Applied Differential Equations 2250

Exam date: Tuesday, 30 October, 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%.

1. (Frame sequences and the 3 properties)
Determine $a, b$ such that the system has a unique solution, infinitely many solutions, or no solution:

$$x + 2y - az = a$$
$$3x + 3by + (3 + a)z = b$$
$$4x + 8y + 3z = 2 + a$$

$$\begin{pmatrix}
1 & 2 & -a \\
3 & 3b & 3+a \\
4 & 8 & 2+a
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 2 & -a \\
0 & 3b - 6 & 3+a \\
4 & 8 & 2+a
\end{pmatrix}$$

combo(1,2,-3)

$$\begin{pmatrix}
1 & 2 & -a \\
0 & 3b - 6 & 3+a \\
0 & 0 & 3+a
\end{pmatrix}$$

combo(1,3,-y)

Case $b = 2$: $y$ = free var

$$\begin{pmatrix}
1 & 2 & -a \\
0 & 0 & 3+a
\end{pmatrix}$$

Case $b \neq 2$: $y \neq$ free var

$$\begin{pmatrix}
1 & 2 & -a \\
0 & 1 & 3+a \\
0 & 0 & 3+a
\end{pmatrix}$$

No sol $b = 2$, $3+a \neq 0$

No sol $b = 2$, $3+a = 0$

No sol: singular matrix

00-many sols: one + free var

Unique sol: zero free vars

00-many sols: one + free var

Unique sol: zero free vars

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2. (vector spaces) Do all three parts.
(a) [20%] The vector space $V$ is the set of all functions $f(x) = c_1 + c_2(1+x) + c_3(1-x) + c_4 \cos x$. Find a subspace $S$ of $V$ of dimension 3 and display a basis for $S$. Don’t justify anything.
(b) [40%] Let $V$ be the vector space of all column vectors \[
\begin{pmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{pmatrix}
\] and let $S$ be the subset of $V$ given by the equations $x_1 = 2x_3$, $2x_2 = 5x_3$. Prove that $S$ is a subspace of $V$.
(c) [40%] Find a basis of vectors for the subspace of $\mathbb{R}^4$ given by the system of equations
\[
\begin{align*}
    x_1 + 2x_2 - 2x_3 + x_4 &= 0, \\
    x_1 + x_2 - 3x_3 + x_4 &= 0, \\
    x_1 + 4x_2 + x_4 &= 0.
\end{align*}
\]

(a) \(V\) spanned by \(1, 1+x, 1-x, \cos x\)

Basis of \(V = 1, 1+x, \cos x\)

Basis of \(S = 1, 1+x, \cos x \Rightarrow S = V\).

(b) Define \(A = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{pmatrix}\). Then \(S = \{ \vec{x} : A\vec{x} = \vec{0} \}\).

Apply the Kernel Theorem (Thm 2, 4.2 Eq 8). Then $S$ is a subspace.

(c) The reduced echelon system is
\[
\begin{align*}
    x_1 - 4x_3 + x_4 &= 0, \quad \text{with} \quad \begin{cases} 
    x_1 = 4t_1 - t_2 \\
    x_2 = -t_1 \\
    x_3 = t_1 \\
    x_4 = t_2
\end{cases} \quad \text{gen scalar} \quad \begin{cases} 
    x_1 = u \\
    x_2 = -t_1 \\
    x_3 = t_1 \\
    x_4 = t_2
\end{cases}
\end{align*}
\]

The basis consists of vector particles \(\partial t_1, \partial t_2\)

Basis = \{ \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \} \}.

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3. (independence) Do only two of the three parts.

(a) [50%] Let \( \mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \). State a test that decides independence of the list of three vectors [20%]. Apply the test and report the result [30%].

(b) [50%] State the pivot theorem [20%]. Then extract from the list below a largest set of independent vectors [30%].

\[
\begin{align*}
\mathbf{v}_1 &= \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad 
\mathbf{v}_2 &= \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix}, \quad 
\mathbf{v}_3 &= \begin{pmatrix} 0 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \quad 
\mathbf{v}_4 &= \begin{pmatrix} 5 \\ -3 \\ 0 \\ 7 \end{pmatrix}, \quad 
\mathbf{v}_5 &= \begin{pmatrix} 3 \\ -1 \\ 0 \\ 7 \end{pmatrix}.
\end{align*}
\]

(c) [50%] [If you did (a) and (b) already, then 100% has been attained: skip this one!]

Assume that \( 3 \times 2 \) matrix \( \mathbf{D} \) has rank 2 and \( \mathbf{D} \mathbf{x} = \mathbf{b} \) has a solution \( \mathbf{x} \) for some vector \( \mathbf{b} \). Prove that there exists unique numbers \( c_1, c_2 \) such that \( \text{rref}(\text{aug}(\mathbf{D}, \mathbf{b})) = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 0 & 0 \end{pmatrix} \).

