Name KEY Classtime ______ 2250 Midterm 2 Ver 2 [10:45, F2007]

Applied Differential Equations 2250

Exam date: Tuesday, 30 October, 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. **(Frame sequences, the 3 properties and symbols a, b)**
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} 4x & + & 8y & + & 3z & = & 2-a \\ x & + & 2y & + & az & = & -a \\ 3x & - & 3by & + & (3-a)z & = & -b \end{array}$$

$$\left(\begin{array}{ccc|c} 4 & 8 & 3 & 2-a \\ 1 & 2 & a & -a \\ 3 & -3b & 3-a & -b \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & a & -a \\ 4 & 8 & 3 & 2-a \\ 3 & -3b & 3-a & -b \end{array}\right) \text{swap}(1,2)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & a & -a \\ 0 & 0 & 3-4a & 2+3a \\ 3 & -3b & 3-a & -b \end{array}\right) \text{combo}(1,2,-4)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & a & -a \\ 3 & -3b & 3-a & -b \\ 0 & 0 & 3-4a & 2+3a \end{array}\right) \text{swap}(2,3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & a & -a \\ 0 & -3b-6 & 3-4a & -b+3a \\ 0 & 0 & 3-4a & 2+3a \end{array}\right) \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & a & -a \\ 0 & -3b-6 & 0 & -2-b \\ 0 & 0 & 3-4a & 2+3a \end{array}\right) \text{combo}(3,2,-1)$$

$-2-b=0 \Rightarrow$ row 2 is all zeros

$b=-2$, $3-4a\ne 0$ ∞-many sols	$b\ne -2$, $3-4a\ne 0$ unique sol
$b=-2$, $3-4a=0$ No sol	$b\ne -2$, $3-4a=0$ No sol

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 2 [10:45, F2007]

2. **(vector spaces)** Do all three parts.

(a) [20%] The vector space V is the set of all functions $f(x) = c_1 + c_2(1-x) + c_3(3+2x) + c_4 \sin x$. Find a subspace S of V of dimension 3 and display a basis for S. Don't justify anything.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 = 2x_3 + x_2$, $2x_2 = 5x_1$, $x_1 - x_3 = 0$, $x_1 + x_3 = 0$. Prove that S is a subspace of V.

(c) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & = & 0, \\ x_1 & + & x_2 & & & + & x_4 & = & 0, \\ x_1 & + & 3x_2 & - & 4x_3 & + & x_4 & = & 0. \end{array}$$

(a) V spanned by $1, 1-x, 3+2x, \sin x$ but $3+2x$ redundant. S = span of basis elements $1, 1-x, \sin x$ has $\dim(S) = 3$.

(b) Define $A = \begin{pmatrix} 1 & -1 & -2 \\ 5 & -2 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 1 \end{pmatrix}$. Then $S = \{\vec{x} : A\vec{x} = \vec{0}\}$.

Apply the kernel Theorem (Thm 2, 4.2 E&P). Then S is a subspace.

(c) $A = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 3 & -4 & 1 \end{pmatrix}$ $\quad \mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

scalar gen sol $\begin{cases} x_1 = -2t_1 - t_2 \\ x_2 = 2t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$ $\quad$ Basis $= \left\{ \begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 2 [10:45, F2007]

3. **(independence)** Do **only two** of the three parts.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \end{pmatrix}$. State a test that decides dependence of the list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] State the pivot theorem [20%]. Then extract from the list below a largest set of independent vectors [30%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 7 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 5 \\ -3 \\ 0 \\ 7 \end{pmatrix}, \mathbf{v}_6 = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 9 \end{pmatrix}.$$

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!] Assume that 4×2 matrix D has rank 2 and $D\mathbf{x} = \mathbf{b}$ has a solution $.x$ for some vector $\mathbf{b}$. Prove that there exists unique numbers c_1, c_2 such that $\mathbf{rref}(\mathbf{aug}(D, \mathbf{b})) = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(a) Test: $\vec{v}_1, \vec{v}_2, \vec{v}_3$ dependent $\iff$ rank$(\text{aug}(\vec{v}_1, \vec{v}_2, \vec{v}_3)) \neq 3$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 3 \\ 1 & 1 & 4 \\ 1 & 1 & 5 \end{pmatrix}. \qquad \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

rank $= 3$ $\boxed{\text{independent}}$

(b) Pivot Theorem.

