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Applied Differential Equations 2250

Exam date: Tuesday, 27 November, 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(ch4)** Complete enough of the following to add to 100%.

(a) [100%] Let V be the vector space of all continuous functions defined on $0 \le x \le 1$. Define S to be the set of all continuously differentiable functions $f(x)$ in V such that $f(0) = 0$ and $f(1) = \int_0^1 x f'(x)dx$. Prove that S is a subspace of V, by using the Subspace Criterion.

(b) [30%] If you solved (a), then skip (b) and (c). Let V be the set of all 3×1 column vectors $\mathbf{x}$ with components x_1, x_2, x_3. Assume the usual $\mathcal{R}^3$ rules for addition and scalar multiplication. Let S be the subset of V defined by the equations $C\mathbf{x} = B\mathbf{x}$, $\mathbf{b}^T\mathbf{x} = 0$, where B and C are 2×3 matrices and $\mathbf{b}$ is a nonzero vector in V. Prove that S is a subspace of V.

(c) [70%] If you solved (a), then skip (b) and (c). Solve for the unknowns x_1, x_2, x_3, x_4 in the system of equations below by showing all details of a frame sequence from the augmented matrix C to **rref**(C). Report the **vector form** of the general solution.

$$\begin{array}{ccccccccc} x_1 & + & 4x_2 & - & 2x_3 & + & 3x_4 & = & 5 \\ & + & 2x_2 & + & 2x_3 & + & & = & 1 \\ x_1 & + & 6x_2 & + & & + & 3x_4 & = & 6 \\ x_1 & + & 10x_2 & + & 4x_3 & + & 3x_4 & = & 8 \end{array}$$

(a) • $\vec{0}$ is in S because $f(x) \equiv 0$ satisfies both restriction equations.

• Let packages $\vec{x}, \vec{y}$ be defined by f, g, resp., satisfying the restriction equations of S. Then $\vec{x}+\vec{y}$ is defined by $f+g$ and

$$(f+g)(0) = f(0)+g(0) = 0+0 = 0 \qquad (f+g)(1) = f(1)+g(1) = f(0)+\int_0^1 x f'(x)dx + g(0) + \int_0^1 x g'(x)dx = (f+g)(0) + \int_0^1 x (f+g)'(x)dx$$

Therefore $\vec{x}+\vec{y}$ is in S

• Let $\vec{x}$ be defined by f satisfying the restriction equations of S. Then

$$(cf)(0) = c f(0) = 0 \qquad (cf)(1) = c f(1) = c\left(f(0)+\int_0^1 x f'(x)dx\right) = (cf)(0) + \int_0^1 x (cf)'(x)dx$$

The proof is complete.

(b) Apply the kernel theorem (Thm 2, 4.2) to matrix A whose last row is b^T and whose first rows are the rows of $C-B$.

(c) C = augmented matrix

$$\text{rref}(C) = \left(\begin{array}{cccc|c} 1 & 0 & -6 & 3 & 3 \\ 0 & 1 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

General Solution $\vec{x} = t_1 \begin{pmatrix} 6 \\ -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1/2 \\ 0 \\ 0 \end{pmatrix}$

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2. **(ch5)** Complete (a), (b) and then either (c) or (d). **Do not do both** (c) and (d).

(a) [30%] Given $4x''(t) + 20x'(t) + 26x(t) = 0$, which represents a damped spring-mass system with $m = 4$, $c = 20$, $k = 26$, **solve** the differential equation [20%] and **classify** the answer as over-damped, critically damped or under-damped [10%].

(b) [20%] Find a particular solution $y_p(t)$ and the homogeneous solution $y_h(x)$ for $y^{iv} + 4y'' = 5 + x$. Reminder: y^{iv} is the fourth derivative.

(c) [50%] Find by undetermined coefficients the steady-state periodic solution for the equation $x'' + 2x' + 5x = \sin(t)$.

(d) [50%] If you did (c) above, then skip this one! Determine the practical resonance frequency ω for the equation $x'' + 2x' + 5x = 10\cos(\omega t)$.

(a) $4r^2 + 20r + 26 = (2r+5)^2 + 1 \Rightarrow r = \frac{-5}{2} \pm \frac{i}{2}$ $\boxed{\text{under-damped}}$

$x(t) = c_1 e^{-5t/2}\cos(t/2) + c_2 e^{-5t/2}\sin(t/2)$

(b) Trial solution $y = x^2(d_1 + d_2 x) \Rightarrow d_1 = \frac{5}{8}, d_2 = \frac{1}{24}$ $\boxed{y_p = \frac{5}{8}x^2 + \frac{x^3}{24}}$

(c) Trial solution $x = d_1 \cos t + d_2 \sin t$

$d_1 = \frac{-1}{10}$, $d_2 = \frac{1}{5}$

(d) $\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{\frac{5}{1} - \frac{2^2}{2\cdot 1^2}} = \boxed{\sqrt{3}}$

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3. **(ch5)** Complete all parts below.

