Contents

Chapter 6

Scalar Linear Differential Equations

Contents

Studied here are linear differential equations of the second order

$$a(x)y'' + b(x)y' + c(x)y = f(x) \tag{1}$$

and corresponding nth order models. Important to the theory is continuity of the **coefficients** $a(x)$, $b(x)$, $c(x)$ and the **non-homogeneous term** $f(x)$, which is also called the **forcing term** or the **input**.

6.1 Linear 2nd Order Constant

Studied is the homogeneous 2nd order equation

$$Ay'' + By' + Cy = 0$$

where $A \neq 0$, B and C are constants. An explicit formula for the general solution is developed. Prerequisites are the quadratic formula, complex numbers, Cramer's rule for 2×2 linear algebraic equations and first order linear differential equations.

Theorem 6.1 (How to Solve Second Order Constant Equations)
In the differential equation $Ay''+By'+Cy=0$, let $A \neq 0$, B and C be real constants. Let r_1, r_2 denote the two roots of the quadratic equation $Ar^2+Br+C=0$. If the roots are complex, then let $r_1 = a+ib$ with $b>0$, and $r_2 = \overline{r_1} = a-ib$. Define solutions $y_1(x)$, $y_2(x)$ of $Ay''+By'+Cy=0$ according to the following three cases, which are organized by the sign of the college algebra discriminant $\mathcal{D} = B^2-4AC$:

Case 1. $\mathcal{D}>0$ (Real distinct) $y_1(x) = e^{r_1 x}$, $y_2(x) = e^{r_2 x}$.

Case 2. $\mathcal{D}=0$ (Real equal) $y_1(x) = e^{r_1 x}$, $y_2(x) = xe^{r_1 x}$.

Case 3. $\mathcal{D}<0$ (Conjugate roots) $y_1(x)=e^{ax}\cos(bx), y_2(x)=e^{ax}\sin(bx)$.

Then each solution of $Ay''+By'+Cy=0$ is obtained, for some specialization of the constants c_1, c_2, from the expression

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Proof on page 437. Examples 6.1–6.3, page 434, consider the three cases.

A **general solution** is an expression that represents all solutions of the differential equation. Theorem 6.1 gives an expression of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

where c_1 and c_2 are *symbols* representing constants and y_1, y_2 are special solutions of the differential equation, determined by the roots of the **characteristic equation** $Ar^2+Br+C=0$ as in Theorem 6.1.

The **initial value problem** for $Ay''+By'+Cy=0$ selects the constants c_1, c_2 in the general solution $y = c_1 y_2 + c_2 y_2$ from **initial conditions** of the form

$$y(x_0) = g_1, \quad y'(x_0) = g_2.$$

In these conditions, x_0 is a given point in $-\infty < x < \infty$ and g_1, g_2 are two real numbers, e.g., $g_1 =$ position, $g_2 =$ velocity at $x = x_0$.

Theorem 6.2 (Picard-Lindelöf Existence-Uniqueness)
Let $A \neq 0$, B, C, x_0, g_1 and g_2 be constants. Then the initial value problem $Ay''+By'+Cy=0$, $y(x_0)=g_1$, $y'(x_0)=g_2$ has one and only one solution, found from the general solution $y = c_1y_1+c_2y_2$ by applying Cramer's rule or the method of elimination. The solution is defined on $-\infty < x < \infty$.

Proof on page 437. Cramer's rule details are in Example 6.4, page 435.

Working Rule to solve $Ay''+By'+Cy=0$.

Find the roots of the characteristic equation $Ar^2+Br+C=0$. Apply Theorem 6.2 to write down y_1, y_2. The general solution is then $y = c_1y_1+c_2y_2$. If initial conditions are given, then determine c_1, c_2 explicitly, otherwise c_1, c_2 remain symbols.

Theorem 6.3 (Superposition)
In differential equation $Ay'' + By' + Cy = 0$, let $A \neq 0$, B and C be constants. Assume y_1, y_2 are solutions and c_1, c_2 are constants. Then $y = c_1 y_1 + c_2 y_2$ is a solution of $Ay'' + By' + Cy = 0$.

A proof appears on page 438. The result is implicitly used in Theorem 6.1, in order to show that a general solution satisfies the differential equation.

Structure of Solutions

The special solutions y_1, y_2 constructed in Theorem 6.1 have the form

$$e^{ax}, xe^{ax}, e^{ax}\cos bx, e^{ax}\sin bx.$$

These functions will be called **Euler solution atoms** or briefly **Atoms**.

Definition 6.1 (Euler Solution Atoms)
Define an **Euler base atom** to be one of the functions

$$e^{ax}, e^{ax}\cos bx, e^{ax}\sin bx,$$

where a, $b > 0$ are real constants with $b > 0$. Define

$$\textbf{Euler solution atom} = x^n(\textbf{base atom}), \quad n = 0, 1, 2, \ldots.$$

L. Euler (1707-1783) discovered these special solutions by substitution of $y = e^{rx}$ into the differential equation $Ay'' + By' + Cy = 0$, which results in the equations

$Ar^2e^{rx} + Bre^{rx} + Ce^{rx} = 0$	**Euler's Substitution** $y = e^{rx}$.
$Ar^2 + Br + C = 0$	**Characteristic equation**, found by canceling e^{rx}.

The same equations can also be found for the substitution $y = xe^{rx}$, called **Euler's substitution**. Together, the equations imply:

Theorem 6.4 (Euler's Exponential Substitution)

Euler atom $y = e^{rx}$ is a solution of $Ay'' + By' + Cy = 0$ if and only if r is a root of characteristic equation $Ar^2 + Br + C = 0$.

Euler atom $y = xe^{rx}$ is a solution of $Ay'' + By' + Cy = 0$ if and only if r is a double root of characteristic equation $Ar^2 + Br + C = 0$.

Euler atoms $y = e^{ax}\cos bx$ and $y = e^{ax}\sin bx$ are real solutions of $Ay'' + By' + Cy = 0$ if and only if $r = a + ib$ and $\overline{r} = a - ib$ are complex roots of characteristic equation $Ar^2 + Br + C = 0$.

Proof on page 439.

Theorem 6.1 may be succinctly summarized as follows.

> The general solution y of a second order linear homogeneous constant-coefficient differential equation is a sum of constants times Euler solution atoms. The atoms are found from Euler's Theorem.

Speed

The time taken to write out the general solution varies among individuals and according to the algebraic complexity of the characteristic equation. Judge your understanding of the *Theorem* by these statistics: most persons can write out the general solution in under 60 seconds. Especially simple equations like $y'' = 0$, $y'' + y = 0$, $y'' - y = 0$, $y'' + 2y' + y = 0$, $y'' + 3y' + 2y = 0$ are finished in less than 30 seconds.

Graphics

Computer programs can produce plots for initial value problems. Computers cannot plot **symbolic solutions** containing unevaluated symbols c_1, c_2 that appear in the general solution.

Errors

Recorded below in Table 1 are some common but fatal errors made in displaying the general solution.

Table 1. Errors in Applying Theorem 6.1.

Bad equation	For $y'' - y = 0$, the correct characteristic equation is $r^2 - 1 = 0$. A common error is to write $r^2 - r = 0$.
Sign reversal	For factored equation $(r + 1)(r + 2) = 0$, the roots are $r = -1$, $r = -2$. A common error is to claim $r = 1$ and/or $r = 2$ is a root.
Miscopy signs	The equation $r^2+2r+2 = 0$ has complex conjugate roots $a \pm bi$, where $a = -1$ and $b = 1$ ($b > 0$ is required). A common error is to miscopy signs on a and/or b.
Copying $\pm i$	The equation $r^2 + 2r + 5 = 0$ has roots $a \pm ib$ where $a = -1$ and $b = 2$. A common mistake is to display $e^{-x}\cos(\pm 2ix)$ and $e^{-x}\sin(\pm 2ix)$. These expressions are not real solutions: neither $\pm$ nor the complex unit i should be copied.

Examples

Example 6.1 (Case 1)
Solve $y'' + y' - 2y = 0$.

Solution: The general solution is $y = c_1 e^x + c_2 e^{-2x}$. Ordering is not important; an equivalent answer is $y = c_1 e^{-2x} + c_2 e^x$. The answer will be justified below, by finding the two solutions y_1, y_2 in Theorem 6.1.

The characteristic equation $r^2 + r - 2 = 0$ is found formally by replacements $y'' \to r^2$, $y' \to r$ and $y \to 1$ in the differential equation $y'' + y' - 2y = 0$.[1]

A college algebra method[2] called **inverse-FOIL** applies to factor $r^2 + r - 2 = 0$ into $(r-1)(r+2) = 0$. The roots are $r = 1$, $r = -2$. Used implicitly here are the college algebra **factor theorem** and **root theorem**[3].

Applying case $\mathcal{D} > 0$ of Theorem 6.1 gives solutions $y_1 = e^x$ and $y_2 = e^{-2x}$.

Example 6.2 (Case 2)
Solve $4y'' + 4y' + y = 0$.

Solution: The general solution is $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$. To justify this formula, find the characteristic equation $4r^2 + 4r + 1 = 0$ and factor it by the **inverse-FOIL** method or **square completion** to obtain $(2r+1)^2 = 0$. The roots are both $-1/2$.

Case $\mathcal{D} = 0$ of Theorem 6.1 gives $y_1 = e^{-x/2}$, $y_2 = xe^{-x/2}$. Then the general solution is $y = c_1 y_1 + c_2 y_2$, which completes the verification.

Example 6.3 (Case 3)
Solve $4y'' + 2y' + y = 0$.

Solution: The solution is $y = c_1 e^{-x/4}\cos(\sqrt{3}x/4) + c_2 e^{-x/4}\sin(\sqrt{3}x/4)$. This formula is justified below, by showing that the solutions y_1, y_2 of Theorem 6.1 are given by $y_1 = e^{-x/4}\cos(\sqrt{3}x/4)$ and $y_2 = e^{-x/4}\sin(\sqrt{3}x/4)$.

The characteristic equation is $4r^2 + 2r + 1 = 0$. The roots by the *quadratic formula* are

$$r = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$
College algebra formula for the roots of the quadratic $Ar^2 + Br + C = 0$.

$$= \frac{-2 \pm \sqrt{2^2 - (4)(4)(1)}}{(2)(4)}$$
Substitute $A = 4$, $B = 2$, $C = 1$.

$$= -\frac{1}{4} \pm \frac{\sqrt{-1}\sqrt{12}}{8}$$
Simplify. Used $\sqrt{(-1)(12)} = \sqrt{-1}\sqrt{12}$.

$$= -\frac{1}{4} \pm i\frac{\sqrt{3}}{4}$$
Convert to complex form, $i = \sqrt{-1}$.

[1] Some history. Euler's formal substitution $y = e^{rx}$ into the differential equation $y'' + y' - 2y = 0$ produces $r^2 + r - 2 = 0$ directly. Formal replacement $y'' \to r^2$, $y' \to r$ and $y \to 1$ gives the same characteristic equation $r^2 + r - 2 = 0$, with a reduction in errors. We prefer the shortcut, to increase the speed.

[2] **FOIL** is an abbreviation for **F**irst=AC, **O**utside=AD, **I**nside=BC, **L**ast=BD in the expansion of the algebraic product $(A+B)(C+D)$.

[3] **Theorem**. $r = r_0$ is a root of $p(r) = 0$ if and only if $(r - r_0)$ is a factor of $p(r)$.

The real part of the root is labeled $a = -1/4$. The two imaginary parts are $\sqrt{3}/4$ and $-\sqrt{3}/4$. Only the positive one is labeled, the other being discarded: $b = \sqrt{3}/4$.

Theorem 6.1 applies in the discriminant case $\mathcal{D} < 0$ to give solutions $y_1 = e^{ax}\cos(bx)$ and $y_2 = e^{ax}\sin(bx)$. Substitution of $a = -1/4$ and $b = \sqrt{3}/4$ results in the formulas $y_1 = e^{-x/4}\cos(\sqrt{3}x/4)$, $y_2 = e^{-x/4}\sin(\sqrt{3}x/4)$. The verification is complete.

The substitutions of a, b are remembered from the following diagram.

$$\begin{array}{cccc}
\boxed{-1/4} & \boxed{\sqrt{3}/4} & \boxed{-1/4} & \boxed{\sqrt{3}/4} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
e^{\boxed{\mathbf{a}}\,x} & \cos\left(\boxed{\mathbf{b}}\,x\right) & e^{\boxed{\mathbf{a}}\,x} & \sin\left(\boxed{\mathbf{b}}\,x\right) \\
\multicolumn{2}{c}{e^{-x/4}\cos\left(\sqrt{3}4x\right),} & \multicolumn{2}{c}{e^{-x/4}\sin\left(\sqrt{3}4x\right)}
\end{array}$$

It is recommended to perform the a, b substitution to find the first atom, which is $e^{-x/4}\cos\left(\sqrt{3}4x\right)$. Then replace $\boxed{\cos}$ by $\boxed{\sin}$ in that expression to obtain the second atom $e^{-x/4}\sin\left(\sqrt{3}4x\right)$.

Example 6.4 (Initial Value Problem)

Solve $y'' + y' - 2y = 0$, $y(0) = 1$, $y'(0) = -2$ and graph the solution on $0 \le x \le 2$.

Solution: The solution to the initial value problem is $y = e^{-2x}$. The graph appears in Figure 1.

Details. The general solution is $y = c_1 e^{x} + c_2 e^{-2x}$, from Example 6.1. The problem of finding c_1, c_2 uses the two equations $y(0) = 1$, $y'(0) = -2$ and the general solution to obtain expanded equations for c_1, c_2. For instance, $y(0) = 1$ expands to $\left.(c_1 e^{x} + c_2 e^{-2x})\right|_{x=0} = 1$, which is an equation for symbols c_1, c_2. The second equation $y'(0) = -2$ expands similarly, to give the two equations

$$\begin{array}{rcrcr}
e^0 c_1 & + & e^0 c_2 & = & 1, \\
e^0 c_1 & - & 2e^0 c_2 & = & -2.
\end{array}$$

The equations will be solved by the method of elimination. Because $e^0 = 1$, the equations simplify. Subtracting them eliminates the variable c_1 to give $3c_2 = 3$. Therefore, $c_2 = 1$ and back-substitution finds $c_1 = 0$. Then $y = c_1 e^{x} + c_2 e^{-2x}$ reduces, after substitution of $c_1 = 0$, $c_2 = 1$, to the equation $y = e^{-2x}$.

Graph. The solution $y = e^{-2x}$ is graphed by a routine application of curve library methods, which appear in the appendices, page **??**. No hand-graphing methods will be discussed here. To produce a computer graphic of the solution, the following code is offered. Calculator plots are similar.

```
plot(exp(-2*x),x=0..2);          Maple
plot2d(exp(-2*x),[x,0,2]);       Maxima
Plot[{exp(-2 x)},{x,0,2}];       Mathematica
plot [0:2] exp(-2*x)             Gnuplot
x=0:0.05:2; plot(x,exp(-2*x)) Matlab and Scilab
```

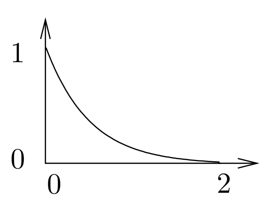

Figure 1. Exponential solution $y = e^{-2x}$. The graph decreases to zero at $x = \infty$.

Example 6.5 (Euler Solution Atoms)
Consider the list

$$1,\ x^2,\ 2,\ 3x+4x^2,\ x^3e^{x/\pi},\ 2x+3\cos x,\ \frac{x}{1+x^2}.$$

Box each entry that is precisely an atom and identify its base atom. Double-box the non-atom list entries that are a sum of constants times atoms.

Solution:
The answers and explanations:

$\boxed{1}$	An atom. Base atom $= 1$.
$\boxed{x^2}$	An atom. Base atom $= 1$.
$\boxed{\boxed{2}}$	$\boxed{\mathbf{X}}$ Not an atom. Constant 2 times the atom 1, which is a linear combination of atoms.
$\boxed{\boxed{3x+4x^2}}$	$\boxed{\mathbf{X}}$ **Not** an atom. Linear combination of atoms x, x^2.
$\boxed{e^{x/\pi}}$	An atom. Base atom $= e^{ax}$ where $a = 1/\pi$.
$\boxed{\boxed{2x+3\cos x}}$	$\boxed{\mathbf{X}}$ **Not** an atom. Linear combination of atoms $x, \cos x$.
$\dfrac{x}{1+x^2}$	$\boxed{\mathbf{X}}$ **Not** an atom. Not a linear combination of atoms.

Example 6.6 (Inverse Problem)
Consider a 2nd order differential equation $Ay'' + By' + Cy = 0$, the coefficients A, B, C initially unknown. Find a set of coefficients for each of the following three examples, given the supplied information about the differential equation.

(a) The characteristic equation is $r^2 + 2r + 5 = 0$.

(b) The characteristic equation has roots $r = -1, 2$.

(c) Two solutions are e^x and xe^x.

Solution:
(a) The characteristic equation of $Ay''+By'+Cy = 0$ is $Ar^2+Br+C = 0$. Comparing terms to $r^2 + 2r + 5 = 0$ implies a differential equation is $y'' + 2y' + 5y = 0$. The substitutions $y \to 1, y' \to r, y'' \to r^2$ are used here in reverse.

(b) The characteristic polynomial $Ar^2 + Br + C$ factors into $A(r - r_1)(r - r_2)$ where r_1, r_2 are the two roots of the quadratic equation. Given $r_1 = -1$ and $r_2 = 2$, then the characteristic equation has to be $A(r - (-1))(r - 2) = 0$ for some number $A \neq 0$. Assume $A = 1$ to find one equation. Multiply out the product $(r + 1)(r - 2)$ to give characteristic equation $r^2 - r - 2 = 0$. This reduces the problem to methods in part (a). Then a differential equation is $y'' - y' - 2y = 0$.

(c) The two given solutions are Euler solution atoms created from root $r = 1$. Consulting Theorem 6.4, these two atoms are solutions of a second order equation with characteristic equation roots $r = 1, 1$ (a double root). The method in (b) is then applied: multiply out the product $(r-1)(r-1)$ to get characteristic equation $r^2 - 2r + 1 = 0$. Then apply the method of (a). A differential equation is $y'' - 2y' + y = 0$.

Proofs and Details

Proof of Theorem 6.1: To show that y_1 and y_2 are solutions is left to the exercises. For the remainder of the proof, assume y is a solution of $Ay'' + By' + Cy = 0$. It has to be shown that $y = c_1 y_1 + c_2 y_2$ for some real constants c_1, c_2.

Algebra background. In college algebra it is shown that the polynomial $Ar^2 + Br + C$ can be written in terms of its roots r_1, r_2 as $A(r - r_1)(r - r_2)$. In particular, the sum and product of the roots satisfy the relations $B/A = -r_1 - r_2$ and $C/A = r_1 r_2$.

Case $\mathcal{D} > 0$. The equation $Ay'' + By' + Cy = 0$ can be re-written in the form $y'' - (r_1 + r_2)y' + r_1 r_2 y = 0$ due to the college algebra relations for the sum and product of the roots of a quadratic equation. The equation *factors* into $(y' - r_2 y)' - r_1(y' - r_2 y) = 0$, which suggests the substitution $u = y' - r_2 y$. Then $Ay'' + By' + Cy = 0$ is equivalent to the first order system

$$\begin{aligned} u' - r_1 u &= 0, \\ y' - r_2 y &= u. \end{aligned}$$

Growth-decay theory, page **??**, applied to the first equation gives $u = u_0 e^{r_1 x}$. The second equation $y' - r_2 y = u$ is then solved by the integrating factor method, as in Example **??**, page **??**. This gives $y = y_0 e^{r_2 x} + u_0 e^{r_1 x}/(r_1 - r_2)$. Therefore, any possible solution y has the form $c_1 e^{r_1 x} + c_2 e^{r_2 x}$ for some c_1, c_2. This completes the proof of the case $\mathcal{D} > 0$.

Case $\mathcal{D} = 0$. The details follow the case $\mathcal{D} > 0$, except that $y' - r_2 y = u$ has a different solution, $y = y_0 e^{r_1 x} + u_0 x e^{r_1 x}$ (exponential factors $e^{r_1 x}$ and $e^{r_2 x}$ cancel because $r_1 = r_2$). Therefore, any possible solution y has the form $c_1 e^{r_1 x} + c_2 x e^{r_1 x}$ for some c_1, c_2. This completes the proof of the case $\mathcal{D} = 0$.

Case $\mathcal{D} < 0$. The equation $Ay'' + By' + Cy = 0$ can be re-written in the form $y'' - (r_1 + r_2)y' + r_1 r_2 y = 0$ as in the case $\mathcal{D} > 0$, even though y is real and the roots are complex. The substitution $u = y' - r_2 y$ gives the same equivalent system as in the case $\mathcal{D} > 0$. The solutions are symbolically the same, $u = u_0 e^{r_1 x}$ and $y = y_0 e^{r_1 x} + u_0 e^{r_1 x}/(r_1 - r_2)$. Therefore, any possible real solution y has the form $C_1 e^{r_1 x} + C_2 e^{r_2 x}$ for some possibly complex C_1, C_2.

