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Differential Equations and Linear Algebra 2250

Midterm Exam 1 Version 1 [7:30]

Tuesday, 25 September 2007

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 75%. The answer counts 25%. Unevaluated integrals will receive partial credit.

1. (Quadrature Equation)
Solve for the general solution $y(x)$ in the equation

$$y' = \frac{1}{x}\cot(2 + \ln|x|) + (\csc x + \cot x)^2 + \frac{1-x}{2+x}.$$

$F_1 = \frac{1}{x}\cot(2+\ln|x|) = \cot(u)du \qquad u = 2+\ln|x|$

$= \frac{d}{du}\ln|\sin u|$

$F_2 = (\csc x + \cot x)^2 = \csc^2 x + \cot^2 x + 2\csc x\cot x$

$= 2\csc^2 x - 1 + 2\csc x\cot x$

$= \frac{d}{dx}(-2\cot x - x - 2\csc x)$

$F_3 = \frac{1-x}{2+x} = \frac{3-(2+x)}{2+x} = -1 + \frac{3}{2+x} = \frac{d}{dx}(-x + 3\ln|x+2|)$

$y' = F_1 + F_2 + F_3$

$\int y'dx = \int F_1 + \int F_2 + \int F_3$ Quadrature method

$$\boxed{\begin{aligned} y = c &+ \ln|\sin(2+\ln|x|)| \\ &+ (-2\cot x - x - 2\csc x) \\ &+ (-x + 3\ln|x+2|) \end{aligned}}$$

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2. (Classification of Equations)

The problem $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [40%] Check ($\boxed{\text{X}}$) the problems that can be put into separable form, but don't supply any details.

☐	$y' = y(2xy+3) + (x-2)y$ $= 2xy^2 + xy + y$	☐	$yy' = (x-1)(y+1) - xy$ $= xy - y + x - 1 - xy$
☒	$y' = 2e^{2x} + e^{2x+y}$ $= e^{2x}(2+e^y)$	☒	$y' + y = \dfrac{1+\pi}{2+\pi}$

(b) [25%] State a test which can verify that an equation $y' = f(x,y)$ is quadrature but not linear.

(c) [35%] Use the separable equation test to show that $y' = (1+x-y)^2$ is not separable.

(b) $\frac{\partial f}{\partial y} = 0$ means Quadrature. $\frac{\partial f}{\partial y}$ independent of y means linear.
There is $\boxed{\text{no such test}}$, every quadrature equation is linear.

(c) Choose $x_0 = y_0 = 0$ and define

$$F(x) = \frac{f(x,0)}{f(0,0)} = \frac{(1+x)^2}{1}, \quad G(y) = f(0,y) = (1-y)^2$$

Then $FG = (1+x)^2(1-y)^2$
$\neq (1+x-y)^2 = f$

and the DE is not separable.

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3. (Solve a Separable Equation)
Given $yy' = \left(\dfrac{\sec^2 x}{\cot x} + \dfrac{x^2+1}{5+x}\right)(y-1)(2-y)$.

Find the non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$\frac{1}{G} = \frac{y}{(y-1)(y-2)} = \frac{-1}{y-1} + \frac{2}{y-2}$$ by Heaviside coverup method

$$= \frac{d}{dy}\left(-\ln|y-1| + 2\ln|y-2|\right)$$

$$\begin{array}{r} x-5 \\ x+5 \,\overline{)\, x^2+1} \\ \underline{x^2+5x} \\ 1-5x \\ \underline{-5x-25} \\ 26 \end{array}$$

$$F = \frac{\sec^2 x}{\cot x} + \frac{x^2+1}{5+x}$$

$$= \sec^2 x \tan x + x - 5 + \frac{26}{x+5}$$

$$= \frac{d}{dx}\left(\frac{\sec^2 x}{2} + \frac{x^2}{2} - 5x + 26\ln|x+5|\right)$$

$$\int \frac{y'}{G(y)}\,dx = \int F\,dx$$ method of quadrature

$$\boxed{-\ln|y-1| + 2\ln|y-2| = \frac{1}{2}\sec^2 x + \frac{1}{2}x^2 - 5x + 26\ln|x+5| + c}$$

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4. (Linear Equations)

(a) [60%] Solve $10x'(t) = -98 + \frac{10}{2t+5}x(t)$, $x(0) = -49$. Show all integrating factor steps.

(b) [20%] Solve the homogeneous equation $\frac{dy}{dx} = -(\cot x)y$. The answer contains symbol c.

(c) [20%] Solve $y' = y + 5$ using the superposition principle $y = y_h + y_p$.

(a) $x' + px = q$, $p = \frac{-1}{2t+5}$ $q = \frac{-98}{10}$

$W = e^{\int p\,dt} = (2t+5)^{-1/2}$ near $t=0$

$\frac{(xW)'}{W} = q$ integrating factor method

$(xW)' = qW$ Ready for quadrature method

$$xW = -\frac{98}{10}\int W\,dt$$
$$= -\frac{98}{10}\int (2t+5)^{-1/2}\,dt$$
$$= -\frac{98}{10}(2t+5)^{1/2} + c$$
$$x = \frac{1}{(2t+5)^{-1/2}}\left(-\frac{98}{10}(2t+5)^{1/2} + c\right)$$
$$x = -\frac{98}{10}(2t+5) + c(2t+5)^{1/2}$$