- The pivot columns of A are linearly independent
- The non-pivot columns of A are linear combinations of the pivot columns of A.

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 & 5 & 3 \\ -1 & 0 & -1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 7 & 2 & 7 & 9 \end{pmatrix} \qquad \text{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & -2 & 3 & -1 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

pivots of $A = 1, 2, 3$

Largest set of independent vectors $= \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

(c) rref(D) must have 2 leading ones hence $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ fill its columns. Consistency of $D\vec{x} = \vec{b}$ implies rref(aug$(D, \vec{b})$) has the cited form where $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is the unique sol of the system $D\vec{x} = \vec{b}$.

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 2 [10:45, F2007]

4. (determinants and elementary matrices) Do all three parts.
(a) [50%] Assume given 3×3 matrices A, B. Suppose $E_5E_4B = E_3E_2E_1A^2$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a combination, a multiply by 3, a combination and a multiply by 2. Assume $\det(A) = 2$. Find $\det(2AB^2)$.
(b) [30%] Let A be a 4×4 matrix such that $(I + 3A)^{-1} = I - 3A$. Find the value of $\det(A - 3A^2)$.
(c) [20%] Let B be a 3×3 matrix and assume that B is the product of elementary swap and combination matrices. Determine all possible values of $\det(B)$.

(a) $\det(2AB^2) = \det((2I)ABB)$
$= \det(2I)\det(A)\det(B)^2$
$= 2^3(2)\det(B)^2$

$\det E_5 \det E_4 \det B = \det E_3 \det E_2 \det E_1 \det(A)^2$
$(2)\ (1)\ \det B = (3)\ (1)\ (-1)\ (2)^2$
$\det B = -6$

$\det(2AB^2) = 2^3(2)(-6)^2$
$= 576$

(b) $C = I - 3A$ and $D = I + 3A$ are given to satisfy $CD = DC = I$
Then $(I - 3A)(I + 3A) = I$
$I - 3A + 3A - 9A^2 = I$
$-9A^2 = 0$
$A^2 = 0$
Then $\det A \det A = 0$ or $\boxed{\det A = 0}$

(c) Because $B = E_k \cdots E_1$ and $\det E = 1$ or -1 for combo or swap, then $\det(B) = 1$ or $\det(B) = -1$,

Use this page to start your solution. Attach extra pages as needed, then staple.

Name KEY Classtime ______ 2250 Midterm 2 Ver 2 [10:45, F2007]

5. **(inverses and Cramer's rule)** Do all three parts.

(a) [20%] Determine all values of x for which A is invertible: $A = \begin{pmatrix} 1 & 2x-1 & 1 \\ 2 & 4 & 0 \\ 1 & 2 & e^{2x} \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 1, column 3 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

(c) [40%] Solve for x_3 in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 6 & 8 & 1 \\ 3 & 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}.$$

(a) A is invertible $\Leftrightarrow \det(A) \neq 0 \Leftrightarrow \begin{vmatrix} 2 & 4 \\ 1 & 2 \end{vmatrix} + e^{2x} \begin{vmatrix} 1 & 2x-1 \\ 2 & 4 \end{vmatrix} \neq 0$

$\Leftrightarrow 0 + e^{2x}(6-4x) \neq 0$

$\Leftrightarrow x \neq 3/2$

(b) $\text{entry}(1,3) = \dfrac{\text{cofactor}(3,1)}{\det(A)} = \dfrac{(-1)^{3+1} \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 0 & 3 \end{vmatrix}}{2} = -4$

(c) $x_3 = \dfrac{\Delta_3}{\Delta}$

$= -1$

$\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 6 & 8 & 1 \\ 3 & 0 & 2 & 0 \end{vmatrix} = -12$

$\Delta_3 = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 6 & -1 & 1 \\ 3 & 0 & 2 & 0 \end{vmatrix} = 12$

Use this page to start your solution. Attach extra pages as needed, then staple.