Name ______________________ Class Time _____

Math 2250 Extra Credit Problems
Chapter 5
September 2007

Due date: See the internet due date for 7.4, which is the due date for these problems. Records are locked on that date and only corrected, never appended.

Submitted work. Please submit one stapled package per problem. Kindly label problems **Extra Credit**. Label each problem with its corresponding problem number, e.g., XC5.2-18. You may attach this printed sheet to simplify your work.

Problem XCL5.2. (maple lab 5, row space)

You may submit this problem only for score increases on maple lab 5.

Let $A = \begin{pmatrix} 1 & 1 & 1 & 2 & 6 \\ 3 & 3 & -2 & 1 & -3 \\ 0 & 1 & -4 & -3 & -15 \\ 3 & 2 & 2 & 4 & 12 \end{pmatrix}$. Find two different bases for the row space of A, using the following three methods.

1. The method of Example 2 in maple lab 5 (see the web site).
2. The `maple` command `rowspace(A)`.
3. The **rref**-method: select rows from **rref**(A).

Two of the methods produce exactly the same basis. **Verify** that the two bases $\mathcal{B}_1 = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $\mathcal{B}_2 = \{\mathbf{w}_1, \mathbf{w}_2\}$ are **equivalent**. This means that each vector in $\mathcal{B}_1$ is a linear combination of the vectors in $\mathcal{B}_2$, and conversely, that each vector in $\mathcal{B}_2$ is a linear combination of the vectors in $\mathcal{B}_1$. See the examples in maple Lab 5, at the web site,

Problem XCL5.3. (maple lab 5, Matrix Equations)

You may submit this problem only for score increases on maple lab 5.

Let $A = \begin{pmatrix} -6 & -4 & 11 \\ 3 & 1 & -3 \\ -4 & -4 & 9 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Let P denote a 3×3 matrix. Assume the following result:

> **Lemma 1**. The equality $AP = PT$ holds if and only if the columns $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of P satisfy $A\mathbf{v}_1 = \mathbf{v}_1$, $A\mathbf{v}_2 = -2\mathbf{v}_2$, $A\mathbf{v}_3 = 5\mathbf{v}_3$. [proved after Example 4, see maple lab 5, web site]

(a) Determine three specific columns for P such that $\det(P) \neq 0$ and $AP = PT$. Infinitely many answers are possible. See Example 4 for the maple method that determines a column of P.

(b) After reporting the three columns, check the answer by computing $AP - PT$ (it should be zero) and $\det(P)$ (it should be nonzero).

Problem XC5.1-all. (Second order DE)

This problem counts as 700 if 5.1 was not submitted and 100 otherwise. Solve the following seven parts.

(a) $y'' + 4y' = 0$ (b) $4y'' + 12y' + 9y = 0$ (c) $y'' + 2y' + 5y = 0$
(d) $21y'' + 10y' + y = 0$ (e) $16y'' + 8y' + y = 0$ (f) $y'' + 4y' + (4 + \pi)y = 0$
(g) Find the differential equation $ay'' + by' + cy = 0$, if e^{-x} and e^x are solutions.

Problem XC5.2-18. (Initial value problems)

Given $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$ has three solutions x, $1/x^2$, $\dfrac{\ln|x|}{x^2}$, prove by the Wronskian test that they are independent and then solve for the unique solution satisfying $y(1) = 1$, $y'(1) = 5$, $y''(1) = -11$.

Problem XC5.2-22. (Initial value problem)

Solve the problem $y'' - 4y = 2x$, $y(0) = 2$, $y'(0) = -1/2$, given a particular solution $y_p(x) = -x/2$.

Problem XC5.3-8. **(Complex roots)**

Solve $y'' - 6y' + 25y = 0$.

Problem XC5.3-10. **(Higher order complex roots)**

Solve $y^{iv} + \pi^2 y''' = 0$.

Problem XC5.3-16. **(Fourth order DE)**

Solve the fourth order homogeneous equation whose characteristic equation is $(r-1)(r^3-1) = 0$.

Problem XC5.3-32. **(Theory of equations)**

Solve $y^{iv} - y''' + y'' - 3y' - 6y = 0$. Use the theory of equations [factor theorem, root theorem, rational root theorem, division algorithm] to completely factor the characteristic equation. You may check answers by computer, but please display the complete details of factorization.

Problem XC5.4-20. **(Overdamped, critically damped, underdamped)**

(a) Consider $2x''(t) + 12x'(t) + 50x(t) = 0$. Classify as overdamped, critically damped or underdamped.

(b) Solve $2x''(t)+12x'(t)+50x(t) = 0$, $x(0) = 0$, $x'(0) = -8$. Express the answer in the form $x(t) = C_1 e^{\alpha_1 t}\cos(\beta_1 t - \theta_1)$.

(c) Solve the zero damping problem $2u''(t) + 50u(t) = 0$, $u(0) = 0$, $u'(0) = -8$. Express the answer in phase-amplitude form $u(t) = C_2 \cos(\beta_2 t - \theta_2)$.

(d) Using computer assist, display on one graphic plots of $x(t)$ and $u(t)$. The plot should showcase the damping effects. A hand-made replica of a computer or calculator graphic" is sufficient.

