Name. KEY Time of your class ______

Differential Equations and Linear Algebra 2250

Midterm Exam 1 Version 2 [10:45]
Tuesday, 25 September 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Unevaluated integrals will receive partial credit.

1. **(Quadrature Equation)**
   Solve for the general solution $y(x)$ in the equation

$$y' = \frac{1}{x}\tan(1+\ln|x|) + (\sec x + \tan x)^2 + \frac{x+5}{4+x}.$$

$$F_1 = \frac{1}{x}\tan(1+\ln|x|)$$
$$= \tan(u)du \qquad u = 1+\ln|x|$$
$$= \frac{d}{du}(-\ln|\cos(u)|)$$

$$F_2 = (\sec x + \tan x)^2$$
$$= \sec^2 x + \tan^2 x + 2\sec x\tan x$$
$$= 2\sec^2 x - 1 + 2\sec x\tan x$$
$$= \frac{d}{dx}(2\tan x - x + 2\sec x)$$

$$F_3 = \frac{x+5}{x+4}$$
$$= 1 + \frac{1}{x+4}$$
$$= \frac{d}{dx}(x+\ln|x+4|)$$

$$\int y'dx = \int F_1 dx + \int F_2 dx + \int F_3 dx$$ Quadrature Step

$$\boxed{\begin{aligned} y = c &+ \ln|\sec(1+\ln|x|)| \\ &+ 2\tan(x) - x + 2\sec(x) \\ &+ x + \ln|x+4| \end{aligned}}$$

← The x-terms cancel.

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2. (Classification of Equations)
The problem $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

☒	$y' = y(2xy+1) + (x-1)y$ $= 2xy^2 + y + xy - y$	☒	$e^x e^y y' = y^2 \cos x + 4\cos x$ $= \cos x\,(y^2+4)$
☐	$y' = e^{3x+2y} + e^{2x+3y}$	☒	$y' + 3y = 13$

(b) [25%] State a test which can verify that an equation $y' = f(x,y)$ is linear but not separable.

(c) [35%] Use the separable equation test to show that $y' = (x+y)^2$ is not separable.

(b) $\frac{\partial f}{\partial y}$ independent of y means linear. Then $f(x,y) = -p(x)y + q(x)$

Given $f(x_0,y_0) \neq 0$ for some x_0, y_0, then

$$\frac{f(x,y_0)}{f(x_0,y_0)} f(x_0,y) \neq f(x,y)$$

implies the DE is not separable.

(c) Define $x_0 = 0,\ y_0 = 1$,

$$F(x) = \frac{f(x,1)}{f(0,1)} = (1+x)^2, \quad G(y) = f(0,y) = y^2$$

$$FG = (1+x)^2 y^2$$
$$\neq (x+y)^2 = f$$

Then $y' = (x+y)^2$ is not separable

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3. (Solve a Separable Equation)

Given $yy' = \left(\dfrac{\csc^2 x}{\tan x} + \dfrac{x+1}{5+x}\right)(y+1)(y+2)$.

Find the non-equilibrium solution in implicit form.

To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{1}{G} = \frac{y}{(y+1)(y+2)} = \frac{-1}{y+1} + \frac{2}{y+2}$$
$$= \frac{d}{dy}\left(-\ln|y+1| + 2\ln|y+2|\right)$$

$$F = \frac{\csc^3 x}{\tan x} + \frac{x+1}{x+5} = \csc^2 x \csc x \cot x + 1 + \frac{-4}{x+5}$$
$$= \frac{d}{dx}\left(-\frac{\csc^3(x)}{3} + x - 4\ln|x+5|\right)$$

$$\int \frac{y'\,dx}{G(y)} = \int F\,dx$$

Quadrature on prepared DE

$$\boxed{-\ln|y+1| + 2\ln|y+2| = -\frac{\csc^3(x)}{3} + x - 4\ln|x+5| + c}$$

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4. (Linear Equations)

(a) [60%] Solve $10x'(t) = -98 + \frac{10}{2t+3}x(t)$, $x(0) = -147/5$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} = -(\tan x)y$. The answer contains symbol c.

(c) [20%] Solve $y' = 5y + 3$ by using the superposition principle $y = y_h + y_p$.

(a) $x' + px = q$ $\quad p = \frac{-1}{2t+3}$ $\quad q = -\frac{98}{10}$

$\frac{(xW)'}{W} = q$ $\quad W = e^{\int p\,dt} = (2t+3)^{-1/2}$ near $t=0$

$(xW)' = qW$ prepared for quadrature

$xW = -\frac{98}{10}\int (2t+3)^{-1/2}\,dt$ quadrature step

$= -\frac{98}{10}(2t+3)^{1/2} + c$

$x = (2t+3)^{1/2}\left(-\frac{98}{10}(2t+3)^{1/2} + c\right)$

$-\frac{147}{5} = -\frac{98}{10}(3) + \sqrt{3}c$, Then $c = 0$

$$\boxed{x(t) = -\frac{98}{10}(2t+3)}$$

(b) $(yW)' = 0$ where $W = e^{\int \tan x\,dx} = e^{-\ln|\cos x|}$. use $W = \sec x$