Taking the real part of both sides of this equation gives $y = c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$ for some real constants c_1, c_2. Details follow.

$y = \mathcal{R}\mathrm{e}(y)$	Because y is real.
$= \mathcal{R}\mathrm{e}(C_1 e^{r_1 x} + C_2 e^{r_2 x})$	Substitute $y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$.
$= e^{ax}\,\mathcal{R}\mathrm{e}(C_1 e^{ibx} + C_2 e^{-ibx})$	Use $e^{u+iv} = e^u e^{iv}$.
$= e^{ax}\,\mathcal{R}\mathrm{e}\,(C_1 \cos bx + iC_1 \sin bx$ $+ C_2 \cos bx - iC_2 \sin bx)$	Use $e^{i\theta} = \cos\theta + i\sin\theta$.
$= e^{ax}\,\mathcal{R}\mathrm{e}(C_1 + C_2))\cos bx$ $+ e^{ax}\,\mathcal{R}\mathrm{e}(iC_1 - iC_2)\sin bx$	Collect on trigonometric factors.
$= c_1 e^{ax}\cos(bx) + c_2 e^{ax}\sin(bx)$	Where $c_1 = \mathcal{R}\mathrm{e}(C_1 + C_2)$ and $c_2 = \mathcal{I}\mathrm{m}(C_2 - C_1)$ are real.

This completes the proof of the case $\mathcal{D} < 0$.

Proof of Theorem 6.2: The left sides of the two requirements $y(x_0) = g_1$, $y'(x_0) = g_2$ are expanded using the relation $y = c_1 y_1 + c_2 y_2$ to obtain the following system of equations for the unknowns c_1, c_2:

$$\begin{array}{rcrcl} y_1(x_0)c_1 &+& y_2(x_0)c_2 &=& g_1, \\ y_1'(x_0)c_1 &+& y_2'(x_0)c_2 &=& g_2. \end{array}$$

If the determinant of coefficients

$$\Delta = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$$

is nonzero, then Cramer's rule says that the solutions c_1, c_2 are given as quotients

$$c_1 = \frac{g_1 y_2'(x_0) - g_2 y_2(x_0)}{\Delta}, \quad c_2 = \frac{y_1(x_0)g_2 - y_1'(x_0)g_1}{\Delta}.$$

The organization of the proof is made from the three cases of Theorem 6.1, using x instead of x_0, to simplify notation. The issue of a unique solution has now been reduced to verification of $\Delta \neq 0$, in the three cases.

Case $\mathcal{D} > 0$. Then

$$\begin{aligned} \Delta &= e^{r_1 x} r_2 e^{r_2 x} - r_1 e^{r_1 x} e^{r_2 x} && \text{Substitute for } y_1,\ y_2. \\ &= (r_2 - r_1)e^{r_1 x + r_2 x} && \text{Simplify.} \\ &\neq 0 && \text{Because } r_1 \neq r_2. \end{aligned}$$

Case $\mathcal{D} = 0$. Then

$$\begin{aligned} \Delta &= e^{r_1 x}(e^{r_1 x} + r_1 x e^{r_1 x}) - r_1 e^{r_1 x} x e^{r_1 x} && \text{Substitute for } y_1,\ y_2. \\ &= e^{2r_1 x} && \text{Simplify.} \\ &\neq 0 \end{aligned}$$

Case $\mathcal{D} < 0$. Then $r_1 = \overline{r_2} = a + ib$ and

$$\begin{aligned} \Delta &= be^{2ax}(\cos^2 bx + \sin^2 bx) && \text{Two terms cancel.} \\ &= be^{2ax} && \text{Use } \cos^2\theta + \sin^2\theta = 1. \\ &\neq 0 && \text{Because } b > 0. \end{aligned}$$

In applications, the method of elimination is sometimes used to find c_1, c_2. In some references, it is called *Gaussian elimination*.

Proof of Superposition Theorem 6.3: The three terms of the differential equation are computed using the expression $y = c_1 y_1 + c_2 y_2$:

$$\begin{aligned} &\text{Term 1:} && cy = cc_1 y_1 + cc_2 y_2 \\ &\text{Term 2:} && by' = b(c_1 y_1 + c_2 y_2)' \\ &&& = bc_1 y_1' + bc_2 y_2' \\ &\text{Term 3:} && ay'' = a(c_1 y_1 + c_2 y_2)'' \\ &&& = ac_1 y_1'' + ac_2 y_2'' \end{aligned}$$

The left side of the differential equation, denoted LHS, is the sum of the three terms. It is simplified as follows:

$$\begin{aligned} \text{LHS} &= c_1[ay_1'' + by_1' + cy_1] && \text{Add terms 1,2 and 3,} \\ &\quad + c_2[ay_2'' + by_2' + cy_2] && \text{then collect on } c_1,\ c_2. \\ &= c_1[0] + c_2[0] && \text{Both } y_1,\ y_2 \text{ satisfy } ay'' + by' + cy = 0. \\ &= \text{RHS} && \text{The left and right sides match.} \end{aligned}$$

Proof of Euler's Theorem 6.4 The substitution $y = e^{rx}$ requires the derivative formulas $y' = re^{rx}$, $y'' = r^2e^{rx}$, which then imply from $Ay'' + By' + Cy = 0$ the relation

$$Ar^2e^{rx} + Bre^{rx} + Ce^{rx} = 0. \tag{2}$$

Assume that $y = e^{rx}$ is a solution of the differential equation. Then relation (2) holds. Cancel e^{rx} to obtain $Ar^2 + Br + C = 0$, then r is a root of the characteristic equation.

Conversely, if r is a root of the characteristic equation, then multiply $Ar^2 + Br + C = 0$ by e^{rx} to give relation (2). Then $y = e^{rx}$ is a solution of the differential equation.

This completes the proof of the first statement in Euler's theorem, in the special case for r real. Examination of the details reveals it is also valid for complex $r = a + ib$, with $y = e^{rx}$ a complex solution.

We go on to prove the third statement in Euler's theorem. A complex exponential solution $y = e^{rx}$, with $r = a + ib$, can be expanded as $y = e^{rx} = e^{ax+ibx} = e^{ax}\cos bx + ie^{ax}\sin bx$, because of Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$. Write $u = e^{ax}\cos bx$ and $v = e^{ax}\sin bx$, then $y = u + iv$ with u, v real. Expand the differential equation $Ay'' + By' + Cy = 0$ using $y = u + iv$ as

$$(Au'' + Bu' + Cu) + i(Av'' + Bv' + Cv) = 0 + 0i.$$

Then A, B, C, u, v all real implies, by equality of complex numbers, the two equations

$$\begin{aligned} Au'' + Bu' + Cu &= 0, \\ Av'' + Bv' + Cv &= 0. \end{aligned}$$

Together, these equations imply that $u = e^{ax}\cos bx$ and $v = e^{ax}\sin bx$ are solutions of the differential equation. Conversely, if both u and v are solutions, then the steps can be reversed to show $y = e^{rx}$ is a solution, which in turn implies $r = a + ib$ is a root of the characteristic equation. Finally, if $a + ib$ is a root and A, B, C are real, then college algebra implies $a - ib$ is a root. This completes the proof of the last statement of Euler's theorem.

The second statement of Euler's theorem will be proved. Substitute $y = xe^{rx}$ into the differential equation using the formulas $y' = e^{rx} + rxe^{rx}$, $y'' = 2re^{rx} + r^2xe^{rx}$ to obtain the relation

$$(Ar^2 + Br + C)xe^{rx} + (2Ar + B)e^{rx} = 0. \tag{3}$$

If $y = xe^{rx}$ is a solution of the differential equation, then relation (3) holds for all x. Cancel e^{rx} to get the polynomial relation

$$(Ar^2 + Br + C)x + (2Ar + B) = 0, \quad \text{for all } x.$$

Substitute $x = 0$ and then $x = 1$ to obtain $2Ar + B = 0$ and $Ar^2 + Br + C = 0$. These equations say that r is a double root of the characteristic equation, because the polynomial $p(t) = At^2 + Bt + C$ then satisfies $p(r) = p'(r) = 0$.

Conversely, suppose that r is a double root of $Ar^2+Br+C = 0$. Then $p(t) = At^2+Bt+C$ must satisfy the relations $p(r) = p'(r) = 0$, which imply $Ar^2+Br+C = 0$ and $2Ar+B = 0$. Then for all x, relation (3) holds, which in turn implies that $y = xe^{rx}$ is a solution. ■

Exercises 6.1

General Solution 2nd Order

Solve the constant equation using Theorem 6.1, page 430. Report the general solution using symbols c_1, c_2. Model the solution after Examples 6.1–6.3, page 434.

1. $y'' = 0$
Ans: $y = c_1 + c_2 x$

2. $3y'' = 0$

3. $y'' + y' = 0$

4. $3y'' + y' = 0$

5. $y'' + 3y' + 2y = 0$

6. $y'' - 3y' + 2y = 0$

7. $y'' - y' - 2y = 0$

8. $y'' - 2y' - 3y = 0$

9. $y'' + y = 0$

10. $y'' + 4y = 0$

11. $y'' + 16y = 0$

12. $y'' + 8y = 0$

13. $y'' + y' + y = 0$

14. $y'' + y' + 2y = 0$

15. $y'' + 2y' + y = 0$

16. $y'' + 4y' + 4y = 0$

17. $3y'' + y' + y = 0$

18. $9y'' + y' + y = 0$

19. $5y'' + 25y' = 0$

20. $25y'' + y' = 0$

21. $2y'' + y' - y = 0$

22. $2y'' - 3y' - 2y = 0$

23. $2y'' + 7y' + 3y = 0$

24. $4y'' + 8y' + 3y = 0$

25. $6y'' + 7y' + 2y = 0$

26. $6y'' + y' - 2y = 0$

27. $y'' + 4y' + 8y = 0$

28. $y'' - 2y' + 4y = 0$

29. $y'' + 2y' + 4y = 0$

30. $y'' + 4y' + 5y = 0$

31. $4y'' - 4y' + y = 0$

32. $4y'' + 4y' + y = 0$

33. $9y'' - 6y' + y = 0$

34. $9y'' + 6y' + y = 0$

35. $4y'' + 12y' + 9y = 0$

36. $4y'' - 12y' + 9y = 0$

Initial Value Problem 2nd Order

Solve the given problem, modeling the solution after Example 6.4.

37. $6y''+7y'+2y = 0$, $y(0) = 0$, $y'(0) = -1$

38. $2y''+7y'+3y = 0$, $y(0) = 5$, $y'(0) = -5$

39. $y'' - 2y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$

40. $y'' + 4y' + 5y = 0$, $y(0) = 1$, $y'(0) = 1$

41. $9y'' - 6y' + y = 0$, $y(0) = 3$, $y'(0) = 1$

42. $4y''+12y'+9y = 0$, $y(0) = 2$, $y'(0) = 1$

Detecting Euler Solution Atoms

A **Euler solution atom** is defined in Definition 6.1 page 432. Box each list entry that is precisely an atom. Double-box non-atom list entries that are a sum of constants times atoms. Follow Example 6.5 page 436.

43. 1, $e^{x/5}$, -1, $e^{1.1x}$, $2e^x$

44. $-x\cos\pi x$, $x^2\sin 2x$, x^3, $2x^3$

45. e^{2x}, $e^{-x^2/2}$, $\cos^2 2x$, $\sin 1.57x$

46. $x^7 e^x \cos 3x$, $x^{10} e^x \sin 4x$

47. $x^7 e^x \cosh 3x$, $x^{10} e^{-x} \sinh 5x$

48. $\cosh^2 x$, $x(1+x)$, $x^{1.5}$, $\sqrt{x}e^{-x}$

49. $x^{1/2}e^{x/2}$, $\dfrac{1}{x}e^x$, $e^x(1+x^2)$

50. $\dfrac{x}{1+x}$, $\dfrac{1}{x}(1+x^2)$, $\ln|x|$

Euler Base Atom

An **Euler base atom** is defined in Definition 6.1 page 432. Find the base atom for each Euler solution atom in the given list.

51. $x\cos\pi x$, x^3, $x^{10}e^{-x}\sin 5x$

52. x^6, x^4e^{2x}, $x^2e^{-x/\pi}$, $x^7e^x\cos 1.1x$

Inverse Problems

Find the homogeneous 2nd order differential equation, given the supplied information. Follow Example 6.6.

53. $e^{-x/5}$ and 1 are solutions.
Ans: $5y'' + y' = 0$.

54. e^{-x} and 1 are solutions.

55. $e^x + e^{-x}$ and $e^x - e^{-x}$ are solutions.

56. $e^{2x} + xe^{2x}$ and xe^{2x} are solutions.

57. x and $2+x$ are solutions.

58. $4e^x$ and $3e^{2x}$ are solutions.

59. The characteristic equation is $r^2+2r+1=0$.

60. The characteristic equation is $4r^2+4r+1=0$.

61. The characteristic equation has roots $r=-2,3$.

62. The characteristic equation has roots $r=2/3,3/5$.

63. The characteristic equation has roots $r=0,0$.

64. The characteristic equation has roots $r=-4,-4$.

65. The characteristic equation has complex roots $r=1\pm 2i$.

66. The characteristic equation has complex roots $r=-2\pm 3i$.

Details of proofs

67. **(Theorem 6.1, Background)** Expand the relation $Ar^2+Br+C = A(r-r_1)(r-r_2)$ and compare coefficients to obtain the sum and product of roots relations

$$\frac{B}{A} = -(r_1+r_2), \qquad \frac{C}{A} = r_1r_2.$$

68. **(Theorem 6.1, Background)**
Let r_1, r_2 be the two roots of $Ar^2+Br+C=0$. The discriminant is $\mathcal{D} = B^2-4AC$. Use the quadratic formula to derive these relations for $\mathcal{D}>0$, $\mathcal{D}=0$, $\mathcal{D}<0$, respectively:

$$\begin{aligned} r_1 &= \tfrac{-B+\sqrt{\mathcal{D}}}{2A}, r_2 = \tfrac{-B-\sqrt{\mathcal{D}}}{2A},\\ r_1 &= r_2 = \tfrac{\sqrt{\mathcal{D}}}{2A}.\\ r_1 &= \tfrac{-B+i\sqrt{-\mathcal{D}}}{2A}, r_2 = \tfrac{-B-i\sqrt{-\mathcal{D}}}{2A}.\end{aligned}$$

69. **(Theorem 6.1, Case 1)**
Let $y_1 = e^{r_1x}$, $y_2 = e^{r_2x}$. Assume $Ar^2+Br+C = A(r-r_1)(r-r_2)$. Show that y_1, y_2 are solutions of $Ay''+By'+Cy=0$.

70. **(Theorem 6.1, Case 2)**
Let $y_1 = e^{r_1x}$, $y_2 = x\,e^{r_1x}$. Assume $Ar^2+Br+C{=}A(r-r_1)(r-r_1)$. Show that y_1, y_2 are solutions of $Ay''+By'+Cy=0$.

71. **(Theorem 6.1, Case 3)**
Let a, b be real, $b>0$. Let $y_1 = e^{ax}\cos bx$, $y_2 = e^{ax}\sin bx$. Assume factorization
$Ar^2{+}Br{+}C{=}A(r{-}a{-}ib)(r{-}a{+}ib)$
then show that y_1, y_2 are solutions of $Ay''+By'+Cy=0$.

6.2 Continuous Coefficient Theory

The existence, uniqueness and structure of solutions for the equation

$$a(x)y'' + b(x)y' + c(x)y = f(x) \tag{1}$$

will be studied, guided in part by the first order theory.

Continuous–Coefficient Equations

The **homogeneous equation** is $a(x)y'' + b(x)y' + c(x)y = 0$ while the **non-homogeneous equation** is $a(x)y'' + b(x)y' + c(x)y = f(x)$. An equation is said to have **constant coefficients** if a, b and c are scalars.

A **linear combination** of two functions y_1, y_2 is $c_1y_1(x) + c_2y_2(x)$, where c_1 and c_2 are constants. The natural domain is the common domain of y_1 and y_2.

The **general solution** of $a(x)y'' + b(x)y' + c(x)y = f(x)$ is an expression which describes all possible solutions of the equation. Exactly how to write such an expression is revealed in the theorems below.

An **initial value problem** is the problem of solving $a(x)y'' + b(x)y' + c(x)y = f(x)$ subject to **initial conditions** $y(x_0) = g_1$, $y'(x_0) = g_2$. It is assumed that x_0 is in the common domain of continuity of the coefficients and that g_1, g_2 are prescribed numbers.

Theorem 6.5 (Superposition)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has the *superposition property*:

> If y_1, y_2 are solutions and c_1, c_2 are constants, then the linear combination $y(x) = c_1y_1(x) + c_2y_2(x)$ is a solution.

Proof on page 445.

Theorem 6.6 (Picard-Lindelöf Existence-Uniqueness)
Let the coefficients $a(x)$, $b(x)$, $c(x)$, $f(x)$ be continuous on an interval J containing $x = x_0$. Assume $a(x) \neq 0$ on J. Let g_1 and g_2 be constants. Then the initial value problem

$$a(x)y'' + b(x)y' + c(x)y = f(x), \quad y(x_0) = g_1, \quad y'(x_0) = g_2$$

has a unique solution $y(x)$ defined on J.

Proof on page 446.

Theorem 6.7 (Homogeneous Structure)
The homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ has a general solution of the form $y_h(x) = c_1y_1(x) + c_2y_2(x)$, where c_1, c_2 are arbitrary constants and $y_1(x)$, $y_2(x)$ are solutions.

Proof on page 447.

Theorem 6.8 (Non-Homogeneous Structure)
The non-homogeneous equation $a(x)y'' + b(x)y' + c(x)y = f(x)$ has general solution $y = y_h + y_p$, where $y_h(x)$ is the general solution of the homogeneous equation $a(x)y''+b(x)y'+c(x)y = 0$ and $y_p(x)$ is a particular solution of the non-homogeneous equation $a(x)y'' + b(x)y' + c(x)y = f(x)$.

Proof on page 447.

Theorem 6.9 (Reduction of Order)
Let $y_1(x)$ be a solution of $a(x)y'' + b(x)y' + c(x)y = 0$ on an interval J. Assume $a(x) \neq 0$, $y_1(x) \neq 0$ on J. Let all coefficients be continuous on J. Select x_0 in J. Then the general solution has the form $y_h(x) = c_1 y_1(x) + c_2 y_2(x)$ where c_1, c_2 are constants and

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{t} (b(r)/a(r))dr}}{y_1^2(t)} dt.$$

Proof on page 448.

Theorem 6.10 (Equilibrium Method)
A non-homogeneous equation

$$ay'' + by' + cy = f$$

has an easily-found particular solution $y_p(x)$ in the special case when all coefficients a, b, c, f are constant. The solution can be found by the *equilibrium method*. The answers:

$$\begin{array}{lll} c \neq 0 & y_p(x) & = \dfrac{f}{c}, \\ c = 0, b \neq 0 & y_p(x) & = \displaystyle\int \frac{f}{b} dx = \frac{f}{b} x, \\ c = b = 0, a \neq 0 & y_p(x) & = \displaystyle\int \left(\int \frac{f}{a} dt \right) dx = \frac{f}{a} \frac{x^2}{2}. \end{array}$$

See Example 6.11 page 445.

Equilibrium Method. The method applies to non-homogeneous equations with constant coefficients $ay'' + by' + cy = f$. The method:

Truncate the LHS of the differential equation to just the lowest order term, then solve the resulting equation by the method of quadrature.

Examples and Methods

Example 6.7 (Superposition)
Verify that $y = c_1 y_1 + c_2 y_2$ is a solution, given equation $y'' + 4y' + 4y = 0$ and solutions $y_1(x) = e^{2x}$, $y_2(x) = xe^{2x}$.

Solution: The answer check details can be simplified as follows.

LHS $= y'' + 4y' + 4y$	Given differential equation LHS.
LHS $= c_1y_1'' + c_2y_2'' + 4(c_1y_1' + c_2y_2') + 4(c_1y_1 + c_2y_2)$	Substitute $y = c_1y_1 + c_2y_2$.
LHS $= c_1(y_1'' + 4y_1' + 4y_1) + c_2(y_2'' + 4y_2' + 4y_2)$	Collect on c_1, c_2.
LHS $= c_1(0) + c_2(0)$	Because y_1, y_2 are solutions of the equation $y'' + 4y' + 4y = 0$.

Then $y = c_1y_1 + c_2y_2$ satisfies $y'' + 4y' + 4y = 0$, as claimed.

Example 6.8 (Continuous Coefficients)
Determine all intervals J of existence of $y(x)$, according to Picard's theorem, for the differential equation $y'' + \frac{1}{1+x}y' + \frac{x}{2+x}y = 0$.

Solution: The challenge is describe the open intervals J where $1+x \neq 0$ and $2+x \neq 0$, because the coefficients are continuous whenever both inequalities hold. The real line is divided by the exceptions $x = -1$, $x = -2$. Then $-\infty < x < -2$, $-2 < x < -1$, $-1 < x < \infty$ are the possible intervals J in Picard's theorem.

Example 6.9 (Recognizing y_h)
Consider $y'' + 4y = x$. Extract from the solution $y = 2\cos 2x + 3\sin 2x + x/4$ a particular solution y_p with fewest terms.