Problem XC5.4-34. **(Inverse problem)**

A body weighing 100 pounds undergoes damped oscillation in a spring-mass system. Assume the differential equation is $mx''+cx'+kx = 0$, with t in seconds and $x(t)$ in feet. Observations give $x(0.4) = 6.1/12$, $x'(0.4) = 0$ and $x(1.2) = 1.4/12$, $x'(1.2) = 0$ as successive maxima of $x(t)$. Then $m = 3.125$ slugs. Find c and k.

Atoms. An **atom** is a term of the form $x^k e^{ax}$, $x^k e^{ax}\cos bx$ or $x^k e^{ax}\sin bx$. The symbol k is a non-negative integer. Symbols a and b are real numbers with $b > 0$. In particular, 1, x, x^2, e^x, $\cos x$, $\sin x$ are atoms. Any distinct list of atoms is linearly independent.

Roots and Atoms. Define **atomRoot**(A) as follows. Symbols α, β, r are real numbers, $\beta > 0$ and k is a non-negative integer.

atom A	**atomRoot**(A)
$x^k e^{rx}$	r
$x^k e^{\alpha x}\cos\beta x$	$\alpha + i\beta$
$x^k e^{\alpha x}\sin\beta x$	$\alpha + i\beta$

The fixup rule for undetermined coefficients can be stated as follows:

> *Compute* **atomRoot**(A) *for all atoms A in the trial solution. Assume r is a root of the characteristic equation of multiplicity k. Search the trial solution for atoms B with* **atomRoot**$(B) = r$*, and multiply each such B by x^k. Repeat for all roots of the characteristic equation.*

Problem Xc5.5-1A. **(AtomRoot Part 1)**

1. Evaluate **atomRoot**(A) for $A = 1$, x, x^2, e^{-x}, $\cos 2x$, $\sin 3x$, $x\cos\pi x$, $e^{-x}\sin 3x$, x^3, e^{2x}, $\cos x/2$, $\sin 4x$, $x^2\cos x$, $e^{3x}\sin 2x$.

2. Let $A = xe^{-2x}$ and $B = x^2 e^{-2x}$. Verify that **atomRoot**$(A) =$ **atomRoot**(B).

Problem Xc5.5-1B. **(AtomRoot Part 2)**

3. Let $A = xe^{-2x}$ and $B = x^2 e^{2x}$. Verify that **atomRoot**$(A) \neq$ **atomRoot**(B).

4. Atoms A and B are said to be **related** if and only if the derivative lists A, A', ... and B, B', ... share a common atom. Prove: atoms A and B are related if and only if **atomRoot**$(A) =$ **atomRoot**(B).

Problem XC5.5-6. **(Undetermined coefficients, fixup rule)**

Find a particular solution $y_p(x)$ for the equation $y^{iv} - 4y'' + 4y = xe^{2x} + x^2e^{-2x}$. Check your answer in `maple`.

Problem XC5.5-12. ()

Find a particular solution $y_p(x)$ for the equation $y^{iv} - 5y'' + 4y = xe^{x} + x^2e^{2x} + \cos x$. Check your answer in `maple`.

Problem XC5.5-22. **(Fixup rule, trial solution)**

Report a trial solution y for the calculation of y_p by the method of undetermined coefficients, after the fixup rule has been applied. To save time, do not do any further undetermined coefficients steps.

$$y^{v} + 2y''' + 2y'' = 5x^3 + e^{-x} + 4\cos x.$$

Hint: Test $r^2(r^3 + 2r + 2) = 0$ when $r =$ **atomRoot**(B) and B is an atom in the initial trial solution. This means a test only for $r = 0, -1, i$.

Problem XC5.5-54. **(Variation of parameters)**

Solve by variation of parameters for $y_p(x)$ in the equation $y'' - 16y = xe^{4x}$. Check your answer in `maple`.

Problem XC5.5-58. **(Variation of parameters)**

Solve by the method of variation of parameters for $y_p(x)$ in the equation $(x^2 - 1)y'' - 2xy' + 2y = x^2 - 1$. Use the fact that $\{x, 1 + x^2\}$ is a basis of the solution space of the homogeneous equation. Apply (33) in the textbook, after division of the leading coefficient $(x^2 - 1)$. Check your answer in `maple`.

Problem XC5.6-4. **(Harmonic superposition)**

Write the general solution $x(t)$ as the superposition of two harmonic oscillations of frequencies 2 and 3, for the initial value problem $x''(t) + 4x(t) = 16\sin 3t$, $x(0) = 0$, $x'(0) = 0$.

Problem XC5.6-8. **(Steady-state periodic solution)**

The equation $x''(t) + 3x'(t) + 3x(t) = 8\cos 10t + 6\sin 10t$ has a unique steady-state periodic solution of period $2\pi/10$. Find it.

Problem XC5.6-18. **(Practical resonance)**

Use the equation $\omega = \sqrt{\dfrac{k}{m} - \dfrac{c^2}{2m^2}}$ to decide upon practical resonance for the equation $mx'' + cx' + kx = F_0\cos\omega t$ when $F_0 = 10$, $m = 1$, $c = 4$, $k = 5$. Sketch the graph of $C(\omega) = \dfrac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$ and mark on the graph the location of the resonant frequency (if any). See Figure 5.6.9 in Edwards-Penney.

End of extra credit problems chapter 5.