Math 2250 Extra Credit Problems  
Chapter 7  
October 2007

Due date: Submit these problems on the first day of final week, under the door 113 JWB, before 9pm. Records are locked on that date and only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Kindly label problems [Extra Credit]. Label each problem with its corresponding problem number, e.g., [Xc7.1-8]. You may attach this printed sheet to simplify your work.

Problem Xc7.1-8. (Transform to a first order system)
Use the position-velocity substitution $u_1 = x(t)$, $u_2 = x'(t)$, $u_3 = y(t)$, $u_4 = y'(t)$ to transform the system below into vector-matrix form $u'(t) = Au(t)$. Do not attempt to solve the system.

$$x'' - 2x' + 5y = 0, \quad y'' + 2y' - 5x = 0.$$

Problem Xc7.1-20a. (Dynamical systems)
Prove this result for system

(1)  
\[
\begin{align*}
x' &= ax + by, \\
y' &= cx + dy.
\end{align*}
\]

**Theorem.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and define $\text{trace}(A) = a + d$. Then $p_1 = -\text{trace}(A)$, $p_2 = \det(A)$ are the coefficients in the determinant expansion

$\det(A - rI) = r^2 + p_1r + p_2$

and $x(t)$ and $y(t)$ in equation (??) are both solutions of the differential equation $u'' + p_1u' + p_2u = 0$.

Problem Xc7.1-20b. (Solve dynamical systems)
(a) Apply the previous problem to solve

\[
\begin{align*}
x' &= 2x - y, \\
y' &= x + 2y.
\end{align*}
\]

(b) Use first order methods to solve the system

\[
\begin{align*}
x' &= 2x - y, \\
y' &= 2y.
\end{align*}
\]

Problem Xc7.2-12. (General solution answer check)
(a) Verify that $x_1(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $x_2(t) = e^{2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are solutions of $x' = Ax$, where

$A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$.

(b) Apply the Wronskian test $\det(\text{aug}(x_1, x_2)) \neq 0$ to verify that the two solutions are independent.
(c) Display the general solution of $x' = Ax$.

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Problem Xc7.2-14. (Particular solution)
(a) Find the constants \( c_1, c_2 \) in the general solution

\[
x(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}
\]

satisfying the initial conditions \( x_1(0) = 4, x_2(0) = -1 \).
(b) Find the matrix \( A \) in the equation \( x' = Ax \). Use the formula \( AP = PD \) and Fourier’s model for \( A \), which is given implicitly in (a) above.

Problem Xc7.3-8. (Eigenanalysis method 2 \times 2)
(a) Find \( \lambda_1, \lambda_2, v_1, v_2 \) in Fourier’s model \( A (c_1 v_1 + c_2 v_2) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \) for

\[
A = \begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}.
\]

(b) Display the general solution of \( x' = Ax \).

Problem Xc7.3-20. (Eigenanalysis method 3 \times 3)
(a) Find \( \lambda_1, \lambda_2, \lambda_3, v_1, v_2, v_3 \) in Fourier’s model \( A (c_1 v_1 + c_2 v_2 + c_3 v_3) = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + c_3 \lambda_3 v_3 \) for

\[
A = \begin{pmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{pmatrix}.
\]

(b) Display the general solution of \( x' = Ax \).

Problem Xc7.3-30. (Brine Tanks)
Consider two brine tanks satisfying the equations

\[
x_1'(t) = -k_1 x_1 + k_2 x_2, \quad x_2'(t) = k_1 x_1 - k_2 x_2.
\]

Assume \( r = 10 \) gallons per minute, \( k_1 = r/V_1, \quad k_2 = r/V_2, \quad x_1(0) = 30 \) and \( x_2(0) = 0 \). Let the tanks have volumes \( V_1 = 50 \) and \( V_2 = 25 \) gallons. Solve for \( x_1(t) \) and \( x_2(t) \).

Problem Xc7.3-40. (Eigenanalysis method 4 \times 4)
Display (a) Fourier’s model and (b) the general solution of \( x' = Ax \) for the 4 \times 4 matrix

\[
A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 5 & 0 \\ 0 & 16 & -4 \end{pmatrix}.
\]

End of extra credit problems chapter 7.