Solution: The homogeneous equation $y'' + 4y = 0$ has characteristic equation $r^2 + 4 = 0$ with complex roots $\pm 2i$ and Euler solution atoms $\cos 2x, \sin 2x$. Then $2\cos 2x + 3\sin 2x$ is a solution y_h of the homogeneous equation and $y = y_h + x/4$. Subtract the homogeneous solution to obtain a particular solution $x/4$. By Theorem 6.8, this is a particular solution y_p. It has the fewest possible terms.

Example 6.10 (Reduction of Order)
Given solution $y_1 = 1$, find an independent solution y_2 of $y'' + 4y' = 0$ by reduction of order.

Solution: The answer is $y_2 = \frac{1}{4}\left(1 - e^{-4x}\right)$. The method is Theorem 6.9.
We apply the theorem by inserting the formula $y_1 = 1$ into

$$y_2(x) = y_1(x) \int_{x_0}^{x} \frac{e^{-\int_{x_0}^{t}(b(r)/a(r))dr}}{y_1^2(t)} dt.$$

Then, using $x_0 = 0$, $a(x) = 1$, $b(x) = 4$, $c(x) = 0$ gives

$$\begin{aligned} y_2(x) &= (1) \int_0^x \frac{e^{-\int_0^t (4/1)dr}}{(1)^2} dt \\ &= (1) \int_0^x \frac{e^{-4t}}{(1)^2} dt \\ &= \frac{e^{-4x} - 1}{-4} \end{aligned}$$

Example 6.11 (Equilibrium Method)
Apply the equilibrium method to find y_p, then find the general solution $y = y_h + y_p$. This method works only for constant coefficients. meaning $a(x)$, $b(x)$, $c(x)$, $f(x)$ in equation (1) are constant.

(a) $y'' + 4y' + 4y = \pi$

(b) $2y'' + 3y' = -5$

(c) $3y'' = 20$

Solution: All equations have constant coefficients, therefore the method applies. The method selects a trial solution for y_p which makes all terms zero except the lowest derivative term. Then solve for the trial solution by quadrature to obtain y_p. The answer should be verified due to the possibility of integration and algebra errors.

(a) Truncate all but the lowest term to obtain $4y = \pi$, then $y_p(x) = \pi/4$. The homogeneous solution y_h is the solution of $y'' + 4y' + 4y = 0$ with characteristic equation $r^2 + 4r + 4 = 0$, factoring into $(r + 2)(r + 2) = 0$. Then the atoms are e^{-2x}, xe^{-2x} and $y_h(x) = c_1 e^{-2x} + c_2 xe^{-2x}$. The general solution is $y(x) = y_h(x) + y_p(x) = c_1 e^{-2x} + c_2 xe^{-2x} + \frac{\pi}{4}$.

(b) Truncate to $3y' = -5$ and integrate to obtain $y_p(x) = \frac{-5}{3}x$. The characteristic equation of $2y'' + 3y' = 0$ is $(2r + 3)r = 0$ with roots $r = 0, -3/2$. The atoms are e^{0x}, $e^{-3x/2}$ and then $y_h(x) = c_1 e^{0x} + c_2 e^{-3x/2}$. The general solution is $y(x) = y_h(x) + y_p(x) = c_1 + c_2 e^{-3x/2} + \frac{-5}{2}x$, because e^{0x} is written as 1.

(c) The quadrature solution is $y_p(x) = \frac{20}{3}\frac{x^2}{2}$. The characteristic equation for $3y'' = 0$ is $3r^2 = 0$ with double root $r = 0, 0$. The atoms are e^{0x}, xe^{0x} and the homogeneous solution is $y_h(x) = c_1 e^{0x} + c_2 xe^{0x} = c_1 + c_2 x$. Then the general solution is $y(x) = y_h(x) + y_p(x) = c_1 + c_2 x + \frac{20}{3}\frac{x^2}{2}$.

Example 6.12 (Equilibrium Method Failure)
The equation $y'' + y' = 2x$ fails to have constant coefficients, meaning $a(x)$, $b(x)$, $c(x)$, $f(x)$ are not all constant. Blind application of the equilibrium method gives $y = x^2$, **not** a solution. Explain.

Solution: The error: $y'' + y' = 2x$ does not have constant coefficients, which is required to apply the equilibrium method. **What went wrong**? The equilibrium method blindly applied gives the equation $0 + y' = 2x$, which by quadrature implies $y(x) = x^2$. It appears to work! Let's test $y = x^2$. Insert $y = x^2$ into $y'' + y' = 2x$, then $(x^2)'' + (x^2)' = 2x$, which implies $2 + 2x = 2x$ and finally the false equation $2 = 0$. Therefore, $y = x^2$ is not a solution of $y'' + y' = 2x$.

Proofs and Details

Proof of Theorem 6.5: The three terms of the differential equation, $c(x)y$, $b(x)y'$ and $a(x)y''$, are computed using the expression $y = c_1 y_1 + c_2 y_2$. The formulas are added to obtain the left hand side LHS of the differential equation:

LHS $= c_1[ay_1'' + by_1' + cy_1]$	Add terms $c(x)y$, $b(x)y'$, $a(x)y''$
$\quad + c_2[ay_2'' + by_2' + cy_2]$	and then collect on c_1, c_2.
$= c_1[0] + c_2[0]$	Both y_1, y_2 satisfy $ay'' + by' + cy = 0$.
$=$ RHS	The left and right sides match.

Proof of Theorem 6.6: The basic ideas for the proof appear already in the proof of the Picard-Lindelöf theorem, page **??**. Additional proof is required, because the solution is supposed to be defined on all of J, whereas the basic Picard-Lindelöf theorem supplies only *local existence*.

Existence. Picard's ideas write the solution $y(x)$ on J as the sum of an infinite series of continuous functions. This is accomplished by using the **Position-Velocity substitution** $x = t$, $X = y(t)$, $Y = y'(t)$ and definitions $t_0 = x_0$, $X_0 = g_1$, $Y_0 = g_2$ to re-write the differential equation and initial conditions in the new form

$$\begin{aligned} &X' = Y, \quad Y' = (f(t) - b(t)Y - c(t)X)/a(t), \\ &X(t_0) = X_0, \quad Y(t_0) = Y_0. \end{aligned}$$

The *Picard iterates* are defined by

$$\begin{aligned} X_n(t) &= \int_{t_0}^{t} Y_{n-1}(x)dx, \\ Y_n(t) &= \int_{t_0}^{t} (f(x) - b(x)Y_{n-1}(x) - c(x)X_{n-1}(x)) \frac{dx}{a(x)}. \end{aligned}$$

The new bit of information provided by these formulas is significant: because X_0 and Y_0 are defined everywhere on J, so also are X_n and Y_n. This explains why the series equality

$$y(x) = X_0 + \sum_{n=1}^{\infty} (X_n(x) - X_{n-1}(x))$$

provides a formula for $y(x)$ on *all of interval* J, instead of on just a local section of the interval.

The demand that the series converge on J creates new technical problems, to be solved by modifying Picard's proof. Suffice it to say that Picard's ideas are sufficient to give series convergence and hence existence of $y(x)$ on J.

Uniqueness. An independent proof of the uniqueness will be given, based upon calculus ideas only.

Let two solutions y_1 and y_2 of the differential equation be given, having the same initial conditions. Then their difference $y = y_1 - y_2$ satisfies the homogeneous differential equation $a(x)y'' + b(x)y' + c(x)y = 0$ and the initial conditions $y(x_0) = y'(x_0) = 0$. Some details:

LHS $= ay'' + by' + cy$	Left side of $ay'' + by' + cy = f$.
$= ay_1'' + by_1' + cy_1$	Substitute $y = y_1 - y_2$.
$\quad - (ay_2'' + by_2' + cy_2)$	
$= f(x) - f(x)$	Both y_1, y_2 satisfy $ay'' + by' + cy = f$.
$= 0$	The homogeneous equation is satisfied.

To prove $y_1 = y_2$, it suffices to show $y(x) \equiv 0$. This will be accomplished by showing that the non-negative function

$$z(t) = (y(t))^2 + (y'(t))^2$$

satisfies $z(t) \le 0$, which implies $z(t) \equiv 0$ and then $y(x) \equiv 0$. The argument depends upon the following inequality.

Lemma. The function $z(t)$ satisfies $|z'| \le Mz$ for some constant $M \ge 0$.

To finish the uniqueness proof, observe first that initial conditions $y(x_0) = y'(x_0) = 0$ imply $z(t_0) = 0$. By the lemma, $|z'| \le Mz$ for some constant M, or equivalently $-Mz \le z' \le Mz$. Multiply $z' \le Mz$ by the *integrating factor* e^{-Mt} to give $(e^{-Mt}z(t))' \le 0$. Integration over $[t_0, t]$ shows $e^{-Mt}z(t) \le 0$. Then $z(t) = 0$ for $t \ge t_0$. Similarly, $-z' \le Mz$ implies $z(t) = 0$ for $t \le t_0$. This concludes the uniqueness proof, except for the proof of the lemma.

Proof of the lemma. Compute the derivative z' as follows, using notation $X = y(t)$ and $Y = y'(t)$ to re-write $z(t) = (y(t))^2 + (y'(t))^2 = X^2 + Y^2$.

$z' = 2XX' + 2YY'$	Power and product rules.
$\quad = 2XY + 2Y(-cX - bY)/a$	Use $X' = Y$ and the homogeneous equation $aY' + bY + cX = 0$.
$\quad = (2 - 2c/a)XY + (-2b/a)Y^2$	Collect terms.

Let $M = 2\max_{A \le x \le B}\{|1 - c(x)/a(x)| + |-2b(x)/a(x)|\}$, where $[A, B]$ is an arbitrary subinterval of J containing x_0. The estimate $|z'| \le Mz$ will be established.

$\lvert z'\rvert = \lvert(2 - 2c/a)XY + (-2b/a)Y^2\rvert$	Estimate modulus of z'.
$\quad \le \lvert 1 - c/a\rvert \lvert 2XY\rvert + \lvert -2b/a\rvert \lvert Y\rvert^2$	Apply $\lvert c + d\rvert \le \lvert c\rvert + \lvert d\rvert$ and $\lvert uv\rvert = \lvert u\rvert\lvert v\rvert$.
$\quad \le (M/2)\lvert 2XY\rvert + (M/2)\lvert Y\rvert^2$	Definition of maximum M applied.
$\quad \le Mz$	Use $\lvert 2XY\rvert \le X^2 + Y^2$, proved from $(\lvert X\rvert - \lvert Y\rvert)^2 \ge 0$.

Proof of Theorem 6.7: To define y_1 and y_2 requires application of Picard's existence-uniqueness Theorem 6.6, page 442. Select them by their initial conditions, $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x_0) = 0$, $y_2'(x_0) = 1$.

To complete the proof, a given solution $y(x)$ must be expressed as a linear combination $y(x) = c_1y_1(x) + c_2y_2(x)$ for some values of c_1, c_2.

Define $c_1 = y(x_0)$, $c_2 = y'(x_0)$. Let $u(x) = y(x) - c_1y_1(x) - c_2y_2(x)$. The equation $y(x) = c_1y_1(x) + c_2y_2(x)$ will be verified by showing $u(x) \equiv 0$.

First, u is a solution of $a(x)y'' + b(x)y' + c(x)y = 0$, by the superposition principle, Theorem 6.5. It has initial conditions $u(x_0) = y(x_0) - c_1(1) - c_2(0) = 0$ and $u'(x_0) = y'(x_0) - c_1(0) - c_2(1) = 0$. By uniqueness of initial value problems, $u(x) \equiv 0$, which completes the proof.

Proof of Theorem 6.8: Let $y_p(x)$ be a given particular solution of $a(x)y'' + b(x)y' + c(x)y = f(x)$. Let $y(x)$ be any other solution of this equation and define $u(x) = y(x) - y_p(x)$. Subtract the two differential equations to verify that u is a solution of the homogeneous equation $a(x)u'' + b(x)u' + c(x)u = 0$. By Theorem 6.7, $u = y_h(x)$ for some choice of constants c_1, c_2. Then $y(x) = u(x) + y_p(x) = y_h(x) + y_p(x)$, as was to be shown, completing the proof.

Proof of Theorem 6.9: . Let $W(x) = e^{-\int_{x_0}^x (b(r)/a(r))dr}$. By the chain rule and the fundamental theorem of calculus, $W' = -bW/a$. Let $u(x) = 1/y_1^2(x)$ to simplify displays. The successive derivatives of y_2 are

$$\begin{aligned}
y_2(x) &= y_1(x)\int_{x_0}^x Wudt && \text{Definition of } y_2,\ u \text{ and } W.\\
y_2'(x) &= \left(y_1(x)\int_{x_0}^x Wudt\right)' && \text{Apply the product rule.}\\
&= y_1'(x)\int_{x_0}^x Wudt + y_1(x)W(x)u(x) && \text{Use } (\textstyle\int_{x_0}^x G(t)dt)' = G(x).\\
&= y_1'(x)\int_{x_0}^x Wudt + \frac{W(x)}{y_1(x)}\\
y_2''(x) &= \left(y_1'(x)\int_{x_0}^x Wudt + \frac{W(x)}{y_1(x)}\right)'\\
&= y_1''(x)\int_{x_0}^x Wudt + y_1'(x)W(x)u(x) && \text{Apply the sum and quotient rules.}\\
&\quad + \frac{W'(x)y_1(x) - W(x)y_1'(x)}{y_1^2(x)}\\
&= y_1''(x)\int_{x_0}^x Wudt + \frac{W'(x)}{y_1(x)} && \text{Simplify non-integral terms.}\\
&= y_1''(x)\int_{x_0}^x Wudt - \frac{b(x)W(x)}{a(x)y_1(x)} && \text{Use } W' = -(b/a)W.
\end{aligned}$$

The derivative formulas are multiplied respectively by c, b and a to obtain an expression $\mathcal{E} = ay_2'' + by_2' + cy_2$, which must be shown to be zero. The details:

$$\begin{aligned}
\mathcal{E} &= cy_2 + by_2' + ay_2''\\
&= c\left(y_1\int Wu\right) + b\left(y_1'\int Wu + W/y_1\right)\\
&\qquad + a\left(y_1''\int Wu - bW/(ay_1)\right)\\
&= (cy_1 + by_1' + ay_1'')\int Wu && \text{Collect all integral terms.}\\
&\quad + bW/y_1 - bW/y_1\\
&= 0 && \text{Because } ay_1'' + by_1' + cy_1 = 0.
\end{aligned}$$

General Solution. To show that $c_1y_1 + c_2y_2$ is the general solution, for this choice of y_1, y_2, let $y(x)$ be a solution of the homogeneous equation and define

$$c_1 = \frac{y(x_0)}{y_1(x_0)}, \quad c_2 = y_1(x_0)(y'(x_0) - c_1y_1'(x_0)).$$

It will be shown that $y(x) = c_1y_1(x) + c_2y_2(x)$ by verifying that $u(x) = y(x) - c_1y_1(x) - c_2y_2(x)$ is zero. Superposition implies u is a solution of the homogeneous equation. It has initial conditions $u(x_0) = 0$, $u'(x_0) = 0$, because $y_2'(x_0) = 1/y_1(x_0)$. Uniqueness of initial value problems implies $u(x) \equiv 0$, completing the proof.

Proof of Theorem 6.10, Equilibrium Method: In the case $c \neq 0$, find an equilibrium solution y = constant by substitution of $y = k$ into the differential equation (the *equilibrium method*). Then $ck = f$ and $y_p(x) = \frac{f}{c}$.

For case $c = 0$, $b \neq 0$, observe that the differential equation in terms of the velocity $v = y'$ is $av' + bv = f$. Apply the equilibrium method to this equation to obtain $v = f/b$ and finally $y = \int vdx = \frac{f}{b}x$.

For the last case $b = c = 0$ and $a \neq 0$, then the equation is in terms of the acceleration $p = y''$ the new equation $ap = f$. Then $p = f/a$ is the quadrature equation $y'' = f/a$ with solution $y_p(x) = \frac{f}{a}\frac{x^2}{2}$.

Exercises 6.2

Continuous Coefficients

Determine all intervals J of existence of $y(x)$, according to Picard's theorem.

1. $y'' + y = \ln|x|$

2. $y'' = \ln|x - 1|$

3. $y'' + (1/x)y = 0$

4. $y'' + \frac{1}{1+x}y' + \frac{1}{x}y = 0$

5. $x^2y'' + y = \sin x$

6. $x^2y'' + xy' = 0$

Superposition

Verify that $y = c_1y_1 + c_2y_2$ is a solution.

7. $y'' = 0$, $y_1(x) = 1$, $y_2(x) = x$

8. $y'' = 0$, $y_1(x) = 1 + x$, $y_2(x) = 1 - x$

9. $y''' = 0$, $y_1(x) = x$, $y_2(x) = x^2$

10. $y''' = 0$, $y_1(x) = 1 + x$, $y_2(x) = x + x^2$

Structure

Verify that $y = y_h + y_p$ is a solution.

11. $y'' + y = 2$, $y_h(x) = c_1 \cos x + c_2 \sin x$, $y_p(x) = 2$

12. $y'' + 4y = 4$, $y_h(x) = c_1 \cos 2x + c_2 \sin 2x$, $y_p(x) = 1$

13. $y'' + y' = 5$, $y_h(x) = c_1 + c_2e^{-x}$, $y_p(x) = 5x$

14. $y'' + 3y' = 5$, $y_h(x) = c_1 + c_2e^{-3x}$, $y_p(x) = 5x/3$

15. $y'' + y' = 2x$, $y_h(x) = c_1 + c_2e^{-x}$, $y_p(x) = x^2 - 2x$

16. $y'' + 2y' = 4x$, $y_h(x) = c_1 + c_2e^{-2x}$, $y_p(x) = x^2 - x$

Initial Value Problems

Solve for constants c_1, c_2 in the general solution $y_h = c_1y_1 + c_2y_2$.

17. $y'' = 0$, $y_1 = 1$, $y_2 = x$, $y(0) = 1$, $y'(0) = 2$

18. $y'' = 0$, $y_1 = 1+x$, $y_2 = 1-x$, $y(0) = 1$, $y'(0) = 2$

19. $y'' + y = 0$, $y_1 = \cos x$, $y_2 = \sin x$, $y(0) = 1$, $y'(0) = -1$

20. $y'' + y = 0$, $y_1 = \sin x$, $y_2 = \cos x$, $y(0) = 1$, $y'(0) = -1$

21. $y'' + 4y = 0$, $y_1 = \cos 2x$, $y_2 = \sin 2x$, $y(0) = 1$, $y'(0) = -1$

22. $y'' + 4y = 0$, $y_1 = \sin 2x$, $y_2 = \cos 2x$, $y(0) = 1$, $y'(0) = -1$

23. $y''+y' = 0$, $y_1 = 1$, $y_2 = e^{-x}$, $y(0) = 1$, $y'(0) = -1$

24. $y''+y' = 0$, $y_1 = 1$, $y_2 = e^{-x}$, $y(0) = 2$, $y'(0) = -3$

25. $y'' + 3y' = 0$, $y_1 = 1$, $y_2 = e^{-3x}$, $y(0) = 1$, $y'(0) = -1$

26. $y'' + 5y' = 0$, $y_1 = 1$, $y_2 = e^{-5x}$, $y(0) = 1$, $y'(0) = -1$

Recognizing y_h

Extract from the given solution y a particular solution y_p with fewest terms.

27. $y'' + y = x$,
$y = c_1 \cos x + c_2 \sin x + x$

28. $y'' + y = x$,
$y = \cos x + x$

29. $y'' + y' = x$,
$y = c_1 + c_2e^{-x} + x^2/2 - x$

30. $y'' + y' = x$,
$y = e^{-x} - x + 1 + x^2/2$

31. $y'' + 2y' + y = 1 + x$,
$y = (c_1 + c_2x)e^{-x} + x - 1$

32. $y'' + 2y' + y = 1 + x$,
$y = e^{-x} + x + xe^{-x} - 1$

Reduction of Order

Given solution y_1, find an independent solution y_2 by reduction of order.

33. $y'' + 2y' = 0$, $y_1(x) = 1$

34. $y'' + 2y' = 0$, $y_1(x) = e^{-2x}$

35. $2y'' + 3y' + y = 0$, $y_1(x) = e^{-x}$

36. $2y'' - y' - y = 0$, $y_1(x) = e^{x}$

Equilibrium Method

Apply the equilibrium method to find y_p, then find the general solution $y = y_h + y_p$.

37. $2y'' = 3$

38. $y'' + 4y' = 5$

39. $y'' + 3y' + 2y = 3$

40. $y'' - y' - 2y = 2$

41. $y'' + y = 1$

42. $3y'' + y' + y = 7$

43. $6y'' + 7y' + 2y = 5$

44. $y'' - 2y' + 4y = 8$

45. $4y'' - 4y' + y = 8$

46. $4y'' - 12y' + 9y = 18$

6.3 Higher Order Linear Constant-Coefficient Equations

Discussed here are structure results for the n-th order linear differential equation

$$a_n y^{(n)} + \cdots + a_0 y = f(x).$$

It is assumed that each coefficient is **constant** and the leading coefficient a_n is **not zero**. The **forcing term** or **input** $f(x)$ is assumed to either be zero, in which case the equation is called **homogeneous**, or else $f(x)$ is nonzero and continuous, and then the equation is called **non-homogeneous**. The **characteristic equation** is

$$a_n r^n + \cdots + a_0 = 0.$$

It is obtained from Euler's substitution $y = e^{rx}$ or by the shortcut substitutions $y^{(k)} \to r^k$. The left side of the characteristic equation is called the **characteristic polynomial**.

