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Applied Differential Equations 2250

Exam date: Tuesday, 30 October, 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Frame sequences and the 3 properties)
Determine a, b such that the system has a unique solution, infinitely many solutions, or no solution:

$$\begin{array}{rcrcrcl} x & + & 2y & - & az & = & a \\ 3x & + & 3by & + & (3+a)z & = & b \\ 4x & + & 8y & + & 3z & = & 2+a \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -a & a \\ 3 & 3b & 3+a & b \\ 4 & 8 & 3 & 2+a \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -a & a \\ 0 & 3b-6 & 3+4a & b-3a \\ 4 & 8 & 3 & 2+a \end{array}\right) \quad \text{combo}(1,2,-3)$$

$$\left(\begin{array}{ccc|c} 1 & 2 & -a & a \\ 0 & 3b-6 & 3+4a & b-3a \\ 0 & 0 & 3+4a & 2-3a \end{array}\right) \quad \text{combo}(1,3,-4)$$

<u>case $b=2$</u> : y = free var

$$\left(\begin{array}{cccc} 1 & 2 & -a & a \\ 0 & 0 & 3+4a & 2-3a \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{\infty\text{-many sols } b=2,\ 3+4a\neq 0}$

$\boxed{\text{No Sol } b=2,\ 3+4a=0}$

<u>case $b\neq 2$</u> : $y \neq$ free var

$$\left(\begin{array}{ccc|c} 1 & 2 & -a & a \\ 0 & 1 & \frac{3+4a}{3b-6} & \frac{b-3a}{3b-6} \\ 0 & 0 & 3+4a & 2-3a \end{array}\right)$$

$\boxed{\text{No sol } b\neq 2,\ 3+4a=0}$

$\boxed{\text{unique sol } b\neq 2,\ 3+4a\neq 0}$

No Sol : Signal equation
∞-many sols: one + free var
unique sol : zero free vars

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2. **(vector spaces)** Do all three parts.

(a) [20%] The vector space V is the set of all functions $f(x) = c_1 + c_2(1+x) + c_3(1-x) + c_4 \cos x$. Find a subspace S of V of dimension 3 and display a basis for S. Don't justify anything.

(b) [40%] Let V be the vector space of all column vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and let S be the subset of V given by the equations $x_1 = 2x_3$, $2x_2 = 5x_3$. Prove that S is a subspace of V.

(c) [40%] Find a basis of vectors for the subspace of $\mathcal{R}^4$ given by the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & = & 0, \\ x_1 & + & x_2 & - & 3x_3 & + & x_4 & = & 0, \\ x_1 & + & 4x_2 & & & + & x_4 & = & 0. \end{array}$$

(a) V spanned by $1,\ 1+x,\ 1-x,\ \cos x$

Basis of V = $1,\ 1+x,\ \cos x$

Basis of S = $1,\ 1+x,\ \cos x \implies S = V$.

(b) Define $A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{pmatrix}$. Then $S = \{ \vec{x} : A\vec{x} = \vec{0} \}$.

Apply the kernel Theorem (Thm 2, 4.2 E&P). Then S is a subspace.

(c) The reduced echelon system is

$$\begin{cases} x_1 - 4x_3 + x_4 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{cases}$$ with gen scalar sol $$\begin{cases} x_1 = 4t_1 - t_2 \\ x_2 = -t_1 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$

The basis consists of vector partials $\partial_{t_1}, \partial_{t_2}$

$$\text{Basis} = \left\{ \begin{pmatrix} 4 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

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3. (independence) Do **only two** of the three parts.

(a) [50%] Let $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 4 \end{pmatrix}$. State a test that decides independence of the list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] State the pivot theorem [20%]. Then extract from the list below a largest set of independent vectors [30%].

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} 5 \\ -3 \\ 0 \\ 7 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 3 \\ -1 \\ 0 \\ 7 \end{pmatrix}.$$

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!] Assume that 3×2 matrix D has rank 2 and $D\mathbf{x} = \mathbf{b}$ has a solution $.x$ for some vector $\mathbf{b}$. Prove that there exists unique numbers c_1, c_2 such that $\mathbf{rref}(\mathbf{aug}(D, \mathbf{b})) = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 0 \end{pmatrix}$.