$yW = c$ or $\boxed{y = c \cdot \cos x}$

(c) $y_p = -3/5$ an equilibrium sol

$y_h = ce^{5x}$ a growth-decay recipe sol for $y' - 5y = 0$

$y = y_h + y_p$

$$\boxed{y = ce^{5x} - \frac{3}{5}}$$

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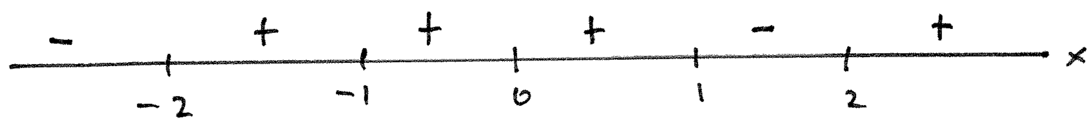
5. (Stability)
(a) [50%] Draw a phase line diagram for the differential equation

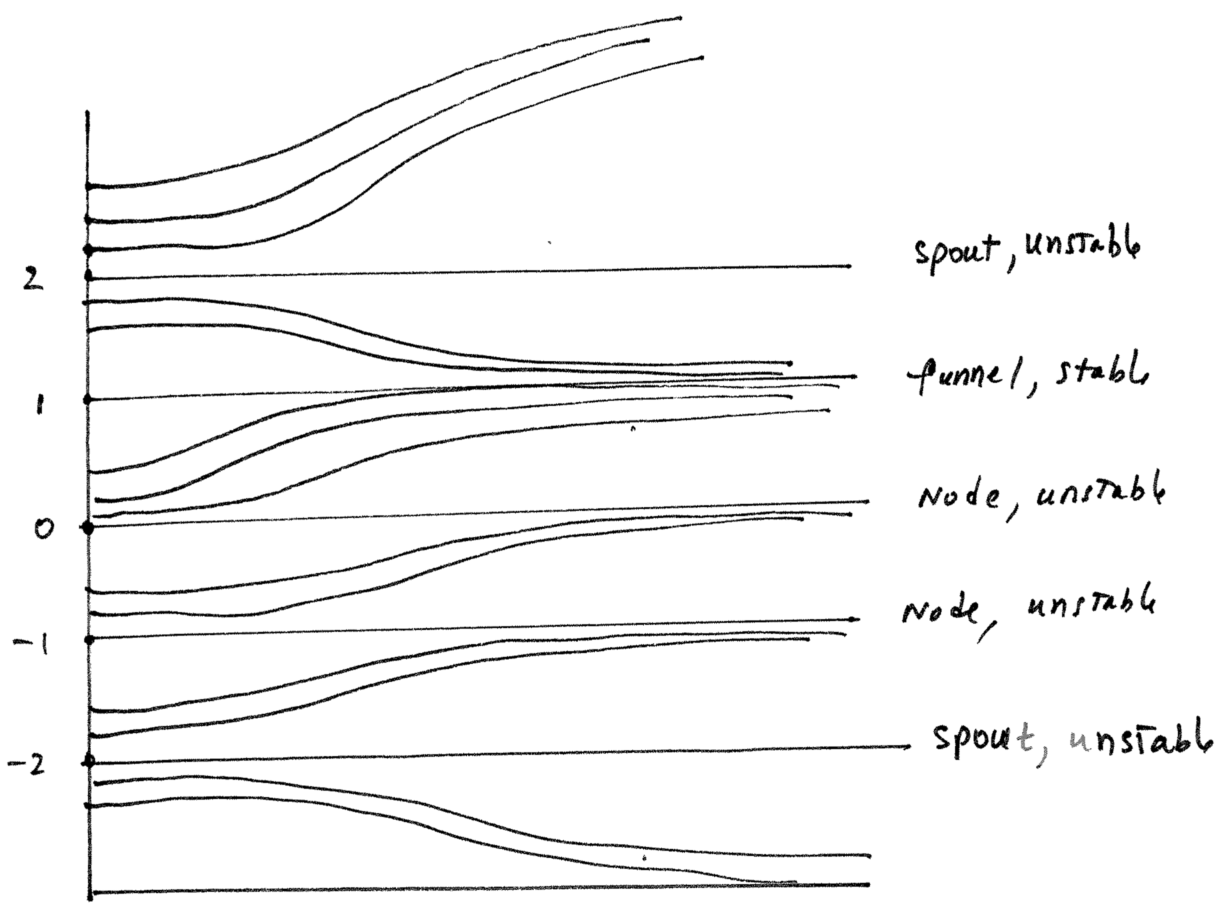
$$\frac{dx}{dt} = \ln(1+x^2)\left(1-\sqrt[3]{|x|}\right)^2(1-x)(4-x^2)(x^2-1)^4.$$

Expected in the phase line diagram are equilibrium points and signs of x'.
(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, neither nor node, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

(a) $f(x) = -\ln(1+x^2)\,(1-\sqrt[3]{|x|})^2(x-1)^5(x+1)^4(2-x)(2+x)$
Roots $=$ $0, \pm 1, \pm 2$, changes sign at $x = 1, 2, -2$, $f > 0$ at ∞ and $f < 0$ at $-\infty$.



(b)



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Def: node = "not a funnel and not a spout"