Picard-Lindelöf Theorem

The foundation of the theory of linear constant coefficient differential equations is the existence-uniqueness result of Picard-Lindelöf, which says that, given constants $g_1, \ldots, g_n$, the initial value problem

$$\begin{aligned} &a_n y^{(n)} + \cdots + a_0 y = f(x), \\ &y(0) = g_1, \ldots, y^{n-1}(0) = g_n, \end{aligned}$$

has a unique solution $y(x)$ defined on each open interval for which $f(x)$ is defined and continuous.

General Solution

A linear homogeneous constant coefficient differential equation has a general solution $y_h(x)$ written in terms of n arbitrary constants $c_1, \ldots, c_n$ and n solutions $y_1(x), \ldots, y_n(x)$ as the linear combination

$$y_h(x) = c_1 y_1(x) + \cdots + c_n y_n(x).$$

Discussed here is one way to define the solutions $y_1, \ldots, y_n$.

Consider the case of $n = 2$, already discussed. The Picard-Lindelöf theorem applies with initial values $y(0) = 1$, $y'(0) = 0$ to define solution $y_1(x)$. The initial values are changed to $y(0) = 0$, $y'(0) = 1$, then Picard-Lindelöf applies again to define solution $y_2(x)$. Solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$ satisfies initial conditions $y(0) = g_1$, $y'(0) = g_2$ when $c_1 = g_1, c_2 = g_2$.

In the $n = 2$ case, solutions y_1, y_2 are defined using initial conditions which form the columns of the 2×2 identity matrix. In a similar way, for general n, solutions $y_1(x), \ldots, y_n(x)$ are defined by applying the Picard-Lindeöf theorem, with initial conditions $g_1, \ldots, g_n$ successively taken as the columns of the $n \times n$ identity matrix.

The expression $y_h(x)$ is called a general solution, because any solution of the differential equation is equal to $y_h(x)$ for a unique specialization of the constants $c_1, \ldots, c_n$.

Solution Structure

An **Euler base atom** is one of the functions

$$e^{ax}, e^{ax} \cos bx, e^{ax} \sin bx$$

where a and b are real numbers, $b > 0$.

An **Euler solution atom** is a power x^n times a base atom, where $n \geq 0$ is an integer.

Complex Numbers and Atoms. An Euler solution atom can alternatively be defined as the nonzero real or imaginary part of $x^n e^{rx}$ where $r = a + ib$ with symbols a and $b \geq 0$ are real and $n \geq 0$ is an integer, provided minus signs are stripped off, leaving coefficient 1. Euler's formula

$$e^{i\theta} = \cos\theta + i \sin\theta$$

facilitates taking real and imaginary parts of the complex exponential term $x^n e^{rx}$. For instance,

$$x^7 e^{(2+3i)x} = x^7 e^{2x} \cos 3x + i x^7 e^{2x} \sin 3x$$

has real and imaginary parts $x^7 e^{2x} \cos 3x$, $x^7 e^{2x} \sin 3x$, which are themselves atoms.

A complete list of all possible atoms appears in the rightmost section of the table below, in which a, b are real, $b > 0$ and $n \geq 0$ is an integer.

$r = 0$	$1,$	$x,$	$x^2,$	$\ldots,$	$x^n,$	$\ldots$
$r = a$	$e^{ax},$	$xe^{ax},$	$x^2 e^{ax},$	$\ldots,$	$x^n e^{ax},$	$\ldots$
$r = ib$	$\cos bx,$	$x \cos bx,$	$x^2 \cos bx,$	$\ldots,$	$x^n \cos bx,$	$\ldots$
$r = ib$	$\sin bx,$	$x \sin bx,$	$x^2 \sin bx,$	$\ldots,$	$x^n \sin bx,$	$\ldots$
$r = a + ib$	$e^{ax} \cos bx,$	$xe^{ax} \cos bx,$	$x^2 e^{ax} \cos bx,$	$\ldots,$	$x^n e^{ax} \cos bx,$	$\ldots$
$r = a + ib$	$e^{ax} \sin bx,$	$xe^{ax} \sin bx,$	$x^2 e^{ax} \sin bx,$	$\ldots,$	$x^n e^{ax} \sin bx,$	$\ldots$

The table only uses $b > 0$, because Euler atoms must have coefficient 1. For instance, $xe^{(1-2i)x} = xe^x \cos 2x - ixe^x \sin 2x$ does not have atoms for real and imaginary parts (coefficient -1 is the problem). Yes, stripping the minus sign gives $xe^x \sin 2x$, which is an atom (coefficient 1).

Detecting Euler Solution Atoms

A term that makes up an atom has coefficient 1, therefore 2 and $2e^x$ are not atoms, but the 2 can be stripped off to expose atoms 1 and e^x. Combinations like $2x + 3x^2$ are not atoms, but individual stripped terms x and x^2 are atoms. Terms like e^{x^2}, $\ln|x|$ and $x/(1+x^2)$ are not atoms, nor are they sums of constants times atoms. The expressions $\cosh x$, $\sinh x$ and $\sin^4 x$ are not atoms, but they are combinations of atoms. Fractional powers may not appear in atoms, for instance, neither x^π nor $x^{5/2}\sin x$ is an atom.

Linear Algebra Background

Borrowed from the subject of linear algebra is the terminology **linear combination**, which in the case of two functions f_1, f_2 is the expression $f = c_1 f_1 + c_2 f_2$. More generally, given functions $f_1, \ldots, f_k$, and constants $c_1, \ldots, c_k$, the expression $f = c_1 f_1 + \cdots + c_k f_k$ is called a linear combination of the functions $f_1, \ldots, f_k$.

A function list $f_1, \ldots, f_k$ is called **linearly independent** provided every linear combination is uniquely represented by the constants $c_1, \ldots, c_k$.

Independence is tested by solving for constants $c_1, \ldots, c_k$ in the equation $c_1 f_1(x) + \cdots + c_k f_k(x) = 0$, assumed satisfied for all x in a common domain of $f_1, \ldots, f_k$. Independence holds if and only if the constants are all zero.

Theorem 6.11 (Independence and Euler Solution Atoms)
A list of finitely many distinct Euler solution atoms is linearly independent.

Outline of the proof on page **??**.

Because subsets of independent sets are independent, then list x^2, x^5, x^8 is independent by virtue of independence of the powers $1, x, \ldots, x^n$.

Solution methods for linear constant differential equations implicitly use Theorem 6.11.

Fundamental Results

Theorem 6.12 (Homogeneous Solution y_h and Atoms)
Linear homogeneous differential equations with constant coefficients have general solution $y_h(x)$ equal to a linear combination of Euler atoms.

Theorem 6.13 (Particular Solution y_p and Atoms)
Linear non-homogeneous differential equations with constant coefficients having forcing term $f(x)$ equal to a linear combination of atoms have a particular solution $y_p(x)$ which is a linear combination of Euler atoms.

Theorem 6.14 (General Solution y and Atoms)
Linear non-homogeneous differential equations with constant coefficients having forcing term $f(x) =$ a linear combination of Euler atoms have general solution

$$y(x) = y_h(x) + y_p(x) = \text{ a linear combination of Euler atoms.}$$

The first result, for the special case of second order differential equations, can be justified from Theorem 6.1, page 430. The solutions $e^{r_1 x}$, $e^{r_2 x}$, $xe^{r_1 x}$, $e^{ax}\cos bx$ and $e^{ax}\sin bx$ in the theorem are Euler atoms.

The third theorem easily follows from the first two. The first and second theorems follow directly from Euler's Theorem 6.15 and the method of undetermined coefficients, *infra*.

How to Solve Equations of Order n

Picard's existence–uniqueness theorem says that $y''' + 2y'' + y = 0$ has general solution y constructed from linear combinations of 3 independent solutions of this differential equation. The general solution of an n-th order linear differential equation is constructed from linear combinations of n independent solutions of the equation.

Linear algebra defines the dimension of the solution set to be this same fixed number n. Once n independent solutions are found for the differential equation, the search for the general solution has ended: the general solution y must be a linear combination of these n independent solutions.

Because of the preceding structure theorems, we have reduced the search for the general solution to the following:

> Find n distinct Euler solution atoms of the nth order differential equation.

Euler's basic result tells us how to find the list of distinct atoms.

Theorem 6.15 (Euler's Theorem)
Assume r_0 is a real or complex root of the characteristic equation. If complex, write $r_0 = a + ib$ with a, b real.

(a) The functions $e^{r_0 x}$, $xe^{r_0 x}$, $\ldots$, $x^k e^{r_0 x}$ are solutions of a linear homogeneous constant–coefficient differential equation if and only if $(r - r_0)^{k+1}$ is a factor of the characteristic polynomial.

(b) Assume $b > 0$. Functions $e^{ax}\cos bx$, $xe^{ax}\cos bx$, $\ldots$, $x^k e^{ax}\cos bx$, $e^{ax}\sin bx$, $xe^{ax}\sin bx$, $\ldots$, $x^k e^{ax}\sin bx$ (a, b real, $b > 0$) are solutions of a linear homogeneous constant–coefficient differential equation if and only if $((r-a)^2 + b^2)^{k+1}$ is a factor of the characteristic polynomial.

Proof on page 459.

Theorem 6.16 (Real and Complex Solutions)
Let $y(x) = u(x) + iv(x)$ be a solution of a linear constant-coefficient differential equation (a_0,...,a_n assumed real), with $u(x)$ and $v(x)$ both real. Then $u(x)$ and $v(x)$ are both real solutions of the differential equation. Briefly stated, the real and imaginary parts of a solution are also solutions.

Proof on page 460

Root Multiplicity

A polynomial equation $p(r) = 0$ is defined in college algebra to have a root $r = r_0$ of multiplicity m provided $(r - r_0)^m$ divides $p(r)$ but $(r - r_0)^{m+1}$ does not. For instance, $(r - 1)^3(r + 2)(r^2 + 4)^2 = 0$ has roots 1, -2, $2i$, $-2i$ of multiplicity 3, 1, 2, 2, respectively.

Atom Lists

Let $r = r_0$ be a real root of the characteristic equation $p(r) = 0$, of multiplicity $k + 1$. Then Euler's theorem finds a base atom solution $e^{r_0 x}$. A total of $k + 1$ solutions are obtained from this base atom by **multiplying the base atom** by the powers 1, x, ..., x^k:

$$e^{r_0 x}, \quad xe^{r_0 x}, \quad \ldots, \quad x^k e^{r_0 x}.$$

A special case occurs when $r_0 = 0$. Then $e^{0x} = 1$ is the base atom and the $k + 1$ solution atoms are the powers

$$1, \quad x, \quad \ldots, \quad x^k.$$

The number of Euler solution atoms expected for a given root $r = r_0$ equals the multiplicity of the root r_0.

Let $r = a + ib$ be a complex root of the characteristic equation $p(r) = 0$, of multiplicity $k + 1$. Euler's Theorem implies that e^{ax+ibx} is a solution, and the theorem on complex solutions implies that the differential equation has two base solution atoms

$$e^{ax}\cos bx, \quad e^{ax}\sin bx.$$

Euler's Theorem implies that we should multiply these base atoms by powers 1, x, ..., x^k to obtain $k + 1$ solution atoms for each of the base atoms, giving the atom list for a complex root

$$\begin{array}{llll} e^{ax}\cos(bx), & xe^{ax}\cos(bx), & \ldots, & x^k e^{ax}\cos(bx), \\ e^{ax}\sin(bx), & xe^{ax}\sin(bx), & \ldots, & x^k e^{ax}\sin(bx). \end{array}$$

A special case occurs when $a = 0$. Then the base atoms are pure harmonics $\cos bx$, $\sin bx$ and the list has no visible exponentials:

$$\begin{array}{llll} \cos(bx), & x\cos(bx), & \ldots, & x^k\cos(bx), \\ \sin(bx), & x\sin(bx), & \ldots, & x^k\sin(bx). \end{array}$$

Shortcut Explained. A remaining mystery is the skipped complex root $a - ib$. We explain why we focused on $a + ib$ with $b > 0$ and ignored its conjugate $a - ib$. Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ using $\theta = rx = ax + ibx$ implies

$$x^j e^{rx} = \left(x^j e^{ax}\cos(bx)\right) + i\left(x^j e^{ax}\sin(bx)\right).$$

The real and imaginary parts of this complex linear combination are Euler atoms. If r is replaced by its complex conjugate $\overline{r} = a - ib$, then the same two atoms are distilled from the linear combination. Picard's Theorem dictates that we find $2k+2$ atoms from the pair of roots $a \pm ib$. Because the process above finds $2k+2$ atoms, the second conjugate root is ignored, as a *shortcut*.

Examples and Methods

Example 6.13 (First Order)
Solve $2y' + 5y = 0$, showing $y_h = c_1 e^{-5x/2}$.

Solution: Euler's Theorem 6.15 will be applied. The characteristic equation is $2r+5 = 0$ with real root $r = -5/2$. The corresponding atom e^{rx} is given explicitly by $e^{-5x/2}$. Because the order of the differential equation is 1, then all atoms have been found. Write the general solution y_h by multiplying the atom list by constant c_1, then $y_h = c_1 e^{-5x/2}$.

Example 6.14 (Second Order Distinct Real Roots)
Solve $y'' + 3y' + 2y = 0$, showing $y_h = c_1 e^{-x} + c_2 e^{-2x}$.

Solution: The factored characteristic equation is $(r + 1)(r + 2) = 0$. The distinct real roots are $r_1 = -1$, $r_2 = -2$. Euler's Theorem 6.15 applies to find the atom list e^{-x}, e^{-2x}. All atoms have been found, because the order of the differential equation is 2. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, then $y_h = c_1 e^{-x} + c_2 e^{-2x}$.

Example 6.15 (Second Order Double Real Root)
Solve $y'' + 2y' + y = 0$, showing $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

Solution: The factored characteristic equation is $(r+1)(r+1) = 0$, with double real root $r = -1, -1$. The root multiplicity is 2, so we must find two atoms for the root $r = -1$. Euler's Theorem 6.15 applies to find a base atom e^{-x}. Multiply the base atom by 1, x to find two solution atoms e^{-x}, xe^{-x}. Because the order of the differential equation is 2, then all atoms have been found. Write the general solution y_h by multiplying the atom list by constants c_1, c_2, then $y_h = c_1 e^{-x} + c_2 x e^{-x}$.

Example 6.16 (Second Order Complex Conjugate Roots)
Solve the differential equation $y'' + 2y' + 5y = 0$, verifying the equation $y_h = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x$.

Solution: The characteristic equation $r^2 + 2r + 5 = 0$ factors into $(r+1)^2 + 4 = 0$, therefore it has complex conjugate roots $r_1 = -1 + 2i$, $r_2 = -1 - 2i$. There are two methods for finding the atoms associated with these roots. We discuss both possibilities.

Method 1. The first statement in Euler's Theorem 6.15 applies to report two complex solutions e^{-x+2xi}, e^{-x-2xi}. These solutions are not atoms, but linear combinations of atoms, from which a list of two atoms is determined. The atoms are $e^{-x}\cos 2x$, $e^{-x}\sin 2x$. This process uses the two identities

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad e^{-i\theta} = \cos\theta - i\sin\theta.$$

Write

$$\begin{aligned} e^{-x+2xi} &= \left(e^{-x}\cos 2x\right) + i\left(e^{-x}\sin 2x\right), \\ e^{-x-2xi} &= \left(e^{-x}\cos 2x\right) - i\left(e^{-x}\sin 2x\right), \end{aligned}$$

then extract the two distinct atoms that appear in these two linear combinations:

$$e^{-x}\cos 2x, \quad e^{-x}\sin 2x.$$

Method 2. The second statement in Euler's Theorem 6.15 is more efficient. Characteristic equation root $r = -1 + 2i$ was found from the factorization $(r+1)^2 + 4 = 0$, which by Euler's theorem implies there are two distinct solution atoms

$$e^{-x}\cos 2x, \quad e^{-x}\sin 2x.$$

General Solution. Because the order of the differential equation is 2, then all atoms have been found. Write the general solution y_h by multiplying the atom list by constants c_1, c_2, then $y_h = c_1 e^{-x}\cos 2x + c_2 e^{-x}\sin 2x$.

The example uses a **shortcut**. Euler's theorem applied to the second conjugate root $-1 - 2i$ will produce no new atoms. The step of finding the distinct atoms can be shortened by observing that the outcome is exactly the real and imaginary parts of the first complex exponential e^{ax+ibx} with $b > 0$. The preferred method for finding the atoms is to use the second statement in Euler's theorem.

Example 6.17 (Third Order Distinct Roots)
Solve $y''' - y' = 0$, showing $y_h = c_1 + c_2 e^x + c_3 e^{-x}$.

Solution: The factored characteristic equation is $r(r-1)(r+1) = 0$ with real roots $r_1 = 0$, $r_2 = 1$, $r_3 = -1$. Euler's Theorem 6.15 applies to report the atom list e^{0x}, e^x, e^{-x}. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3, giving $y_h = c_1 e^{0x} + c_2 e^x + c_3 e^{-x}$. Convention replaces e^{0x} by 1 in the final equation.

Example 6.18 (Third Order with One Double Root)
Solve $y''' - y'' = 0$, verifying that $y_h = c_1 + c_2 x + c_3 e^x$.

Solution: The characteristic equation is $r^3 - r^2 = 0$. It factors into $r^2(r-1) = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$. Euler's Theorem 6.15 applies to find the base atom list e^{0x}, e^x. Because root $r = 0$ has multiplicity 2, we must multiply base atom e^{0x} by 1 and x to find the required 2 atoms e^{0x}, xe^{0x}. Then the completed list of 3 atoms is 1, x, e^x. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3 to give $y_h = c_1 + c_2 x + c_3 e^x$.

Example 6.19 (Fourth Order)
Solve $y^{iv} - y'' = 0$, showing $y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Solution: Notation: Define $y^{iv} = \dfrac{d^4y}{dx^4}$, the fourth derivative of y. The factored characteristic equation is $r^2(r-1)(r+1) = 0$ with real roots $r_1 = 0$, $r_2 = 0$, $r_3 = 1$, $r_4 = -1$. Euler's Theorem 6.15 applies to obtain the base atom list e^{0x}, e^x, e^{-x}. The first base atom e^{0x} comes from root $r = 0$, which has multiplicity 2. Euler's Theorem requires that this base atom be multiplied by 1, x. The atom list of 4 atoms is then 1, x, e^x, e^{-x}. All atoms have been found, because the order of the differential equation is 4. The general solution y_h is written by multiplying the atom list by constants c_1, c_2, c_3, c_4 to obtain the general solution $y_h = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$.

Example 6.20 (Tenth Order)
A linear homogeneous constant coefficient differential equation has characteristic equation

$$r^2(r-1)^2(r^2-1)(r^2+1)^2 = 0.$$

Solve the differential equation, showing that

$$\begin{aligned} y_h &= c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x + c_6 e^{-x} \\ &\quad + c_7 \cos x + c_8 x \cos x + c_9 \sin x + c_{10} x \sin x. \end{aligned}$$

Solution: The factored form of the characteristic equation is

$$r^3(r-1)^2(r-1)(r+1)(r-i)^2(r+i)^2 = 0.$$

The roots, listed according to multiplicity, make the *list of roots*

$$L = \{0, 0, \quad 1, 1, 1, \quad -1, \quad i, i, \quad -i, -i\}.$$

There are two methods for finding the atoms from list L.

Method 1. The first statement in Euler's theorem gives the exponential-type solutions

$$e^{0x}, xe^{0x}, e^x, xe^x, x^2e^x, e^{-x}, e^{ix}, xe^{ix}, e^{-ix}, xe^{-ix}.$$

The first six in the list are atoms, but the last four are not. Because $e^{ix} = \cos x + i \sin x$, we can distill from the complex exponentials the additional four atoms $\cos x$, $x \cos x$, $\sin x$, $x \sin x$. Then the *list of* 10 *distinct atoms* is

$$1, x, e^x, xe^x, x^2e^x, e^{-x}, \cos x, x\cos x, \sin x, x \sin x.$$

Method 2. The above list can be obtained directly from the second statement in Euler's theorem. The real exponential atoms are obtained from the first statement in Euler's theorem:

$$1, x, e^x, xe^x, x^2e^x, e^{-x}.$$

The second statement of Euler's theorem applies to the complex factor $(r^2+1)^2$ to obtain the trigonometric atoms

$$\cos x, x \cos x, \sin x, x \sin x.$$

General Solution. Then y_h is a linear combination of the 10 atoms:

$$\begin{aligned} y_h &= c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x + c_6 e^{-x} \\ &\quad + c_7 \cos x + c_8 x \cos x + c_9 \sin x + c_{10} x \sin x. \end{aligned}$$

Example 6.21 (Differential Equation from General Solution)
A linear homogeneous constant coefficient differential equation has general solution

$$y_h = c_1 + c_2 x + c_3 e^x + c_4 x e^x + c_5 x^2 e^x + c_6 \cos x + c_7 \sin x.$$

Find the differential equation.