(a) [60%] A non-homogeneous linear differential equation with constant coefficients has right side $f(x) = x^2e^x + (x+1)(x^2+3) + x\cos 2x$ and characteristic equation of order 10 with roots $0, 0, 0,\ 1,\ -1, -1,\ 2i, -2i,\ 2i, -2i$, listed according to multiplicity. Determine the **corrected** trial solution for y_p according to the method of undetermined coefficients and the **fixup rule**. To save time, **do not** evaluate the undetermined coefficients and **do not** find $y_p(x)$! Undocumented detail or guessing earns no credit.

(b) [20%] Write out the general solution of the homogeneous linear constant coefficient equation whose sixth order characteristic equation has roots $1, 1, 1,\ 0,\ 2+i,\ 2-i$.

(c) [20%] Write out the general solution of the homogeneous linear constant coefficient equation whose characteristic equation is $(r^3 - r^2)(r^2 - r)^2(r^2 + 2r)^2 = 0$.

(a) $y = x^{s_1}(d_1 + d_2 x + d_3 x^2 + d_4 x^3)$
$\quad + x^{s_2}(d_5 e^x + d_6 xe^x + d_7 x^2 e^x)$
$\quad + x^{s_3}(d_8 \cos 2x + d_9 \sin 2x + d_{10} x \cos 2x + d_{11} x \sin 2x)$

s_1 = root count in char. eq. for root atomRoot$(x^3) = 0$

$\boxed{s_1 = 3}$

s_2 = root count in char. eq. for atomRoot$(e^x) = 1$

$\boxed{s_2 = 1}$

s_3 = root count in char eq for atomRoot$(\cos 2x) = 2i$

$\boxed{s_3 = 2}$

(b) $y = (c_1 + c_2 x + c_3 x^2)e^x$
$\quad + c_4 e^{0x}$
$\quad + (c_5 \cos x + c_6 \sin x)e^{2x}$

(c) $r^2(r-1)(r-1)^2 r^2 r^2 (r+2)^2 = 0$

$r^6 (r-1)^3 (r+2)^2 = 0$

$r = 0, 0, 0, 0, 0, 0, 1, 1, 1, -2, -2$

$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5$
$\quad + (c_7 + c_8 x + c_9 x^2)e^x$
$\quad + (c_{10} + c_{11} x)e^{-2x}$

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4. **(ch6)** Complete all of the items below.

(a) [30%] Find the eigenvalues of the matrix $A = \begin{pmatrix} -2 & 7 & 1 & 12 \\ -1 & 6 & -3 & 15 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & -2 & 3 \end{pmatrix}$. To save time, **do not** find eigenvectors!

(b) [40%] Given $A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, assume there exists an invertible matrix P and a diagonal matrix D such that $AP = PD$. Circle all possible columns of P from the list below.

$$\begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -11 \\ -11 \end{pmatrix}.$$

(c) [30%] Find all eigenpairs for the matrix $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then display Fourier's model for A.

(a) Cofactor expansion of $\det(A-\lambda I)$ along row 4 gives $((3-\lambda)^2+4)(\lambda^2-4\lambda-5)=0$ and then $\boxed{\lambda = -1, 5, 3 \pm 2i}$

(b) Test each given vector $\vec{v}$ by the requirement that $\vec{v}$ be an eigenvector, that is, $A\vec{v} = \lambda\vec{v}$ for some λ.

(c) $\det(A-\lambda I)=0$ is the equation $-\lambda(2-\lambda)=0 \Rightarrow \boxed{\lambda = 0, 2}$

Frame sequences give eigenpairs

$$\left(0, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right), \left(2, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\right)$$

Then Fourier's model is

$$A\left(c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) = c_1(0)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2(2)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

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5. **(ch6)** Complete all parts below.

Consider the 3×3 matrix

$$E = \begin{pmatrix} 5 & 2 & -2 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

Already computed is the eigenpair

$$\left(3, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}\right).$$

Corrected at exam time $\lambda_1 = 3$

(a) [50%] Find the remaining eigenpairs of E.

(b) [25%] Suppose a 2×2 matrix A has eigenpairs $\left(2, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right), \left(-3, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$. Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

(c) [25%] Assume the vector general solution $\mathbf{x}(t)$ of the linear differential system $\mathbf{x}' = A\mathbf{x}$ is given by

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Display Fourier's model for the 2×2 matrix A.

(a) $\det(A - \lambda I) = 0$ is $(3-\lambda)(5-\lambda)^2 = 0 \Longrightarrow \boxed{\lambda = 3, 5, 5}$

The frame sequence for $\lambda = 5$ has scalar general solution obtained from $\text{rref} = \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ as $\begin{cases} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = t_2 \end{cases} \Rightarrow \partial_{t_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \partial_{t_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$,

Eigenpairs remaining are $\boxed{\left(5, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), \left(5, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)}$

(b) $D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$, $P = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$

(c) $A\left(c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right) = c_1 (0) \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 (2) \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

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