatrix}}{\begin{vmatrix} s^2-1 & 3 \\ -3 & s^2 \end{vmatrix}} = \boxed{\frac{-3}{s^2(s^2-1)+9}}$$

ch10 ⓑ $\mathcal{L}(f) = \mathcal{L}(f_1) + \mathcal{L}(f_2) + \mathcal{L}(f_3) \Rightarrow f = f_1 + f_2 + f_3$

$$\mathcal{L}(f_1) = \left(\frac{d}{ds}\right)^2 \mathcal{L}(t^3 e^t \sin 3t) \Big|_{s \to s+2}$$

$$= \mathcal{L}(t^5 e^t \sin 3t) \Big|_{s \to s+2}$$

$$= \mathcal{L}(t^5 e^{-t} \sin 3t) \Rightarrow \boxed{f_1 = t^5 e^{-t} \sin 3t}$$

$$\mathcal{L}(f_2) = \frac{s^2+s+2}{(s-1)^3}$$

$u = s-1$

$s^2+s+2 = (u+1)^2 + u+1+2$

$= u^2+2u+1+u+3$

$= u^2+3u+4$

$$= \frac{s^2+3s+4}{s^3} \Big|_{s \to s-1}$$

$$= \mathcal{L}((1+3t+2t^2)e^t) \Rightarrow \boxed{f_2 = (1+3t+2t^2)e^t}$$

$$\mathcal{L}(f_3) = \mathcal{L}(e^t \sin(2t)\cos(2t)) \Rightarrow \boxed{f_3 = e^t \sin(2t)\cos(2t)}$$

ch10 ⓒ $\mathcal{L}(f) = \frac{1}{s} - \frac{6}{s+2} - \frac{5}{(s+2)^2} + \frac{3s-1}{s^2+1}$

$$= \mathcal{L}(1 - 6e^{-2t} - 5te^{-2t} + 3\cos t - \sin t)$$

$$\boxed{f = 1 - 6e^{-2t} - 5te^{-2t} + 3\cos t - \sin t}$$

ch10 ⓓ $\boxed{\mathcal{L}(f) = \ln|s-2| - \frac{1}{2}\ln\left((s-2)^2+1\right)}$