Solution: Take the partial derivative of y_h with respect to the symbols $c_1, \ldots, c_7$ to give the atom list

$$1, x, \quad e^x, xe^x, x^2 e^x, \quad \cos x, \sin x.$$

This atom list is constructed from exponential solutions obtained from Euler's theorem, applied to the root list

$$0, 0, \quad 1, 1, 1, \quad i, -i.$$

There are 7 roots, hence by the root-factor theorem of college algebra the characteristic polynomial has individual factors r, r, $r-1$, $r-1$, $r-1$, $r-i$, $r+i$. Then the differential equation is of order 7 with characteristic polynomial

$$\begin{aligned} p(r) &= (r-0)^2(r-1)^3(r-i)(r+i) \\ &= r^6 - 2r^5 + 2r^4 - 2r^3 + r^2. \end{aligned}$$

The differential equation is obtained by the translation $r^j \to y^{(j)}$:

$$y^{(6)} - 2y^{(5)} + 2y^{(4)} - 2y''' + y'' = 0.$$

Proofs and Details

Proof of Euler's Theorem 6.15: The first statement will be proved for $n = 2$. The details for the general case are left as an exercise.

Let $y = e^{rx}$. Then

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}.$$

Substitute into the differential equation to obtain the following.

$$\begin{aligned} &a_2 y'' + a_1 y' + a_0 y = 0 \\ &a_2 r^2 e^{rx} + a_1 r e^{rx} + a_0 e^{rx} = 0 \\ &\left(a_2 r^2 + a_1 r + a_0\right) e^{rx} = 0 \end{aligned}$$

Then $y = e^{rx}$ is a solution if and only if $a_2 r^2 + a_1 r + a_0 = 0$, that is, the characteristic equation is satisfied.

To prove the second statement, assume a differential equation of order n

$$a_n y^{(n)} + \cdots + a_0 y = 0.$$

Perform a change of variables $y = e^{cx} z$, which changes dependent variable y into z. If y is a solution, then

$$y = e^{cx} z, \quad y' = ce^{cx} z + e^{cx} z', \quad y'' = c^2 e^{cx} z + 2ce^{cx} z' + e^{cx} z'', \cdots$$

Because each derivative of y is a multiple of e^{cx}, then, after substitution of the relations into the differential equation, the common factor e^{cx} cancels, giving a new constant coefficient differential equation for z.

To illustrate, in the case $n = 2$, the new differential equation for z is

$$a_2 z'' + (2a_2 c + a_1)z' + (a_2 c^2 + a_1 c + a_0)z = 0.$$

The coefficients of the z-equation are the Taylor series coefficients $\dfrac{p^k(0)}{k!}$ of the characteristic polynomial $p(r) = a_2 r^2 + a_1 r + a_0$:

$$\begin{aligned} a_2 &= \frac{p''(c)}{2!}, \\ (2a_2 c + a_1) &= \frac{p'(c)}{1!}, \\ (a_2 c^2 + a_1 c + a_0) &= \frac{p(c)}{0!}. \end{aligned}$$

By induction, the change of variables $y = e^{cx}z$ produces from $a_n y^{(n)} + \cdots + a_0 y = 0$ a new constant-coefficient differential equation $b_n z^{(n)} + \cdots + b_0 z = 0$ whose coefficients are given by

$$b_k = \frac{p^k(c)}{k!}.$$

Assume now characteristic polynomial $p(r) = a_n r^n + \cdots + a_0$ and let $r = c$ be a root of $p(r) = 0$ of algebraic multiplicity $k+1$. Then $p(c) = p'(c) = \cdots = p^{(k)}(c) = 0$. This means that $b_0 = \cdots = b_k = 0$. Therefore, the z-equation is a differential equation in the variable $v = z^{(k+1)}$. Because the selections $z = 1, x, \ldots, x^k$ all imply $v = 0$, then the polynomials 1, x, …, x^k are solutions of the z-equation. Hence, $y = e^{cx}z$ implies e^{cx}, xe^{cx}, …, $x^k e^{cx}$ are solutions of the y-equation.

Conversely, assume that e^{cx}, xe^{cx}, …, $x^k e^{cx}$ are solutions of the y-equation. We will verify that $r = c$ is a root of $p(r) = 0$ of algebraic multiplicity $k+1$. First, 1, …, x^k are solutions of the z-equation. Setting $z = 1$ implies $b_0 = 0$ Then setting $z = x$ implies $b_1 = 0$ (because $b_0 = 0$ already). Proceeding in this way, $b_0 = \cdots = b_k = 0$. Therefore, the characteristic polynomial of the z-equation is

$$q(r) = b_n r^n + \cdots + b_{k+1} r^{k+1}.$$

The reader can prove the following useful result; see the exercises.

Lemma 6.1 (Kümmer's Lemma) Under the change of variables $y = e^{cx}z$, the characteristic polynomials $p(r)$, $q(r)$ of the y-equation and the z-equation, respectively, satisfy the relation $q(r) = p(r + c)$.

Assuming Kümmer's Lemma, we can complete the proof. Already, we know that r^{k+1} divides $q(r)$. Then r^{k+1} divides $p(r + c)$, or equivalently, $(r - c)^{k+1}$ divides $p(r)$. This implies $r = c$ is a root of $p(r) = 0$ of algebraic multiplicity $k+1$. ■

Proof of Theorem 6.16: Substitute $y = u + iv$ into the differential equation and separate terms as follows:

$$(a_n u^{(n)} + \cdots + a_0 u) + i(a_n v^{(n)} + \cdots + a_0 v) = 0.$$

For each x, the left side of the preceding relation is a complex number $a + ib$ with a, b real. The right side is $0 + 0i$. By equality of complex numbers, $a = 0$ and $b = 0$, which implies

$$a_n u^{(n)} + \cdots + a_0 u = 0,$$
$$a_n v^{(n)} + \cdots + a_0 v = 0.$$

Therefore, u and v are real solutions of the differential equation. ■

Exercises 6.3

Constant Coefficients

Solve for $y(x)$. Proceed as in Examples 6.13–6.20.

1. $3y' - 2y = 0$

2. $2y' + 7y = 0$

3. $y'' - y' = 0$

4. $y'' + 2y' = 0$

5. $y'' - y = 0$

6. $y'' - 4y = 0$

7. $y'' + 2y' + y = 0$

8. $y'' + 4y' + 4y = 0$

9. $y'' + 3y' + 2y = 0$

10. $y'' - 3y' + 2y = 0$

11. $y'' + y = 0$

12. $y'' + 4y = 0$

13. $y'' + y' + y = 0$

14. $y'' + 2y' + 2y = 0$

15. $y'' = 0$

16. $y''' = 0$

17. $\frac{d^4y}{dx^4} = 0$

18. $\frac{d^5y}{dx^5} = 0$

19. $y''' + 2y'' = 0$

20. $y''' + 4y' = 0$

21. $\frac{d^4y}{dx^4} + y'' = 0$

22. $\frac{d^5y}{dx^5} + y''' = 0$

Detecting Atoms

Decompose each atom into a base atom times a power of x. If the expression fails to be an atom, then explain the failure.

23. $-x$

24. x

25. $x^2 \cos \pi x$

26. $x^{3/2} \cos x$

27. $x^{1000} e^{-2x}$

28. $x + x^2$

29. $\dfrac{x}{1 + x^2}$

30. $\ln |xe^{2x}|$

31. $\sin x$

32. $\sin x - \cos x$

Higher Order

A homogeneous linear constant-coefficient differential equation can be defined by (1) coefficients, (2) the characteristic equation, (3) roots of the characteristic equation. In each case, solve the differential equation.

33. $y''' + 2y'' + y' = 0$

34. $y''' - 3y'' + 2y' = 0$

35. $y^{(4)} + 4y'' = 0$

36. $y^{(4)} + 4y''' + 4y'' = 0$

37. Order 5, $r^2(r - 1)^3 = 0$

38. Order 5, $(r^3 - r^2)(r^2 + 1) = 0$.

39. Order 6, $r^2(r^2 + 2r + 2)^2 = 0$.

40. Order 6, $(r^2-r)(r^2+4r+5)^2=0$.

41. Order 10, $(r^4+r^3)(r^2-1)^2(r^2+1)=0$.

42. Order 10, $(r^3+r^2)(r-1)^3(r^2+1)^2=0$.

43. Order 5, roots $r=0,0,1,1,1$.

44. Order 5, roots $r=0,0,1,i,-i$.

45. Order 6, roots $r=0,0,i,-i,i,-i$.

46. Order 6, roots $r=0,-1,1+i,1-i,2i,-2i$.

47. Order 10, roots $r=0,0,0,1,1,-1,-1,-1,i,-i$.

48. Order 10, roots $r=0,0,1,1,1,-1,i,-i,i,-i$.

Initial Value Problems

Given in each case is a set of independent solutions of the differential equation. Solve for the coefficients c_1, c_2, ... in the general solution, using the given initial conditions.

49. e^x, e^{-x}, $y(0)=0$, $y'(0)=1$

50. xe^x, e^x, $y(0)=1$, $y'(0)=-1$

51. $\cos x, \sin x$, $y(0)=-1$, $y'(0)=1$

52. $\cos 2x, \sin 2x$, $y(0)=1$, $y'(0)=0$

53. $e^x, \cos x, \sin x$, $y(0)=-1$, $y'(0)=1$, $y''(0)=0$

54. $1, \cos x, \sin x$, $y(0)=-1$, $y'(0)=1$, $y''(0)=0$

55. $e^x, xe^x, \cos x, \sin x$, $y(0)=-1$, $y'(0)=1$, $y''(0)=0$, $y'''(0)=0$

56. $1, x, \cos x, \sin x$, $y(0)=1$, $y'(0)=-1$, $y''(0)=0$, $y'''(0)=0$

57. $1, x, x^2, x^3, x^4$, $y(0)=1$, $y'(0)=2$, $y''(0)=1$, $y'''(0)=3$, $y^{(4)}(0)=0$

58. $e^x, xe^x, x^2e^x, 1, x$, $y(0)=1$, $y'(0)=0$, $y''(0)=1$, $y'''(0)=0$, $y^{(4)}(0)=0$

Inverse Problem

Find a linear constant-coefficient homogeneous differential equation from the given information. Follow Example 6.21.

59. The characteristic equation is $(r+1)^3(r^2+4)=0$.

60. The general solution is a linear combination of the Euler solution atoms $e^x, e^{2x}, e^{3x}, \cos x, \sin x$.

61. The roots of the characteristic polynomial are $0, 0, 2+3i, 2-3i$.

62. The equation has order 4. Known solutions are $e^x+4\sin 2x$, xe^x.

63. The equation has order 10. Known solutions are $\sin 2x$, x^7e^x.

64. The equation is $my''+cy'+ky=0$ with $m=1$ and c,k positive. A solution is $y(x)=e^{-x/5}\cos(2x-\theta)$ for some angle θ.

Independence of Euler Atoms

65. Apply the independence test page **??** to atoms 1 and x: form equation $0=c_1+c_2x$, then solve for $c_1=0$, $c_2=0$. This proves Euler atoms $1, x$ are independent.

66. Show that Euler atoms $1, x, x^2$ are independent using the independence test page **??**,

67. A Taylor series is zero if and only if its coefficients are zero. Use this result to give a complete proof that the list $1, \ldots, x^k$ is independent. Hint: a polynomial is a Taylor series.

68. Show that Euler atoms e^x, xe^x, x^2e^x are independent using the independence test page **??**.

Wronskian Test

Establish independence of the given lists of functions by using the Wronskian test page **??**:

Functions $f_1, f_2, \ldots, f_n$ are independent if $W(x_0)\neq 0$ for some x_0, where $W(x)$ is the $n\times n$ determinant

$$\begin{vmatrix} f_1(x) & \cdots & f_n(x) \\ & \vdots & \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}$$

69. $1, x, e^x$

70. $1, x, x^2, e^x$

71. $\cos x, \sin x, e^x$

72. $\cos x, \sin x, \sin 2x$

Kümmer's Lemma

73. Compute the characteristic polynomials $p(r)$ and $q(r)$ for

$$\begin{aligned}&y'' + 3y' + 2y = 0 \text{ and}\\ &z'' + z' = 0.\end{aligned}$$

Verify the equations are related by $y = e^{-x}z$ and $p(r-1) = q(r)$.

74. Compute the characteristic polynomials $p(r)$ and $q(r)$ for

$$\begin{aligned}&ay'' + by' + cy = 0 \text{ and}\\ &az'' + (2ar_0 + b)z' +\\ &\quad (ar_0^2 + br_0 + c)z = 0.\end{aligned}$$

Verify the equations are related by $y = e^{r_0 x}z$ and $p(r + r_0) = q(r)$.

6.4 Variation of Parameters

The **Method of Variation of Parameters** applies to solve

$$a(x)y'' + b(x)y' + c(x)y = f(x). \tag{1}$$

Continuity of a, b, c and f is assumed, plus $a(x) \neq 0$. The method is important because it solves the largest class of equations. Specifically **included** are functions $f(x)$ like $\ln|x|$, $|x|$, e^{x^2}, $x/(1+x^2)$, which are excluded in the method of undetermined coefficients.

Homogeneous Equation

The method of variation of parameters uses facts about the homogeneous differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{2}$$

Success in the method depends upon a general solution expression for (2). Assumed are two *known solutions* y_1, y_2, Symbols c_1, c_2 represent arbitrary constants. The general solution:

$$y = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

If a, b, c are constants, then Theorem 6.1, page 430, applied to (2) implies y_1 and y_2 can be selected as *independent Euler solution atoms.*

Independence

Two solutions y_1, y_2 of (2) are called **independent** if neither is a constant multiple of the other. The term **dependent** means *not independent*, in which case either $y_1(x) = cy_2(x)$ or $y_2(x) = cy_1(x)$ holds for all x, for some constant c. Independence can be tested through the **Wronskian determinant** of y_1, y_2, defined by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

Theorem 6.17 (Wronskian and Independence)
The Wronskian of two solutions satisfies $a(x)W' + b(x)W = 0$, which implies **Abel's identity**

$$W(x) = W(x_0)e^{-\int_{x_0}^{x}(b(t)/a(t))dt}.$$

Two solutions of (2) are independent if and only if $W(x) \neq 0$.

Proof on page 466.

Niels Henrik Abel (1802–1829) was born in Nedstrand, Norway. He made major contributions to mathematics, especially *elliptic functions*, dying from tuberculosis at age 26.

Theorem 6.18 (Variation of Parameters Formula)
Let a, b, c, f be continuous near $x = x_0$ and $a(x) \neq 0$. Let y_1, y_2 be two independent solutions of homogeneous equation $a(x)y'' + b(x)y' + c(x)y = 0$ and let $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$. Then the non-homogeneous differential equation

$$a(x)y'' + b(x)y' + c(x)y = f$$

has a particular solution

$$y_p(x) = \left(\int \frac{y_2(x)(-f(x))}{a(x)W(x)}dx\right) y_1(x) + \left(\int \frac{y_1(x)f(x)}{a(x)W(x)}dx\right) y_2(x). \tag{4}$$

If both integrals have limits x_0 and x, then $y_p(x_0) = 0$.

Proof on page 467.

History of Variation of Parameters

The solution y_p was discovered by varying the constants c_1, c_2 in the homogeneous solution $y_h = c_1y_1 + c_2y_2$, assuming c_1, c_2 depend on x. This results in formulas $c_1(x) = \int C_1F$, $c_2(x) = \int C_2F$ where $F(x) = f(x)/a(x)$, $C_1(t) = \dfrac{-y_2(t)}{W(t)}$,$C_2(t) = \dfrac{y_1(t)}{W(t)}$; see the historical details on page 467. Then

$y = c_1y_1(x) + c_2y_2(x)$	Formula for y_h.
$y = \left(\int C_1F\right) y_1(x) + \left(\int C_2F\right) y_2(x)$	Substitute for c_1, c_2.
$= \left(\int -y_2\frac{F}{W}\right) y_1(x) + \left(\int y_1\frac{F}{W}\right) y_2(x)$	Use (**??**) for C_1, C_2.
$= \int (y_2(x)y_1(t) - y_1(x)y_2(t))\frac{F(t)}{W(t)}dt$	Collect on F/W.
$= \int \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)}F(t)dt$	Expand $W = y_1y_2' - y_1'y_2$.

Any one of the last three equivalent formulas is called a **Classical variation of parameters formula**. The fraction in the last integrand is called Cauchy's kernel. We prefer the first, equivalent to equation (4), for ease of use.

Examples and Methods

Example 6.22 (Independence)
Consider $y'' - y = 0$. Show the two solutions $\sinh(x)$ and $\cosh(x)$ are independent using Wronskians.

Solution: Let $W(x)$ be the Wronskian of $\sinh(x)$ and $\cosh(x)$. The calculation below shows $W(x) = -1$. By Theorem 6.17, the solutions are independent.

Background. The calculus *definitions* for hyperbolic functions are $\sinh x = (e^x - e^{-x})/2$, $\cosh x = (e^x + e^{-x})/2$. Their derivatives are $(\sinh x)' = \cosh x$ and $(\cosh x)' = \sinh x$. For instance, $(\cosh x)'$ stands for $\frac{1}{2}(e^x + e^{-x})'$, which evaluates to $\frac{1}{2}(e^x - e^{-x})$, or $\sinh x$.

Wronskian detail. Let $y_1 = \sinh x$, $y_2 = \cosh x$. Then

$$\begin{aligned} W &= y_1(x)y_2'(x) - y_1'(x)y_2(x) && \text{Definition of Wronskian } W. \\ &= \sinh(x)\sinh(x) - \cosh(x)\cosh(x) && \text{Substitute for } y_1,\, y_1',\, y_2,\, y_2'. \\ &= \tfrac{1}{4}(e^x - e^{-x})^2 - \tfrac{1}{4}(e^x + e^{-x})^2 && \text{Apply exponential definitions.} \\ &= -1 && \text{Expand and cancel terms.} \end{aligned}$$

Example 6.23 (Wronskian)
Given $2y'' - xy' + 3y = 0$, verify that a solution pair y_1, y_2 has Wronskian $W(x) = W(0)e^{x^2/4}$.

Solution: Let $a(x) = 2$, $b(x) = -x$, $c(x) = 3$. The Wronskian is a solution of $W' = -(b/a)W$, hence $W' = xW/2$. The solution is $W = W(0)e^{x^2/4}$, by the linear integrating factor method or the homogeneous equation shortcut.

Example 6.24 (Variation of Parameters)
Solve $y'' + y = \sec x$ by variation of parameters, verifying $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos(x) \ln|\cos(x)|$.

Solution:
Homogeneous solution y_h. Theorem 6.1 is applied to the constant equation $y''+y = 0$. The characteristic equation $r^2 + 1 = 0$ has roots $r = \pm i$ and then $y_h = c_1 \cos x + c_2 \sin x$.

Wronskian. Suitable independent solutions are $y_1 = \cos x$ and $y_2 = \sin x$, taken from the formula for y_h. Then $W(x) = \cos^2 x + \sin^2 x = 1$.

Calculate y_p. The variation of parameters formula (4) is applied. The integration proceeds near $x = 0$, because $\sec(x)$ is continuous near $x = 0$.

$$\begin{aligned} y_p(x) &= -y_1(x)\textstyle\int y_2(x)\sec(x)dx + y_2(x)\int y_1(x)\sec x dx && \boxed{1} \\ &= -\cos x \textstyle\int \tan(x)dx + \sin x \int 1dx && \boxed{2} \\ &= x \sin x + \cos(x)\ln|\cos(x)| && \boxed{3} \end{aligned}$$

Details: $\boxed{1}$ Use equation (4). $\boxed{2}$ Substitute $y_1 = \cos x$, $y_2 = \sin x$. $\boxed{3}$ Integral tables applied. Integration constants set to zero.

Proofs and Details

Proof of Theorem 6.17: The function $W(t)$ given by Abel's identity is the unique solution of the growth-decay equation $W' = -(b(x)/a(x))W$; see page **??**. It suffices then to show that W satisfies this differential equation. The details:

$W' = (y_1 y_2' - y_1' y_2)'$	Definition of Wronskian.
$\quad = y_1 y_2'' + y_1' y_2' - y_1'' y_2 - y_1' y_2'$	Product rule; $y_1' y_2'$ cancels.
$\quad = y_1(-b y_2' - c y_2)/a - (-b y_1' - c y_1) y_2/a$	Both y_1, y_2 satisfy (2).
$\quad = -b(y_1 y_2' - y_1' y_2)/a$	Cancel common $c y_1 y_2/a$.
$\quad = -bW/a$	Verification completed.

The independence statement will be proved from the contrapositive: $W(x) = 0$ for all x if and only if y_1, y_2 are not independent. Technically, independence is defined relative to the common domain of the graphs of y_1, y_2 and W. Henceforth, *for all* x means for all x in the common domain.