$-49 = -\frac{98}{10}(5) + c\sqrt{5}$ implies $c = 0$

$$\boxed{x(t) = -\frac{98}{10}(2t+5)}$$

(b) $(yW)' = 0$, $W = e^{\int \cot x\,dx} = e^{\ln|\sin x|} = |\sin x|$. Choose $+$,

Then $W = \sin(x)$, $y = \frac{c}{W} = \frac{c}{\sin(x)}$ $\boxed{y = c\cdot\csc(x)}$

(c) Equilibrium solution $y_p = -5$

homogeneous solution $y_h = ce^x$ by Growth-Decay recipe

$$\boxed{y = -5 + ce^x}$$

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5. (Stability)

(a) [50%] Draw a phase line diagram for the differential equation

$$\frac{dx}{dt} = \ln(1+2x^2)\left(1-\sqrt[3]{|x|}\right)^2(1+x)(4-x^2)(x^2-1)^5.$$

Expected in the phase line diagram are equilibrium points and signs of x'.

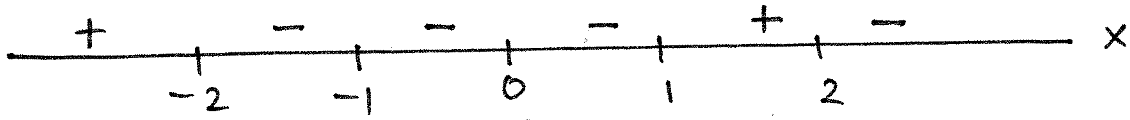
(b) [50%] Draw a phase diagram using the phase line diagram of (a). Add these labels as appropriate: funnel, spout, neither spout nor node, stable, unstable. Show at least 8 threaded curves. A direction field is not expected nor required.

(a) $f(x) = \ln(1+2x^2)\left(1-\sqrt[3]{|x|}\right)^2(1+x)(4-x^2)(x^2-1)^5$

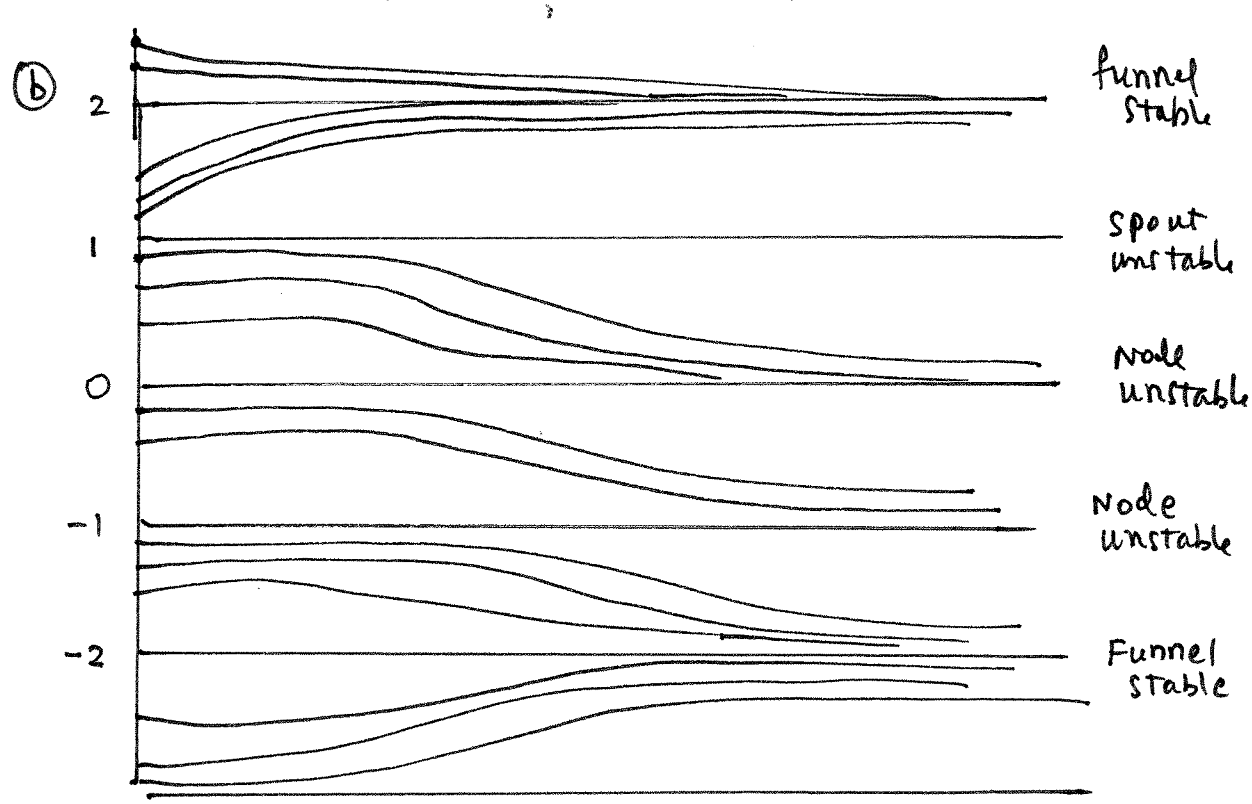
has 5 roots $x = 0, \pm 1, \pm 2$

$f(x) = \ln(1+2x^2)\left(1-\sqrt[3]{|x|}\right)^2(1+x)^6(x-1)^5(2-x)(2+x)$

changes sign at $x = 1, 2, -2$ and Negative at $x = \infty$



(b)



Def: Node ≡ "Not funnel and not spout"

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