Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Unevaluated integrals will receive partial credit.

1. (Quadrature Equation)

Solve for the general solution \( y(x) \) in the equation

\[
y' = \frac{1}{x} \tan(1 + \ln|x|) + (\sec x + \tan x)^2 + \frac{x + 5}{4 + x}.
\]

\[
F_1 = \frac{1}{x} \tan \left(1 + \ln|x|\right)
\]

\[
= \tan(u) \, du \quad \quad u = 1 + \ln|x|
\]

\[
F_2 = (\sec x + \tan x)^2
\]

\[
= \sec^2 x + \tan^2 x + 2 \sec x \tan x
\]

\[
= 2 \sec^2 x - 1 + 2 \sec x \tan x
\]

\[
= \frac{d}{dx} \left(2 \tan x - x + 2 \sec x\right)
\]

\[
F_3 = \frac{x + 5}{x + 4}
\]

\[
= 1 + \frac{1}{x+4}
\]

\[
= \frac{d}{dx} \left(x + \ln|x+4|\right)
\]

\[
\int y' \, dx = \int F_1 \, dx + \int F_2 \, dx + \int F_3 \, dx \quad \quad \text{Quadrature Step}
\]

\[
y = C + \ln|\sec(1 + \ln|x|)|
\]

\[
+ 2 \tan(x) - x + 2 \sec(x)
\]

\[
+ x + \ln|x+4|
\]

\[
\text{\textcircled{5}} \quad \text{The x-terms cancel.}
\]

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2. (Classification of Equations)

The problem \( y' = f(x, y) \) is defined to be separable provided \( f(x, y) = F(x)G(y) \) for some functions \( F \) and \( G \).

(a) [40%] Check the problems that can be put into separable form, but don’t supply any details.

- \( y' = y(2xy + 1) + (x - 1)y \)
- \( e^{xy}y' = y^2 \cos x + 4 \cos x \\
\quad = \cos x \left( y^2 + 4 \right) \)
- \( y' = e^{3x+2y} + e^{2x+3y} \)
- \( y' + 3y = 13 \)

(b) [25%] State a test which can verify that an equation \( y' = f(x, y) \) is linear but not separable.

(c) [35%] Use the separable equation test to show that \( y' = (x + y)^2 \) is not separable.

\[ \frac{\partial f}{\partial y} \text{ independent of } y \Rightarrow \text{ linear} \]

Given \( f(x_0, y_0) = 0 \) for some \( x_0, y_0 \), then

\[ \frac{f(x, y_0)}{f(x_0, y_0)} \neq f(x, y) \]

implies the DE is not separable.

\[ \begin{array}{c}
\text{Define } x_0 = 0, y_0 = 1 , \\
F(x) = \frac{f(x, 1)}{f(0, 1)} = (1+x)^2, \quad G(y) = f(0, y) = y^2
\end{array} \]

\[ FG = (1+x)^2 y^2 \]

\[ \neq (x+y)^2 = f \]

Thus \( y' = (x+y)^2 \) is not separable.
3. (Solve a Separable Equation)

Given \( y' = \left( \frac{\csc^2 x}{\tan x} + \frac{x+1}{5+x} \right) (y+1)(y+2) \).

Find the non-equilibrium solution in implicit form. To save time, do not solve for \( y \) explicitly and do not solve for equilibrium solutions.

\[
\frac{1}{G} = \frac{y}{(y+1)(y+2)} = \frac{-1}{y+1} + \frac{2}{y+2} = \frac{d}{dy} \left( -\ln|y+1| + 2 \ln|y+2| \right)
\]

\[
F = \frac{\csc^2 x}{\tan x} + \frac{x+1}{x+5} = \csc^2 x \csc x \cot x + 1 + \frac{-4}{x+5}
\]

\[
= \frac{d}{dx} \left( -\frac{\csc^3(x)}{3} + x - 4 \ln|x+5| \right)
\]

\[
\int \frac{y'dx}{G(y)} = \int F dx \quad \text{Quadrature on prepared DE}
\]

\[
- \ln|y+1| + 2 \ln|y+2| = -\frac{\csc^3(x)}{3} + x - 4 \ln|x+5| + c
\]
4. (Linear Equations)
   (a) [60%] Solve \(10x'(t) = -98 + \frac{10}{2t+3}x(t)\), \(x(0) = -147/5\). Show all integrating factor steps.
   (b) [20%] Solve the homogeneous equation \(\frac{dy}{dx} = -(\tan x)y\). The answer contains symbol \(c\).
   (c) [20%] Solve \(y' = 5y + 3\) by using the superposition principle \(y = y_h + y_p\).

\[\begin{align*}
   \alpha' + p\alpha &= q \\
   \left(\frac{\alpha W}{w}\right)' &= q \\
   \alpha W &= e^{\int p \, dt} = (2t+3)^{-1/2} \quad \text{near } t = 0 \\
   \alpha W &= -\frac{98}{10} \int (2t+3)^{-1/2} \, dt \\
   \alpha &= \left(2t+3\right)^{1/2} \left(-\frac{98}{10} (2t+3)^{1/2} + c\right) \\
   -\frac{147}{5} &= -\frac{98}{10} (3) + \sqrt{3}c, \quad \text{Then } c = 0 \\
   \alpha(t) &= -\frac{98}{10} (2t+3)
\end{align*}\]

\[\begin{align*}
   \begin{cases}
   (yw)' = 0 & \text{where } W = e^{\int \tan x \, dx} = e^{-\ln(\cos x)} \\
   y W = c & \text{on } y = c \cdot \cos x
   \end{cases}
\end{align*}\]

\[\begin{align*}
   y_p &= -3/5 \quad \text{an equilibrium sol} \\
   y_h &= c e^{5x} \quad \text{a growth-decay recipe sol for } y' - 5y = 0 \\
   \frac{dy}{dx} &= y_h + y_p \\
   \frac{dy}{dx} &= c e^{5x} - \frac{3}{5}
\end{align*}\]

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5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

\[ \frac{dx}{dt} = \ln(1 + x^2) \left( 1 - \sqrt[3]{|x|} \right)^2 (1 - x)(4 - x^2)(x^2 - 1)^4. \]

Expected in the phase line diagram are equilibrium points and signs of \( x' \).
(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, neither node, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

(a) \( f(x) = -\ln(1 + x^2) \left( 1 - \sqrt[3]{|x|} \right)^2 (x-1)(x+1)^4(2-x)(2+x) \)

Roots: 0, ±1, ±2, changes sign at \( x=0, 1, 2, -2 \), \( f > 0 \) at \( \infty \) and \( f < 0 \) at \( -\infty \).

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**Def:** node = "not a funnel and not a spout"