Let y_1, y_2 be two solutions of (2), not independent. By re-labelling as necessary, $y_1(x) = c y_2(x)$ holds for all x, for some constant c. Differentiation implies $y_1'(x) = c y_2'(x)$. Then the terms in $W(x)$ cancel, giving $W(x) = 0$ for all x.

Conversely, let $W(x) = 0$ for all x. If $y_1 \equiv 0$, then $y_1(x) = c y_2(x)$ holds for $c = 0$ and y_1, y_2 are not independent. Otherwise, $y_1(x_0) \neq 0$ for some x_0. Define $c = y_2(x_0)/y_1(x_0)$. Then $W(x_0) = 0$ implies $y_2'(x_0) = c y_1'(x_0)$. Define $y = y_2 - c y_1$. By linearity, y is a solution of (2). Further, $y(x_0) = y'(x_0) = 0$. By uniqueness of initial value problems, $y \equiv 0$, that is, $y_2(x) = c y_1(x)$ for all x, showing y_1, y_2 are not independent.

Proof of Theorem 6.18: Let $F(t) = f(t)/a(t)$, $C_1(x) = -y_2(x)/W(x)$, $C_2(x) = y_1(x)/W(x)$. Then y_p as given in (4) can be differentiated twice using the product rule and the fundamental theorem of calculus rule $(\int g)' = g$. Because $y_1 C_1 + y_2 C_2 = 0$ and $y_1' C_1 + y_2' C_2 = 1$, then y_p and its derivatives are given by

$$\begin{aligned} y_p(x) &= y_1 \textstyle\int C_1 F dx + y_2 \int C_2 F dx, \\ y_p'(x) &= y_1' \textstyle\int C_1 F dx + y_2' \int C_2 F dx, \\ y_p''(x) &= y_1'' \textstyle\int C_1 F dx + y_2'' \int C_2 F dx + F(x). \end{aligned}$$

Let $F_1 = a y_1'' + b y_1' + c y_1$, $F_2 = a y_2'' + b y_2' + c y_2$. Then

$$a y_p'' + b y_p' + c y_p = F_1 \int C_1 F dx + F_2 \int C_2 F dx + aF.$$

Because y_1, y_2 are solutions of the homogeneous differential equation, then $F_1 = F_2 = 0$. By definition, $aF = f$. Therefore,

$$a y_p'' + b y_p' + c y_p = f.$$

■

Historical Details. The original variation ideas, attributed to Joseph Louis Lagrange (1736-1813), involve substitution of $y = c_1(x) y_1(x) + c_2(x) y_2(x)$ into (1) plus imposing an extra unmotivated condition on the unknowns c_1, c_2:

$$c_1' y_1 + c_2' y_2 = 0.$$

The product rule gives $y' = c_1' y_1 + c_1 y_1' + c_2' y_2 + c_2 y_2'$, which then reduces to the two-termed expression $y' = c_1 y_1' + c_2 y_2'$. Substitution into (1) gives

$$a(c_1' y_1' + c_1 y_1'' + c_2' y_2' + c_2 y_2'') + b(c_1 y_1' + c_2 y_2') + c(c_1 y_1 + c_2 y_2) = f$$

which upon collection of terms becomes

$$c_1(a y_1'' + b y_1' + c y_1) + c_2(a y_2'' + b y_2' + c y_2) + a y_1' c_1' + a y_2' c_2' = f.$$

The first two groups of terms vanish because y_1, y_2 are solutions of the homogeneous equation, leaving just $ay_1'c_1' + ay_2'c_2' = f$. There are now two equations and two unknowns $X = c_1'$, $Y = c_2'$:

$$\begin{array}{rcrcl} ay_1'X & + & ay_2'Y & = & f, \\ y_1X & + & y_2Y & = & 0. \end{array}$$

Solving by elimination,

$$X = \frac{-y_2 f}{aW}, \quad Y = \frac{y_1 f}{aW}.$$

Then c_1 is the integral of X and c_2 is the integral of Y, which completes the historical account of the relations

$$c_1(x) = \int \frac{-y2(x)f(x)}{a(x)W(x)}dx, \quad c_2(x) = \int \frac{y_1(x)f(x)}{a(x)W(x)}dx.$$

Exercises 6.4

Independence: Constant Equation

Find solutions y_1, y_2 of the given homogeneous differential equation using Theorem 6.1 page 430. Then apply the Wronskian test page 464 to prove independence, following Example 6.22.

1. $y'' - y = 0$

2. $y'' - 4y = 0$

3. $y'' + y = 0$

4. $y'' + 4y = 0$

5. $4y'' = 0$

6. $y'' = 0$

7. $4y'' + y' = 0$

8. $y'' + y' = 0$

9. $y'' + y' + y = 0$

10. $y'' - y' + y = 0$

11. $y'' + 8y' + 2y = 0$

12. $y'' + 16y' + 4y = 0$

Independence for Euler's Equation

Change variables, $x = e^t$, $u(t) = y(x)$ in $Ax^2y''(x) + Bxy'(x) + Cy(x) = 0$ to obtain a constant-coefficient equation $A\left(\frac{d^2u}{dt^2} - \frac{du}{dt}\right) + B\frac{du}{dt} + Au = 0$. Solve for $u(t)$ and then substitute $t = \ln|x|$ to obtain $y(x)$. Find two solutions y_1, y_2 which are independent by the Wronskian test page 464.

13. $x^2y'' + y = 0$

14. $x^2y'' + 4y = 0$

15. $x^2y'' + 2xy' + y = 0$

16. $x^2y'' + 8xy' + 4y = 0$

Wronskian

Compute the Wronskian, up a constant multiple, without solving the differential equation: Example 6.23 page 466.

17. $y'' + y' - xy = 0$

18. $y'' - y' + xy = 0$

19. $2y'' + y' + \sin(x)y = 0$

20. $4y'' - y' + \cos(x)y = 0$

21. $x^2y'' + xy' - y = 0$

22. $x^2y'' - 2xy' + y = 0$

Variation of Parameters

Find the general solution $y_h + y_p$ by applying a variation of parameters formula: Example 6.24 page 466.

23. $y'' = x^2$

24. $y'' = x^3$

25. $y'' + y = \sin x$

26. $y'' + y = \cos x$

27. $y'' + y' = e^x$

28. $y'' + y' = -e^x$

29. $y'' + 2y' + y = e^{-x}$

30. $y'' - 2y' + y = e^x$

6.5 Undetermined Coefficients

The **method of undetermined coefficients** applies to solve constant-coefficient differential equations

(1) $$ay'' + by' + cy = f(x).$$

It finds a particular solution y_p *without* the integration steps present in variation of parameters. The method's importance is argued from its direct applicability to second order differential equations in mechanics and circuit theory. Requirements for $f(x)$ appear below.

Everything said here for second order differential equations applies unchanged to higher order differential equations

$$y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y = f(x).$$

Definition 6.2 (Euler Solution Atom)
The term **atom** is an abbreviation for the phrase *Euler solution atom of a constant-coefficient linear homogeneous differential equation*. Assume symbols a and b are real constants with $b > 0$. Define an **Euler base atom** as one of the functions

$$e^{ax}, \quad e^{ax}\cos bx, \quad e^{ax}\sin bx.$$

Define an **Euler solution atom** as a power x^m times a base atom, for integers $m = 0, 1, 2, \ldots$:

$$\textbf{Euler solution atom} = x^m(\textbf{base atom}).$$

Requirements

The method of undetermined coefficients has special requirements:

- Equation $ay'' + by' + cy = f(x)$ has constant coefficients a, b, c.
- The function $f(x)$ is a sum of constants times Euler solution atoms.

Method of Undetermined Coefficients

Step 1. Define the list of k Euler atoms in a trial solution using Rule I and Rule II [details below]. Multiply these atoms by **undetermined coefficients** d_1, $\ldots$, d_k, then add to define **trial solution** y.

Step 2. Substitute y into the differential equation.

Step 3. Match coefficients of Euler atoms left and right to write out linear algebraic equations for unknowns d_1, d_2, $\ldots$, d_k. Solve the equations.

Step 4. The trial solution y with evaluated coefficients d_1, d_2, $\ldots$, d_k becomes the particular solution y_p.

The Trial Solution Method

Central to the method of undetermined coefficients is the concept of a **trial solution** y, which is formally a linear combination of functions with coefficients yet to be determined. The method uses a *guess* of the form of a particular solution, then finds it explicitly *without actually solving the differential equation.* Knowing one particular solution y_p is enough to give the general solution of the differential equation (1), due to the superposition principle

$$y = y_h + y_p.$$

Example 6.25 (Trial Solution Illustration)

Consider the equation $y'' = 6x + e^x$ and a trial solution

$$y = d_1 x^3 + d_2 e^x.$$

Derive the equation

$$y_p = x^3 + e^x,$$

by calculating the **undetermined coefficients** d_1, d_2.

Solution: We first discuss how to solve the differential equation, because this background is needed to understand how the trial solution method works.

Answer check. The method of quadrature also applies to find $y = c_1 + c_2 x + x^3 + e^x$ instead of $y = x^3 + e^x$. Superposition $y = y_h + y_p$ implies that the shortest answer for a particular solution is $y_p = x^3 + e^x$, obtained by dropping the homogeneous solution $c_1 + c_2 x$.

Details.

We will show how to find d_1, d_2 in the trial solution $y = d_1 x^3 + d_2 e^x$ *without solving the differential equation*. The idea is to **substitute the trial solution into the differential equation**. This gives from equation $y'' = 6x + e^x$ the successive relations

$$\begin{aligned} (d_1 x^3 + d_2 e^x)'' &= 6x + e^x \\ 6d_1 x + d_2 e^x &= 6x + e^x \end{aligned}$$

The last relation implies, by independence of the atoms x, e^x, the coefficient-matching equations [4]

$$\begin{aligned} 6d_1 &= 6, \\ d_2 &= 1. \end{aligned}$$

The solution to this 2×2 linear system of equations is $d_1 = d_2 = 1$. Then the trial solution is

$$y = d_1 x^3 + d_2 e^x = x^3 + e^x.$$

We write $y_p = x^3 + e^x$.

That y_p is actually a solution of $y'' = 6x + e^x$ can be justified by computing the second derivative of $x^3 + e^x$.

[4] Euler atoms are independent in the sense of linear algebra. See Theorem 6.11, page 453. Independence means unique representation of linear combinations, which provides coefficient matching.

Why the Trial Solution has only Atoms x^3 and e^x

The differential equation $y'' = 6x + e^x$ can also be solved by answering this question:

> What expression y is differentiated twice to obtain $6x + e^x$?

Calculus suggests differentiating some cubic polynomial and some expression containing e^x. This is the central idea behind choosing a trial solution. Any trial solution, when substituted into the left side y'' of the differential equation, has to produce the terms in $6x + e^x$. Therefore, Euler atoms in the trial solution must have base atoms which appear in terms of the right side $6x + e^x$.

Explained is why terms in the trial solution $y = d_1x^3 + d_2e^x$ are limited to base atoms 1 and e^x.

Unexplained is why atoms 1, x, x^2 were not included in the trial solution. Insight can be gained by substitution of a combination $d_3 + d_4x + d_5x^2$ into the differential equation. Consider these steps:

$$\begin{array}{lcl}
(\text{trial solution})'' &=& 6x + e^x \\
(d_3 + d_4x + d_5x^2)'' &=& 6x + e^x \\
d_3(1)'' + d_4(x)'' + d_5(x^2)'' &=& 6x + e^x \\
d_3(0) + d_4(0) + d_5(2) &=& 6x + e^x
\end{array}$$

The coefficients d_3 and d_4 are multiplied by zero, because 1, x are solutions of the homogeneous equation $y'' = 0$. In general, homogeneous solution terms should not be added to a trial solution, because upon substitution these terms vanish from the left side of the differential equation. More succinctly, the missing variables d_3, d_4 are **free variables** in the language of linear algebra. We would choose $d_3 = d_4 = 0$ for simplicity. Term $2d_5$ is a multiple of base atom $1 = e^{0x}$. Because that atom does not appear on the right side $6x + e^x$, then $d_5 = 0$. The conclusion for this experiment: the trial solution $y = d_3 + d_4x + d_5x^2$ has three useless terms which do not contribute to terms on the right side of $y'' = 6x + e^x$.

Euler Solution Atoms in the General Solution

Superposition $y = y_h + y_p$ is used to describe the structure of solutions in differential equations solved by the method of undetermined coefficients. The homogeneous solution y_h of $ay'' + by' + cy = 0$ is constructed from atoms found by Euler's theorem. Therefore, y_h is a sum of constants times atoms. For the nonhomogeneous equation $ay'' + by' + cy = f(x)$, the method of undetermined coefficients finds y_p as a sum of constants times atoms. The plan here is to describe completely the atoms in solutions y_h and y_p.

Theorem 6.19 (Solution Structure)
A differential equation $ay'' + by' + cy = f(x)$ with constant coefficients a, b, c and right side $f(x)$ a sum of constants times Euler atoms has general solution $y = y_h + y_p$ which is a sum of constants times Euler atoms. In the language of linear algebra:

Solutions $y(x)$ of $ay'' + by' + cy = f(x)$ are a linear combination of Euler atoms.

Euler Atoms in the Homogeneous Solution

The atoms in y_h are found from Euler's theorem applied to the characteristic equation $ar^2+br+c=0$. To illustrate, the characteristic equation $r^2+2r+1=0$ has double root -1, -1 and the corresponding atoms are e^{-x}, xe^{-x}.

Euler atoms can be extracted from a general solution $y_h = c_1e^x + c_2xe^x$ by taking partial derivatives on the symbols c_1, c_2. Conversely, two distinct Euler atoms are sufficient to form the general solution y_h. Euler atoms for the homogeneous equation can therefore be prescribed by any one of the following means:

1. The characteristic equation $ar^2 + br + c = 0$.

2. The roots of the characteristic equation.

3. The general solution expression y_h, with symbols c_1, c_2.

Euler Atoms in a Particular Solution y_p

The Euler atoms that appear in y_p may be assumed to not duplicate any atoms in y_h. The logic is that y_p can be shortened in length by moving any homogeneous solution into the terms of y_h, due to superposition $y = y_h + y_p$.

Explained below is how to construct the k atoms in y_p directly from the right side $f(x)$ of the differential equation. This is done by two rules, called **Rule I** and **Rule II**. We always proceed under the assumption that Rule I will work, and if it fails, then we go on to apply Rule II.

Undetermined Coefficients Rule I

Assume $f(x)$ in the equation $ay'' + by' + cy = f(x)$ is a sum of constants times Euler atoms. For each atom A appearing in $f(x)$, extract all distinct atoms that appear in A, A', A'', ..., then collect all computed atoms into a list of k distinct Euler atoms.

Test for a Valid Trial Solution

If the list **contains no solution of the homogeneous differential equation**, then multiply the k Euler atoms by undetermined coefficients d_1, ..., d_k to form the trial solution

$$y_p = d_1(\text{atom 1}) + d_2(\text{atom 2}) + \cdots + d_k(\text{atom k}).$$

Undetermined Coefficients Rule II

Assume Rule I constructed a list of k atoms, but Rule I **FAILED** the **TEST**. The particular solution y_p is still a sum of constants times k Euler atoms. Rule II changes some or all of the k atoms, by repeated multiplication by x.

The k-atom list is subdivided into groups with the same base atom, called **group 1**, **group 2**, and so on. Each group is tested for a solution of the homogeneous differential equation. If found, then multiply each Euler atom in the group by factor x. Repeat until **no group contains a solution of the homogeneous differential equation**. The final set of k Euler atoms is used to construct

$$y_p = d_1(\text{atom 1}) + d_2(\text{atom 2}) + \cdots + d_k(\text{atom k}).$$

Grouping Atoms

The Rule I process of finding derivatives A, A', A'',... can be replaced by the simpler task of forming **the group of each atom** A. The idea can be seen from the example $A = x^2e^x$. Each differentiation $A, A'A'', \ldots$ causes one lower power of x to appear, then we can predict that the distinct atoms that appear in the derivatives of A are

$$e^x, \quad xe^x, \quad x^2e^x.$$

This set is called the **group of Euler atom** A. In this example, $B = e^x$ is the **base atom** for atom $A = x^2e^x$ and the group is base atom B multiplied by the powers 1, x, x^2.

Assume Euler atom A is base atom B times a power x^m, for some integer $m \geq 0$. The **Group of Euler atom** A is the base atom B multiplied successively by the $m + 1$ powers 1, x, $\ldots$, x^m. The group starts with the base atom B and ends with the atom $A = x^mB$.

$$\begin{aligned} B &= \text{any base atom} \\ \text{group of } x^mB &\equiv B, \quad xB, \quad x^2B, \ldots, x^mB. \end{aligned}$$

Differentiation of an atom A with a sine or cosine factor produces two groups, not one. For example, $A = x^2 \sin x$ upon differentiation produces two groups

$$\begin{array}{ll} \textbf{cosine group}: & \cos x, \quad x\cos x, \quad x^2\cos x \\ \textbf{sine group}: & \sin x, \quad x\sin x, \quad x^2\sin x. \end{array}$$

Key Examples of Atom Grouping

1. The atom x^2e^{0x} has base atom $e^{0x} = 1$ and group 1, x, x^2. The group size is 3.
2. The atom $e^{-\pi x}$ has base atom $e^{-\pi x}$ and group $e^{-\pi x}$. A base atom has group size 1.
3. Atom $x^3e^x\cos x$ has base atom $e^x\cos x$ and two 4-element groups:
   $e^x\cos x$, $xe^x\cos x$, $x^2e^x\cos x$, $x^3e^x\cos x$ and
   $e^x\sin x$, $xe^x\sin x$, $x^2e^x\sin x$, $x^3e^x\sin x$.

4. Atom x^2e^x has base atom e^x. The group is the set of 3 atoms e^x, xe^x, x^2e^x.

5. If $A = xe^x \cos 2x$, then the Rule I process of extracting atoms from A, A', A'', ... causes two groups to be formed, **group 1**: $e^x \cos 2x$, $xe^x \cos 2x$ and **group 2**: $e^x \sin 2x$, $xe^x \sin 2x$. A shortcut for writing the second group is to change cosine to sine in the first group.

Undetermined Coefficient Method Details

The undetermined coefficients trial solution y uses **Rule I** and **Rule II**. Then a particular solution, according to the method, is

$$y_p = \text{a linear combination of atoms.}$$

The discussion here is restricted to second order equations $n = 2$.

Superposition. The relation $y = y_h + y_p$ suggests solving $ay'' + by' + cy = f(x)$ in two stages:

(a) Apply **Euler's Theorem** to find y_h as a sum of constants times atoms.

(b) Apply the **method of undetermined coefficients** to find y_p as a sum of constants times atoms.

Symbols. The symbols c_1, c_2 are reserved for use as arbitrary constants in the general solution y_h of the homogeneous equation. Symbols d_1, d_2, ... are reserved for use in the trial solution y of the non-homogeneous equation. Abbreviations: $c =$ constant, $d =$ determined.

Expect to find two arbitrary constants c_1, c_2 in the solution y_h, but in contrast, no arbitrary constants appear in y_p. The literature's terminology *undetermined coefficients* is misleading, because in fact symbols d_1, d_2, ... are *determined.*

Algebra Background. The trial solution method requires background in the solution of simultaneous linear algebraic equations, as is often taught in college algebra. A linear algebra background will make the details seem even easier.

Example 6.26 (Undetermined Coefficients Illustration)

Solve the differential equation $y'' - y = x + xe^x$ by the method of undetermined coefficients, verifying

$$y_h = c_1 e^x + c_2 e^{-x}, \quad y_p = -x - \frac{1}{4} x e^x + \frac{1}{4} x^2 e^x.$$

Solution:

Homogeneous Solution. The homogeneous equation

$$y'' - y = 0$$

has characteristic equation $r^2 - 1 = 0$. The roots $r = \pm 1$ produce by Euler's theorem the list of atoms e^x, e^{-x}. Then the homogeneous solution is a linear combination of the Euler atoms: $y_h = c_1 e^x + c_2 e^{-x}$.

Trial Solution. The shortest trial solution is

$$y = (d_1 + d_2 x) + (d_3 x e^x + d_4 x^2 e^x),$$

to be justified below.

Rule I. Let $f(x) = x + xe^x$. The derivatives $f, f', f'', \ldots$ are linear combinations of the four Euler atoms $1, x, e^x, xe^x$. Because e^x is a solution of the homogeneous equation $y'' - y = 0$, then Rule I FAILS the TEST.

Rule II. Divide the list $1, x, e^x, xe^x$ into two groups with identical base atom:

Group	Euler Atoms	Base Atom
group 1 :	$1, x$	1
group 2 :	e^x, xe^x	e^x

Group 1 contains no solution of the homogeneous equation $y'' - y = 0$, therefore Rule II changes nothing. **Group 2** contains solution e^x of the homogeneous equation. Rule II says to multiply group 2 by x, until the modified group contains no solution of the homogeneous differential equation $y'' - y = 0$ .Then

Group	Euler Atoms	Action
New group 1 :	$1, x$	no change
New group 2 :	xe^x, x^2e^x	multiplied once by x

In **New Group 2**, xe^x is not a solution of the homogeneous problem, because if it is, then 1 is a double root of the characteristic equation $r^2 - 1 = 0$ [it isn't].