(a) The vectors v_1, v_2, v_3 are independent $\Longleftrightarrow$ rank $(\text{aug}(v_1, v_2, v_3)) = 3$

$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 1 & 4 \end{pmatrix}$ $\quad$ $\text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad$ rank$(A) = 2$ $\quad$ $\boxed{\text{Dependent}}$

(b) pivot Theorem.

- The pivot columns of A are independent
- Non-pivot columns of A are linear combinations of the pivot columns of A.

$A = \begin{pmatrix} 1 & 2 & 0 & 5 & 3 \\ -1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 2 & 7 & 7 \end{pmatrix}$ $\quad$ $\text{rref}(A) = \begin{pmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Pivots of A are 1, 2, 5. Largest independent set $= \{v_1, v_2, v_5\}$

(c) Rank 2 implies 2 leading ones in rref(D), so $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ appear in rref$(\text{aug}(D, \vec{b}))$ and this forces the form claimed, due to consistency of the equation $D\vec{x} = \vec{b}$.

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4. **(determinants and elementary matrices)** Do all three parts.
(a) [50%] Assume given 3×3 matrices A, B. Suppose $E_5E_4B = E_3E_2E_1A^2$ and E_1, E_2, E_3, E_4, E_5 are elementary matrices representing respectively a swap, a combination, a multiply by 5, a swap and a multiply by 2. Assume $\det(A) = 3$. Find $\det(3AB^2)$.
(b) [30%] Let A be a 4×4 matrix such that $(I + 2A)^{-1} = I - 2A$. Find the value of $\det(A)$.
(c) [20%] Let A be a 3×3 matrix such that $\det(A) = 0$. Prove that A is not the product of elementary matrices.

(a) $\det(3AB^2) = \det(3I)\det(A)\det(B)^2$ by the det. prod. Thm.

$\det E_5 \det E_4 \det B = \det E_3 \det E_2 \det E_1 \det(A)^2$

$(2) \quad (-1) \det B = (5) \quad (1) \quad (-1) \quad (3^2)$

$\det(B) = \frac{(5)(9)}{2}$

$\det(3AB^2) = \boxed{3^3 \cdot 3 \cdot \left(\frac{(5)(9)}{2}\right)^2}$

(b) Given $(I+2A)(I-2A) = I$ (inverses satisfy $AB = BA = I$)

Then $I + 2A - 2A - 4A^2 = I$, which implies $-4A^2 = 0$.

Then $A^2 = 0$ and $\det A \det A = \det 0 = 0$. So

$\boxed{\det(A) = 0}$

(c) If $A = E_k \cdots E_1$, then $\det(A) = \det E_k \cdots \det E_1$. But each elementary matrix has nonzero determinant. Then $\det A \neq 0$. This proves A is not the product of elementary matrices.

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5. (inverses and Cramer's rule) Do all three parts.

(a) [20%] Determine all values of x for which A is invertible: $A = \begin{pmatrix} 1 & 2x-1 & 0 \\ 2 & 4 & 0 \\ e^x & -5x & e^{2x} \end{pmatrix}$.

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 3, column 4 of A^{-1}, given A below. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

(c) [40%] Solve for x_3 in $A\mathbf{u} = \mathbf{b}$ by Cramer's rule. Other methods are not acceptable.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 6 & 8 & 1 \\ 3 & 0 & 2 & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

(a) $\det A = e^{2x} \begin{vmatrix} 1 & 2x-1 \\ 2 & 4 \end{vmatrix} = e^{2x}(6-4x)$. Matrix A is invertible if and only if $\det A \neq 0$, which is $\boxed{x \neq 3/2}$

(b) $\text{entry}(3,4) = \dfrac{\text{cofactor}(4,3)}{\det(A)} = \dfrac{(-1)^{4+3} \begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{2} = \dfrac{-2}{2} = \boxed{-1}$

(c) $x_3 = \dfrac{\Delta_3}{\Delta}$, $\Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 6 & 8 & 1 \\ 3 & 0 & 2 & 0 \end{vmatrix} = -12$, $\Delta_3 = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 6 & -1 & 1 \\ 3 & 0 & 1 & 0 \end{vmatrix} = 6$

$\boxed{x_3 = \frac{-1}{2}}$

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