\( A \) The vectors \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) are independent \( \iff \text{rank}(\text{aug}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)) = 3 \)
\[
A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 1 & 4 \end{pmatrix} \quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{rank}(A) = 2 \quad \boxed{\text{Dependent}}
\]

\( B \) pivot Theorem.
- The pivot columns of \( A \) are independent
- Non-pivot columns of \( A \) are linear combinations of \( \mathcal{P} \) pivot columns of \( A \).

\[
A = \begin{pmatrix} -1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 2 & -3 & -1 \\ 1 & 4 & 2 & 7 & 4 \end{pmatrix} \quad \text{rref}(A) = \begin{pmatrix} 1 & 0 & -2 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

Pivots of \( A \) are 1, 2, 5. Largest independent set = \( \{ \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5 \} \)

\( C \) Rank 2 implies 2 leading ones in \( \text{rref}(\mathbf{D}) \), so \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) appear in \( \text{rref}(\text{aug}(\mathbf{D}, \mathbf{b})) \) and this forces \( \mathbf{b} \) form consistent, due to consistency of \( \mathbf{D} \) equations \( \mathbf{D} \mathbf{x} = \mathbf{b} \).

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4. (Determinants and elementary matrices) Do all three parts.

(a) [50%] Assume given $3 \times 3$ matrices $A$, $B$. Suppose $E_5 E_4 E_3 E_1 A^2$ and $E_1$, $E_2$, $E_3$, $E_4$, $E_5$ are elementary matrices representing respectively a swap, a combination, a multiply by 5, a swap and a multiply by 2. Assume $\det(A) = 3$. Find $\det(3AB^2)$.

(b) [30%] Let $A$ be a $4 \times 4$ matrix such that $(I + 2A)^{-1} = I - 2A$. Find the value of $\det(A)$.

(c) [20%] Let $A$ be a $3 \times 3$ matrix such that $\det(A) = 0$. Prove that $A$ is not the product of elementary matrices.

\[ \det(3AB^2) = \det(3I) \det(A) \det(B)^2 \quad \text{by the determinant product theorem.} \]
\[
\begin{align*}
\det E_5 \det E_4 \det B &= \det E_3 \det E_2 \det E_1 \det(A)^2 \\
(2) \quad (-1) \quad \det B &= (5) \quad (1) \quad (-1) \quad (3^2) \\
\det(B) &= \frac{(5)(9)}{2} \\
\det(3AB^2) &= 3^3 \cdot \frac{(5)(9)^2}{2}
\end{align*}
\]

(b) Given $(I+2A)(I-2A) = I$ (inverses satisfy $AB = BA = I$).

Then $I + 2A - 2A - 4A^2 = I$, which implies $-4A^2 = 0$.

Then $A^2 = 0$ and $\det A \cdot \det A = \det 0 = 0$. So $\det(A) = 0$.

(c) If $A = E_k \cdots E_1$, then $\det(A) = \det E_k \cdots \det E_1$. But each elementary matrix has a nonzero determinant. Then $\det A \neq 0$.

This proves $A$ is not the product of elementary matrices.

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5. (inverses and Cramer's rule) Do all three parts.

(a) [20%] Determine all values of x for which A is invertible: \( A = \begin{pmatrix} 1 & 2x - 1 & 0 \\ 2 & 4 & 0 \\ e^x & -5x & e^{2x} \end{pmatrix} \).

(b) [40%] Apply the adjugate formula for the inverse to find the value of the entry in row 3, column 4 of \( A^{-1} \), given \( A \) below. Other methods are not acceptable.

\[
A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 3 \end{pmatrix}
\]

(c) [40%] Solve for \( x_3 \) in \( Au = b \) by Cramer's rule. Other methods are not acceptable.

\[
A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 6 & 8 & 1 \\ 3 & 0 & 2 & 0 \end{pmatrix}, \quad u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}
\]

\( \text{det} \ A = e^{2x} \begin{vmatrix} 1 & 2x - 1 \\ 2 & 4 \end{vmatrix} = e^{2x} (6 - 4x) \). Matrix \( A \) is invertible if and only if \( \text{det} \ A \neq 0 \), which is \( x \neq \frac{3}{2} \).

\( \Delta_3 = \frac{\text{cofactor}(4,3)}{\text{det}(A)} = \frac{\text{cofactor}(4,3)}{2} = \frac{-2}{2} = -1 \)

\( x_3 = \frac{\Delta_3}{\Delta} \), \( \Delta = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 6 & 8 & 1 \\ 3 & 0 & 2 & 0 \end{vmatrix} = -12 \), \( \Delta_3 = \begin{vmatrix} 1 & 2 & 0 & 0 \\ 3 & 0 & 4 & 0 \\ 5 & 6 & -1 & 1 \\ 3 & 0 & 1 & 0 \end{vmatrix} = 6 \)

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