The final groups have been found in Rule II. The shortest trial solution is

$$\begin{aligned} y &= \text{linear combination of Euler atoms in the new groups} \\ &= d_1 + d_2 x + d_3 x e^x + d_4 x^2 e^x. \end{aligned}$$

Equations for the undetermined coefficients. Substitute $y = d_1 + d_2 x + d_3 x e^x + d_4 x^2 e^x$ into $y'' - y = x + xe^x$. The details:

$\text{LHS} = y'' - y$	Left side of the equation.
$= [y_1'' - y_1] + [y_2'' - y_2]$	Let $y = y_1 + y_2$, $y_1 = d_1 + d_2 x$, $y_2 = d_3 x e^x + d_4 x^2 e^x$.
$= [0 - y_1] +$ $[2d_3 e^x + 2d_4 e^x + 4d_4 x e^x]$	Use $y_1'' = 0$ and $y_2'' = y_2 + 2d_3 e^x + 2d_4 e^x + 4d_4 x e^x$.
$= (-d_1)1 + (-d_2)x +$ $(2d_3 + 2d_4)e^x + (4d_4)xe^x$	Collect on distinct Euler atoms.

Then $y'' - y = f(x)$ simplifies to

$$(-d_1)1 + (-d_2)x + (2d_3 + 2d_4)e^x + (4d_4)xe^x = f(x).$$

Write out a 4×4 **system**. Because $f(x) = x + xe^x$, the last display gives the expansion below, which has been written with each side a linear combination of the atoms 1, x, e^x, xe^x.

$$\begin{aligned} &(-d_1)1 + (-d_2)x + \\ &(2d_3 + 2d_4)e^x + (4d_4)xe^x = (0)1 + (1)x + (0)e^x + (1)xe^x. \end{aligned} \tag{2}$$

Equate coefficients of matching atoms 1, x, e^x, xe^x left and right to give the system of equations

$$(3)\qquad \begin{array}{rrrrcll} -d_1 & & & & = & 0, & \text{match on } 1 \\ & -d_2 & & & = & 1, & \text{match on } x \\ & & 2d_3 & +2d_4 & = & 0, & \text{match on } e^x \\ & & & 4d_4 & = & 1. & \text{match on } xe^x \end{array}$$

Atom matching effectively removes x and changes the equation into a 4×4 linear algebraic nonhomogeneous system of equations for d_1, d_2, d_3, d_4.

The technique is **independence**. To explain, linear independence of atoms means that a linear combination of atoms is uniquely represented. Then two such equal representations must have matching coefficients. Relation (2) says that two linear combinations of the same list of atoms are equal. Then coefficients of 1, x, e^x, xe^x left and right in (2) must match, giving system (3).

Solve the equations. The 4×4 system by design always has a unique solution. In the language of linear algebra, there are zero free variables. In the present case, the system is triangular, solved by back-substitution to give the unique solution $d_1 = 0$, $d_2 = -1$, $d_4 = 1/4$, $d_3 = -1/4$.

Report y_p. The trial solution $y = d_1+d_2x+d_3xe^x+d_4x^2e^x$ with determined coefficients $d_1 = 0$, $d_2 = -1$, $d_3 = -1/4$, $d_4 = 1/4$ becomes the particular solution

$$y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

General solution. Superposition implies the general solution is $y = y_h + y_p$. From above, $y_h = c_1e^x + c_2e^{-x}$ and $y_p = -x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x$. Then $y = y_h + y_p$ is given by

$$y = c_1e^x + c_2e^{-x} - x - \frac{1}{4}xe^x + \frac{1}{4}x^2e^x.$$

Answer Check. Computer algebra system `maple` is used.

```
yh:=c1*exp(x)+c2*exp(-x);
yp:=-x-(1/4)*x*exp(x)+(1/4)*x^2*exp(x);
de:=diff(y(x),x,x)-y(x)=x+x*exp(x):
odetest(y(x)=yh+yp,de); # Success is a report of zero.
```

Further examples: pages 481, 482, 484, 484.

Constructing Euler Atoms from Roots

An Euler atom is constructed from a real number a or a complex number $a+ib$. The number used for the construction is called a **root** for the atom. Euler's theorem page 454 provides the rules:

> Real root $r = a$ constructs the exponential base atom e^{ax}. If $a = 0$, then the base atom is $e^{0x} = 1$.
>
> For a complex root $r = a + ib$, $b > 0$, construct two base atoms $e^{ax}\cos bx$ and $e^{ax}\sin bx$.

Atoms constructed from roots a or $a+ib$ using Euler's multiplicity theorem gives the complete list of all possible atoms:

Root	Euler Atoms ($b>0$)				
$r=a$	e^{ax},	xe^{ax},	x^2e^{ax},	$\ldots$,	x^ne^{ax}
$r=a+ib$	$e^{ax}\cos bx$,	$xe^{ax}\cos bx$,	$x^2e^{ax}\cos bx$,	$\ldots$,	$x^ne^{ax}\cos bx$
$r=a+ib$	$e^{ax}\sin bx$,	$xe^{ax}\sin bx$,	$x^2e^{ax}\sin bx$,	$\ldots$,	$x^ne^{ax}\sin bx$

Constructing Roots from Euler Atoms

An Euler atom can be viewed as having been constructed from a unique real root a or a unique pair of complex roots $a \pm ib$. The reverse process considers an atom and finds the possible root (or roots) used for its construction plus the root's multiplicity. Details in the following table:

Euler Atom	Base Atom	Root	Multiplicity
x^ne^{ax}	e^{ax}	a	$n+1$
$x^ne^{ax}\cos(bx)$	$e^{ax}\cos(bx)$	$a\pm ib$	$n+1$
$x^ne^{ax}\sin(bx)$	$e^{ax}\sin(bx)$	$a\pm ib$	$n+1$

Examples of Atoms and Roots

The atoms for root $r=0$ of multiplicity 4 are $1, x, x^2, x^3$. The atoms for root $r=2+3i$ of multiplicity 3 are

$$\begin{array}{lll} e^{2x}\cos(3x), & xe^{2x}\cos(3x), & x^2e^{2x}\cos(3x) \\ e^{2x}\sin(3x), & xe^{2x}\sin(3x), & x^2e^{2x}\sin(3x). \end{array}$$

The roots for atom x^3 are $r=0,0,0,0$ (quad root). The roots for atom xe^x are $r=1,1$. The roots for atom $x\cos x$ are $r=i,-i,i,-i$ (double complex root).

Polynomials and Root Multiplicity

In college algebra, roots of polynomials are studied through the theory of equations, which includes the root and factor theorem, the rational root theorem, the division algorithm and Descarte's rule of signs.

The multiplicity of a polynomial root $r=r_0$ is defined in college algebra to be the unique integer m such that $(r-r_0)^m$ divides the polynomial, but $(r-r_0)^{m+1}$ does not.

The algebra topic is enriched by calculus:

Theorem 6.20 (Multiplicity of a Root)
Let $p(r)$ be the characteristic polynomial for a given linear homogeneous differential equation with constant coefficients. The **Multiplicity** of a root $r=r_0$ of $p(r)=0$

can be determined by calculus as follows.

$$\begin{array}{ll}
\text{Multiplicity } 1 & p(r_0) = 0,\ p'(r_0) \neq 0 \\
\text{Multiplicity } 2 & p(r_0) = p'(r_0) = 0,\ p''(r_0) \neq 0 \\
\text{Multiplicity } 3 & p(r_0) = p'(r_0) = p''(r_0) = 0,\ p'''(r_0) \neq 0 \\
\vdots & \qquad\qquad\qquad \vdots \\
\text{Multiplicity } m & p(r_0) = \cdots = p^{(m-1)}(r_0) = 0,\ p^{(m)}(r_0) \neq 0
\end{array}$$

Factorization of the characteristic polynomial may be possible. If so, then the roots and their multiplicities are all known at once. Factorization is not needed at all to test if $r = r_0$ is a root, and only basic calculus is required to determine the multiplicity of a root.

Computing the Shortest Trial Solution

Described here is are two alternatives to Rule I and Rule II, to construct the shortest trial solution in the method of undetermined coefficients. The first method uses Laplace theory. The second method uses differential operator techniques, presented here assuming minimal background.

Laplace's Method

Readers who are unfamiliar with Laplace theory should skip this subsection and go on to the next.

The idea will be communicated by example, which hopefully is enough for a reader already familiar with Laplace theory. Suppose we are going to solve the equation

$$\frac{d^2y}{dt^2} + y = t + e^t$$

using the theory of undetermined coefficients. Then Rule I applies and we don't need Rule II, giving $y = y_h + y_p$ where

$$y_h = c_1 \cos t + c_2 \sin t, \quad y_p = d_1 + d + 2t + d_3 e^t.$$

Laplace theory can quickly find y_p by assuming zero initial data $y(0) = y'(0) = 0$, in which case another candidate y for y_p is found by the transfer function method:

$$\mathcal{L}(y) = (\text{Transfer function})(\text{Laplace of } t + e^t) = \frac{s^2 + s - 1}{s^2(s-1)(s^2+1)}.$$

Partial fraction theory applies:

$$\mathcal{L}(y) = \frac{a + bs}{s^2 + 1} + \frac{c}{s} + \frac{d}{s^2} + \frac{f}{s-1} = \mathcal{L}(a \sin t + b \cos t + c + dt + f e^t).$$

Lerch's theorem applies:

$$y = a\sin t + b\cos t + c + dt + fe^t.$$

The term $a\sin t + b\cos t$ represents a solution y_h of the homogeneous problem $y'' + y = 0$. Remove the homogeneous solution, then report a particular solution as having the form

$$y_p = c + dt + fe^t.$$

This is the shortest trial solution, obtained by Laplace theory.

The Method of Annihilators

Suppose that $f(x)$ is a sum of constants times Euler atoms. The **Annihilator** of $f(x)$ is the unique minimal-order homogeneous constant-coefficient higher order differential equation of leading coefficient one which has $f(x)$ as a particular solution.

For example, if $f(x) = x + e^x$, then the annihilator of $f(x)$ is the third order constant-coefficient homogeneous differential equation [details in examples below]

$$y''' - y'' = 0$$

Required is that $f(x)$ is a *particular solution* of the differential equation, related to the general solution $y(x)$ by specialization of constants.

Examples of annihilators: The differential equation $y''+y = 0$ is the annihilator for $\sin x$, but also the annihilator for $2\cos x - \sin x$. The differential equation $y''' + 4y' = 0$ is the annihilator for any of $\sin 2x$, $1 + \cos 2x$, $7 - 5\sin 2x$.

An annihilator can be given by its characteristic equation, e.g., $r^3 + 4r = 0$ generates annihilator $y''' + 4y' = 0$.

Characteristic Polynomial of the Annihilator

Let $f(x)$ be a given linear combination of atoms. The algorithm:

1. Determine the list of atoms for $f(x)$.

2. Find the root(s) for each base atom B. Then find the corresponding highest power real factors in the characteristic equation, using Euler's theorem.

3. The characteristic polynomial is the product of the highest power distinct factors so found.

For instance, $f(x) = 2e^x + \cos 3x - x - x^3$ has base atoms e^x, $\cos 3x$, 1 with corresponding roots 1, $\pm 3i$, $0, 0, 0, 0$, listed according to multiplicity. By Euler's theorem, the corresponding factors with highest powers are $r-1$, r^2+9, $(r-0)^4$, which implies the characteristic polynomial is $(r-1)(r^2+9)r^4$.

Annihilator Method Algorithm

Assume that the non-homogeneous differential equation of order n has constant coefficients and the right side $f(x)$ is a linear combination of atoms. The method arises by applying the annihilator of f, as a differential operator, to the non-homogeneous differential equation

$$y^{(n)} + \sum_{k=0}^{n-1} a_k y^{(k)} = f(x).$$

Because the annihilator applied to $f(x)$ is zero, then any solution $y = y_p(x)$ satisfies a higher-order homogeneous equation, whose characteristic equation is known [see item 3 below].

1. Find the homogeneous equation characteristic polynomial $p(r)$.
2. Find the characteristic polynomial $q(r)$ for the annihilator of $f(x)$.
3. The shortest trial solution is a linear combination of the atoms obtained from $p(r)q(r) = 0$, after removing those atoms which correspond to the roots of $p(r) = 0$.

Further examples pages 486, 486.

Further study

The trial solution method is enriched by developing a **Library of Special Methods** for finding y_p, which includes Kümmer's method; see page **??**. The library provides an independent justification of the trial solution method. The only background required is college algebra and polynomial calculus. The trademark of the library method is the *absence of linear algebra, tables or special cases*, that can be found in other literature on the subject.

Examples

Example 6.27 (Polynomial Trial Solution)

Solve for y_p in $y'' = 2 - x + x^3$ using the method of undetermined coefficients, verifying $y_p = x^2 - \frac{1}{6}x^3 + \frac{1}{20}x^5$.

Solution:
Homogeneous solution. The homogeneous equation $y'' = 0$ has characteristic equation $r^2 = 0$ with roots $r = 0, 0$. Euler's theorem generates the two atoms 1, x. Then the homogeneous solution is $y_h = c_1 + c_2 x$.

Trial solution. Let's justify the selection of the trial solution

$$y = d_1 x^2 + d_2 x^3 + d_3 x^4 + d_4 x^5.$$

Rule I applied to the right side $f(x) = 2 - x + x^3$ gives a single group of four atoms

$$\textbf{group 1}: \quad 1, x, x^2, x^3.$$

Because 1 is a solution of the homogeneous equation $y'' = 0$, then Rule I **FAILS** the **TEST**. Rule II is applied to **group 1**, which modifies the group by multiplication by x. The correction by x-multiplication must be applied twice, because both 1 and x are solutions of the homogeneous differential equation $y'' = 0$. Then the new group is

$$\textbf{New group 1}: \quad x^2, x^3, x^4, x^5.$$

The trial solution is then a linear combination of four Euler atoms from the new group, $y = d_1x^2 + d_2x^3 + d_3x^4 + d_4x^5$.

Equations for the undetermined coefficients. The details:

$$\begin{aligned} 2 - x + x^3 &= y'' && \text{Reverse sides.} \\ &= 2d_1 + 6d_2x + 12d_3x^2 + 20d_4x^3 && \text{Substitute } y. \end{aligned}$$

Equate coefficients of Euler atoms on each side of the equal sign to obtain the system of equations

$$\begin{array}{rcl} 2d_1 &=& 2, \\ 6d_2 &=& -1, \\ 12d_3 &=& 0, \\ 20d_4 &=& 1. \end{array}$$

Solve the equations. This is a triangular system of linear equations for unknowns d_1, d_2, d_3, d_4. Solving gives $d_1 = 1$, $d_2 = -1/6$, $d_3 = 0$, $d_4 = 1/20$.

Report y_p. The trial solution expression $y = d_1x^2 + d_2x^3 + d_3x^4 + d_4x^5$ after substitution of the values found for d_1 to d_4 gives the particular solution

$$y_p = x^2 - \frac{1}{6}x^3 + \frac{1}{20}x^5.$$

Example 6.28 (Undetermined Coefficient Method)

Solve $y'' + y = 2 + e^x + \sin(x)$ by the trial solution method, verifying $y = c_1\cos(x) + c_2\sin(x) + 2 + \frac{1}{2}e^x - \frac{1}{2}x\sin x$.

Solution:

Homogeneous solution. The characteristic equation for the homogeneous equation $y'' + y = 0$ is $r^2 + 1 = 0$. It has roots $r = \pm i$ and atom list $\cos x, \sin x$. Then y_h is a linear combination of the two atoms:

$$y_h = c_1\cos x + c_2\sin x.$$

Symbols c_1 and c_2 are arbitrary constants.

Rule I. The right side $f(x) = 2 + e^x + \sin x$ of the differential equation is differentiated a few times to discover the atom list $1, e^x, \cos x, \sin x$. Because $\cos x$ is a solution of the homogeneous equation $y'' + y = 0$, then Rule I **FAILS** the **TEST**.

Rule II. The Euler atoms are grouped by equal base atom as follows.

Group	Euler Atoms	Rule II action	New Group
group 1:	1	no change	1
group 2:	e^x	no change	e^x
group 3:	$\cos x$	multiply once by x	$x\cos x$
group 4:	$\sin x$	multiply once by x	$x\sin x$

Group 1 and **Group 2** are unchanged by Rule II, because they do not contain a solution of the homogeneous equation $y'' + y = 0$. **Group 3** and **Group 4** do contain a homogeneous solution, therefore each group is multiplied by x. The resulting new groups 3 and 4 do not contain a homogeneous solution. It is expected, in general, to iterate on x-multiplication on a group until the first time that the new group contains no solution of the homogeneous equation.

The trial solution is a linear combination of the four Euler atoms in the new groups:

$$y = d_1 + d_2e^x + d_3x\cos x + d_4x\sin x.$$

Equations for the undetermined coefficients.

$\mathsf{LHS} = y'' + y$	Left side of the equation $y'' + y = 2 + e^x + \sin(x)$.
$= d_1 + 2d_2e^x - 2d_3\sin(x) + 2d_4\cos(x)$	Substitute trial solution y.

The equation $y'' + y = 2 + e^x + \sin(x)$ becomes

$$d_1 + 2d_2e^x - 2d_3\sin(x) + 2d_4\cos(x) = 2 + e^x + \sin(x).$$

Equating coefficients of atoms left and right implies the equations

$$\begin{aligned} d_1 &= 2, \\ 2d_2 &= 1, \\ -2d_3 &= 1, \\ 2d_4 &= 0. \end{aligned}$$

Solve the equations. There are no details, because the system is diagonal. The displayed answers are $d_1 = 2$, $d_2 = 1/2$, $d_3 = -1/2$, $d_4 = 0$.

Particular solution y_p. The particular solution is obtained from the trial solution $y = d_1 + d_2e^x + d_3x\cos x + d_4x\sin x$ by replacing the undetermined coefficients d_1 to d_4 by their values determined above:

$$y_p = 2 + \frac{1}{2}e^x - \frac{1}{2}x\cos(x).$$

General Solution. Add y_h and y_p to obtain the general solution

$$y = c_1\cos(x) + c_2\sin(x) + 2 + \frac{1}{2}e^x - \frac{1}{2}x\cos(x).$$

Answer check. Computer algebra system `maple` checks the answer as follows.

```
dsolve(diff(y(x),x,x)+y(x)=2+exp(x)+sin(x),y(x));
# y(x) = 2+1/2*exp(x)-1/2*cos(x)*x+_C1*cos(x)+_C2*sin(x)
```

Example 6.29 (Two Methods)

Solve $y'' - y = e^x$ by undetermined coefficients and by variation of parameters. Explain any differences in the answers.

Solution: The general solution is reported to be $y = y_h + y_p = c_1 e^x + c_2 e^{-x} + xe^x/2$. Details follow.

Homogeneous solution. The characteristic equation $r^2 - 1 = 0$ for $y'' - y = 0$ has roots ± 1 with atom list e^x, e^{-x}. The homogeneous solution is $y_h = c_1 e^x + c_2 e^{-x}$.

Undetermined Coefficients Summary. The right side of the differential equation, $f(x) = e^x$, contains only the single atom e^x, therefore the Rule I atom list is e^x. Rule I FAILS the TEST, because e^x is a solution of the homogeneous equation. Rule II applies, then x multiplies the group e^x to obtain the new group xe^x. This atom is not a solution of the homogeneous equation, therefore the trial solution is $y = d_1 xe^x$. Substitute it into $y'' - y = e^x$ to obtain $2d_1 e^x + d_1 xe^x - d_1 xe^x = e^x$. Match coefficients of e^x to compute $d_1 = 1/2$. Then $y_p = xe^x/2$.

Variation of Parameters Summary. The homogeneous solution $y_h = c_1 e^x + c_2 e^{-x}$ found above implies $y_1 = e^x$, $y_2 = e^{-x}$ is a suitable independent pair of solutions. Their Wronskian is

$$W = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2.$$

The variation of parameters formula (6.18) applies:

$$y_p(x) = \left(\int \frac{-e^{-x}}{-2} e^x dx \right) e^x + \left(\int \frac{e^x}{-2} e^x dx \right) e^{-x}.$$

Integration with zero constants of integration gives $y_p(x) = xe^x/2 - e^x/4$.

Differences. The two methods give respectively $y_p = xe^x/2$ and $y_p = xe^x/2 - e^x/4$. The solutions $y_1 = xe^x/2$ and $y_2 = xe^x/2 - e^x/4$ differ by the homogeneous solution $y_h = y_2 - y_1 = -xe^x/4$. In both cases, the general solution is $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} xe^x$, because homogeneous solution terms can be absorbed into the arbitrary constants c_1, c_2.

Example 6.30 (Sine–Cosine Trial solution)

Verify for $y'' + 4y = \sin x - \cos x$ that $y_p(x) = 5\cos x + 3\sin x$, using trial solution $y = A\cos x + B\sin x$.

Solution: Let's justify the trial solution. Rule I differentiates $f(x) = \sin x - \cos x$ to determine the atom list $\cos x$, $\sin x$. Because $\cos x$ and $\sin x$ are not solutions of the homogeneous equation $y'' + 4y = 0$, then Rule I succeeds and the trial solution is $y = d_1 \cos x + d_2 \sin x$. Replace d_1, d_2 by symbols A, B to agree with the given trial solution.

Equations for the undetermined coefficients. Substitute $y = A\cos x + B\sin x$ into the differential equation and use $u'' = -u$ for $u = \sin x$ or $u = \cos x$ to obtain the relation

$$\begin{aligned} \sin x - \cos x &= y'' + 4y \\ &= (-A + 4)\cos x + (-B + 4)\sin x. \end{aligned}$$

Matching coefficients of sine and cosine terms on the left and right gives the system of equations

$$\begin{aligned} -A + 4 &= -1, \\ -B + 4 &= 1. \end{aligned}$$

Solve the equations. The system is diagonal, therefore $A = 5$ and $B = 3$.

Report y_p. The trial solution $y = A\cos x + B\sin x$ after substitution of found values for A, B becomes the particular solution $y_p(x) = 5\cos x + 3\sin x$.

Generally, the method of undetermined coefficients applied to similar second order problems produces linear algebraic equations that must be solved by linear algebra techniques. Sometimes, the most convenient is Cramer's 2×2 rule.

Example 6.31 (Exponential Trial Solution)

Solve for y_p in

$$y'' - 2y' + y = (1 + x - x^2)e^x$$

by the method of undetermined coefficients, verifying that

$$y_p = \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x - \frac{1}{12}x^4e^x.$$

Solution:

Homogeneous solution. The homogeneous equation is $y'' - 2y' + y = 0$. The characteristic equation $r^2 - 2r + 1 = 0$ has a double root $r = 1$ and by Euler's theorem the corresponding atom list is e^x, xe^x. Then the homogeneous general solution is $y_h = c_1e^x + c_2xe^x$, where c_1 and c_2 are arbitrary constants.

Trial solution. Let's apply Rule I. The derivatives of $f(x) = (1 + x - x^2)e^x$ are combinations of the list of distinct Euler atoms e^x, xe^x, x^2e^x. Because the first two atoms are solutions of the homogeneous equation, then Rule I FAILS the TEST. Rule II applies: the list of atoms for $f(x)$ has just one group:

$$\textbf{group 1}: \quad e^x, \quad xe^x, \quad x^2e^x.$$

Rule II modifies the list of three atoms by x-multiplication. It is applied twice, because both e^x and xe^x are solutions of the homogeneous differential equation. The new group of three atoms is

$$\textbf{New group 1}: \quad x^2e^x, \quad x^3e^x, \quad x^4e^x.$$

A trial solution according to Rule II is a linear combination of the new group atoms:

$$y = d_1x^2e^x + d_2x^3e^x + d_3x^4e^x.$$

Equations for the undetermined coefficients. Substitute the trial solution $y = d_1x^2e^x + d_2x^3e^x + d_3x^4e^x$ solution into $y'' - 2y' + y = (1 + x - x^2)e^x$, in order to find the undetermined coefficients d_1, d_2, d_3. To present the details, let $q(x) = d_1x^2 + d_2x^3 + d_3x^4$, then $y = q(x)e^x$ implies

$$\begin{array}{rcl}
\text{LHS} & = & y'' - 2y' + y \\
& = & [q(x)e^x]'' - 2[q(x)e^x]' + q(x)e^x \\
& = & q(x)e^x + 2q'(x)e^x + q''(x)e^x - 2q'(x)e^x - 2q(x)e^x + q(x)e^x \\
& = & q''(x)e^x \\
& = & [2d_1 + 6d_2x + 12d_2x^2]e^x.
\end{array}$$

Matching coefficients of Euler atoms left and right gives the 3×3 system of equations

$$\begin{array}{rcl}
2d_1 & = & 1, \\
6d_2 & = & 1, \\
12d_3 & = & -1.
\end{array}$$

Solve the equations. The 3×3 system is diagonal and needs no further analysis: $d_1 = 1/6$, $d_2 = 1/6$, $d_3 = -1/12$.

Report y_p. The trial solution after substitution of found coefficients d_1, d_2, d_3 becomes the particular solution

$$y_p = \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x - \frac{1}{12}x^4e^x.$$

General solution. The superposition relation $y = y_h + y_p$ is the general solution

$$y = c_1e^x + c_2xe^x + \frac{1}{2}x^2e^x + \frac{1}{6}x^3e^x - \frac{1}{12}x^4e^x.$$

Answer check. The `maple` code:

```
de:=diff(y(x),x,x)-2*diff(y(x),x)+y(x)=(1+x-x^2)*exp(x);
dsolve(de,y(x));
# y(x) = 1/2*exp(x)*x^2 + 1/6*exp(x)*x^3
#       -1/12*exp(x)*x^4+_C1*exp(x)+_C2*exp(x)*x
```

Example 6.32 (Annihilator)

Find the annihilator for $f(x) = x - 4\sin 3x$.

Solution: First, identify $f(x) = x - 4\sin 3x$ as a linear combination of the atoms x, $\sin 3x$. Euler's theorem implies that the characteristic polynomial must have roots $0, 3i, -3i$. Then the characteristic polynomial must contain these factors:

Roots $r = 0, 0$	Atoms $1, x$	Factor r^2
Roots $\pm 3i$	Atoms $\cos 3x, \sin 3x$	Factors $r - 3i$, $r + 3i$

Multiply the factors r^2 and $(r - 3i)(r + 3i) = r^2 + 9$ to generate the characteristic polynomial

$$(\text{factor } r^2) \text{ times } (\text{factor } r^2 + 9) = r^4 + 9r^2.$$

The annihilator is $y^{(4)} + 9y'' = 0$, obtained by translation of characteristic equation $r^4 + 9r^2 = 0$ into a differential equation.

Example 6.33 (Annihilator)

Find the annihilator for $f(x) = e^x(x^2 - 2\cos 3x)$.

Solution: Function $f(x) = e^x(x^2 - 2\cos 3x)$ is a linear combination of the atoms x^2e^x, $e^x \cos 3x$. Euler's theorem implies that the roots are $r = 1, 1, 1, 1 \pm 3i$. Then the characteristic polynomial must contain factors as follows.

Roots	Atoms	Factor
$r = 1, 1, 1$	e^x, xe^x, x^2e^x	$(r-1)^3$
$1 \pm 3i$	$e^x \cos 3x, e^x \sin 3x$	$(r - 1 - 3i)(r - 1 + 3i)$

Multiply the factors $(r - 1)^3$ and $(r - 1)^2 + 9$ to generate the characteristic equation

$$(r - 1)^3((r - 1)^2 + 9) = 0.$$

Expanding, the characteristic polynomial is $r^5 - 5\,r^4 + 19\,r^3 - 37\,r^2 + 32\,r - 10$. In applications, we would stop here, with the characteristic polynomial. If we continue, then the annihilator is the differential equation $y^{(5)} - 5y^{(4)} + 19y''' - 37y'' + 32y' - 10y = 0$.

Example 6.34 (Annihilator Method)
Find the shortest trial solution for the differential equation $y'' - y = x + xe^x$ using the Method of Annihilators.

Solution: The example was solved previously using Rule I and Rule II with answer

$$y_p = d_1 + d_2 x + d_3 x e^x + d_4 x^2 e^x.$$

Homogeneous equation. The characteristic polynomial for homogeneous equation $y'' - y = 0$ is $p(r) = r^2 - 1$. It has roots $r = 1$, $r = -1$ and corresponding atoms e^x, e^{-x}.
Annihilator for $f(x)$. The right side of the differential equation is $f(x) = x + xe^x$. We compute the characteristic polynomial $q(r)$ of an annihilator of $f(x)$. The atoms for $f, f', f'', \ldots$ are $1, x, e^x, xe^x$ with corresponding roots $0, 0, 1, 1$. The factors of the characteristic polynomial $q(r)$ are then r^2, $(r-1)^2$, by Euler's theorem. Specifically, we used these specialized conclusions from Euler's theorem:

1. Root $r = 0$ of $q(r) = 0$ has multiplicity 2 if and only if r^2 is a factor of $q(r)$;
2. Root $r = 1$ of $q(r) = 0$ has multiplicity 2 if and only if $(r-1)^2$ is a factor of $q(r)$.

The conclusion of this analysis is that $q(r) =$ product of the factors $= r^2(r-1)^2$.
Trial solution. Let

$$w(r) = p(r)q(r) = (r^2 - 1)r^2(r-1)^2 = r^2(r+1)(r-1)^3.$$

Then y_p must be a solution of the differential equation with characteristic equation $w(r) = 0$, which implies that y_p is a linear combination of the atoms

$$1, x, e^{-x}, e^x, xe^x, x^2 e^x.$$

Atoms e^{-x} and e^x are solutions of the homogeneous equation, therefore they are removed. The shortest trial solution is a linear combination of the Euler atoms

$$1, x, xe^x, x^2 e^x.$$

Then

$$y_p = d_1 + d_2 x + d_5 x e^x + d_6 x^2 e^x,$$

which agrees with the shortest trial solution obtained by Rule I and Rule II.

Exercises 6.5

Polynomial Solutions

Determine a polynomial solution y_p for the given differential equation.

1. $y'' = x$

2. $y'' = x - 1$

3. $y'' = x^2 - x$

4. $y'' = x^2 + x - 1$

5. $y'' - y' = 1$

6. $y'' - 5y' = 10$

7. $y'' - y' = x$

8. $y'' - y' = x - 1$

9. $y'' - y' + y = 1$

10. $y'' - y' + y = -2$

11. $y'' + y = 1 - x$

12. $y'' + y = 2 + x$

13. $y'' - y = x^2$

14. $y'' - y = x^3$

Polynomial-Exponential Solutions

Determine a solution y_p for the given differential equation.

15. $y'' + y = e^x$

16. $y'' + y = e^{-x}$

17. $y'' = e^{2x}$

18. $y'' = e^{-2x}$

19. $y'' - y = (x+1)e^{2x}$

20. $y'' - y = (x-1)e^{-2x}$

21. $y'' - y' = (x+3)e^{2x}$

22. $y'' - y' = (x-2)e^{-2x}$

23. $y'' - 3y' + 2y = (x^2+3)e^{3x}$

24. $y'' - 3y' + 2y = (x^2-2)e^{-3x}$

Sine and Cosine Solutions

Determine a solution y_p for the given differential equation.

25. $y'' = \sin(x)$

26. $y'' = \cos(x)$

27. $y'' + y = \sin(x)$

28. $y'' + y = \cos(x)$

29. $y'' = (x+1)\sin(x)$

30. $y'' = (x+1)\cos(x)$

31. $y'' - y = (x+1)e^x \sin(2x)$

32. $y'' - y = (x+1)e^x \cos(2x)$

33. $y'' - y' - y = e^x \sin(2x)$

34. $y'' - y' - y = (x^2+x)e^x \cos(2x)$

Undetermined Coefficients Algorithm

Determine a solution y_p for the given differential equation.

35. $y'' = x + \sin(x)$

36. $y'' = 1 + x + \cos(x)$

37. $y'' + y = x + \sin(x)$

38. $y'' + y = 1 + x + \cos(x)$

39. $y'' + y = \sin(x) + \cos(x)$

40. $y'' + y = \sin(x) - \cos(x)$

41. $y'' = x + xe^x + \sin(x)$

42. $y'' = x - xe^x + \cos(x)$

43. $y'' - y = \sinh(x) + \cos^2(x)$

44. $y'' - y = \cosh(x) + \sin^2(x)$

45. $y'' + y' - y = x^2 e^x$

46. $y'' + y' - y = xe^x \sin(2x)$

Roots and Related Atoms

Euler atoms A and B are said to be **related** if and only if the derivative lists A, A', ... and B, B', ... share a common Euler atom.

47. Find the roots, listed according to multiplicity, for the atoms 1, x, x^2, e^{-x}, $\cos 2x$, $\sin 3x$, $x\cos \pi x$, $e^{-x}\sin 3x$.

48. Find the roots, listed according to multiplicity, for the atoms 1, x^3, e^{2x}, $\cos x/2$, $\sin 4x$, $x^2 \cos x$, $e^{3x}\sin 2x$.

49. Let $A = xe^{-2x}$ and $B = x^2e^{-2x}$. Verify that A and B are related.

50. Let $A = xe^{-2x}$ and $B = x^2e^{2x}$. Verify that A and B are not related.

51. Prove that atoms A and B are related if and only if their base atoms have the same roots.

52. Prove that atoms A and B are related if and only if they are in the same **group**. See page 474 for the definition of a group of atoms.

Modify a Trial Solution

Apply Rule II to modify the given Rule I trial solution into the shortest trial solution.

53. The characteristic equation has factors r^3, $(r^3 + 2r^2 + 2)$, $(r - 1)^2$, $(r + 1)$, $(r^2 + 4)^3$ and the Rule I trial solution is constructed from atoms 1, x, e^x, xe^x, e^{-x}, $\cos 2x$, $\sin 2x$, $\cos x$, $\sin x$.

54. The characteristic equation has factors r^2, $(r^3 + 3r^2 + 2)$, $(r + 1)$, $(r^2 + 4)^3$ and the Rule I trial solution is constructed from atoms 1, x, e^x, xe^x, e^{-x}, $\cos 2x$, $\sin 2x$.

Annihilators and Laplace Theory

Laplace theory can construct the annihilator of $f(t)$. The example $y'' + 4y = t + 2t^3$ is used to discuss the techniques. Formulas to be justified: $p(s) = \mathcal{L}(f)/\mathcal{L}(y)$ and $q(s) = \mathbf{denom}(\mathcal{L}(f(t)))$.

55. **(Transfer Function)** Find the characteristic polynomial $q(r)$ for the homogeneous equation $y'' + 4y = 0$. The transfer function for $y'' + 4y = f(t)$ is $\mathcal{L}(y)/\mathcal{L}(f)$, which equals $1/q(s)$.

56. **(Laplace of $y_p(t)$)**
The Laplace of $y(t)$ for problem $y'' + 4y = f(t)$, $y(0) = y'(0) = 0$ must equal the Laplace of $f(t)$ times the transfer function. Justify and explain what it has to do with finding y_p.

57. **(Annihilator of $f(t)$)**
Let $g(t) = t + 2t^3$. Verify that $\mathcal{L}(g(t)) = \dfrac{s^2 + 12}{s^4}$, which is a proper fraction with denominator s^4. Then explain why one annihilator of $g(t)$ has characteristic polynomial r^4. The result means that $y = g(t) = t + 2t^3$ is a solution of $y'''' = 0$.

58. **(Laplace Theory finds y_p)**
Show that the problem $y'' + 4y = t + 2t^3$, $y(0) = y'(0) = 0$ has Laplace transform

$$\mathcal{L}(y) = \frac{s^2 + 12}{(s^2 + 4)s^4}.$$

Explain why $y(t)$ must be a solution of the constant-coefficient homogeneous differential equation having characteristic polynomial $w(r) = (r^2 + 4)r^4$.

Annihilator Method Justified

The method of annihilators can be justified by successive differentiation of a nonhomogeneous differential equation, then forming a linear combination of the resulting formulas. It is carried out here, for exposition efficiency, for the nonhomogeneous equation $y'' + 4y = x + 2x^3$. The right side is $f(x) = x + 2x^3$ and the homogeneous equation is $y'' + 4y = 0$.

59. **(Homogeneous equation)**
Verify that $y'' + 4y = 0$ has characteristic polynomial $q(r) = r^2 + 4$.

60. **(Annihilator)**
Verify that $y^{(4)} = 0$ is an annihilator for $f(x) = x + 2x^3$, with characteristic polynomial $q(r) = r^4$.

61. **(Composite Equation)**
Differentiate four times across the equation $y'' + 4y = f(x)$ to obtain $y^{(6)} + 4y^{(4)} = f^{(4)}(x)$. Argue that $f^{(4)}(x) = 0$ because $y^{(4)} = 0$ is an annihilator of $f(x)$. This proves that y_p is a solution of higher order equation $y^{(6)} + 4y^{(4)} = 0$. Then argue that $w(r) = r^4(r^2 + 4)$ is the characteristic polynomial of the equation $y^{(6)} + 4y^{(4)} = 0$.

62. **(General Solution)**
Solve the homogeneous composite equation $y^{(6)} + 4y^{(4)} = 0$ using its characteristic polynomial $w(r) = r^4(r^2 + 4)$.

63. **(Extraneous Atoms)**
Argue that the general solution from the previous exercise contains two terms constructed from atoms derived from roots of the polynomial $q(r) = r^2 + 4$. Remove these terms to obtain the shortest expression for y_p and explain why it works.

64. **(Particular Solution)**
Report the form of the shortest particular solution of $y'' + 4y = f(x)$, according to the previous exercise.

6.6 Undamped Mechanical Vibrations

The study of vibrating mechanical systems begins here with examples for undamped systems with one degree of freedom. The main example is a mass on a spring. The undamped, unforced cases are considered in a number of physical examples, which include the following: simple pendulum, compound pendulum, swinging rod, torsional pendulum, shockless auto, sliding wheel, rolling wheel.

Simple Harmonic Motion

Consider the spring-mass system of Figure 2, where x measures the signed distance from the equilibrium position of the mass. The spring is assumed to exert a force under both compression and elongation. Such springs are commonly used in automotive suspension systems, notably coil springs and leaf springs. In the case of coil springs, there is normally space between the coils, allowing the spring to exert bidirectional forces.

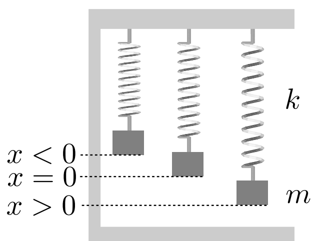

Figure 2. An Undamped Spring-Mass System.
Compression, equilibrium and elongation of the spring are shown with corresponding positions of the mass m.

Hooke's Law. The basic physical law to be applied is:

> The linear restoring force F exerted by a spring is proportional to the signed elongation X, briefly, $F = -kX$.

The number k is called **Hooke's constant** for the spring. In the model of Figure 2, $X = x(t)$ and $k > 0$. The minus sign accounts for the action of the force: the spring tries to **restore** the mass to the equilibrium state, so the vector force is directed toward the equilibrium position $x = 0$.

Newton's Second Law. Specialized to the model in Figure 2, Newton's second law says:

> The force F exerted by a mass m attached to a spring is $F = ma$ where $a = d^2x/dt^2$ is the acceleration of the mass.

The **Weight** $W = mg$ is defined in terms of the **Gravitational Constant** $g = 32$ ft/s^2, 9.8 m/s^2 or 980 cm/s^2 where the mass m is given respectively in **slugs**, **kilograms** or **grams**. The weight is the force due to gravity and it has the appropriate units for a force: **pounds** in the case of the fps system of units.

Method of Force Competition

Hooke's law $F = -kx(t)$ and Newton's second law $F = mx''(t)$ give two independent equations for the force acting on the system. Equating competing forces implies that the signed displacement $x(t)$ satisfies the **Free Vibration** equation

$$mx''(t) + kx(t) = 0.$$

It is also called the **Harmonic Oscillator** in its equivalent form

$$x''(t) + \omega^2 x(t) = 0, \quad \omega^2 = \frac{k}{m}.$$

In this context, ω is the **Natural Frequency** of the free vibration. The harmonic oscillator is said to describe a **Simple Harmonic Motion** $x(t)$. By Theorem 6.1 page 430:

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

Background: Fundamental Trigonometric Identities

The identities used repeatedly in differential equations applications are:

$\cos^2\theta + \sin^2\theta = 1$ $1 + \tan^2\theta = \sec^2\theta$ $\cot^2\theta + 1 = \csc^2\theta$	**Pythagorean identities.** $\boxed{1}$
$\sin(-\theta) = -\sin(\theta)$ $\cos(-\theta) = \cos(\theta)$	**Odd-even identities.** $\boxed{2}$

$\boxed{1}$: Divide the first by $\cos^2\theta$ or $\sin^2\theta$ to derive the others.

$\boxed{2}$: Identities like $\tan(-\theta) = -\tan(\theta)$ can be derived as needed from these two identities, e.g., $\tan\theta = \sin\theta/\cos\theta$.

$\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$ $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$	**Sum identities.** $\boxed{3}$
$\sin(a-b) = \sin(a)\cos(b) - \sin(b)\cos(a)$ $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$	**Difference identities.** $\boxed{4}$

$\boxed{3}$: Obtain the second from the first by differentiation on symbol a, holding b constant.

$\boxed{4}$: Both follow from the sum identities by replacing symbol b by $-b$, then apply the even-odd relations.

Background: Harmonic Motion

It is known from trigonometry that

$$x(t) = A\cos(\omega t - \alpha)$$

has **Amplitude** A, **Period** $2\pi/\omega$ and **Phase shift** α/ω. A full period is called a **Cycle** and a half-period a **Semicycle**. The **Frequency** $\omega/(2\pi)$ is the number of complete cycles per second, or the reciprocal of the period.

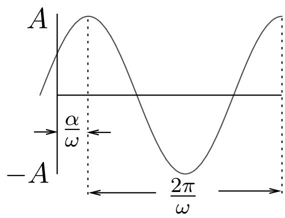

Figure 3. Simple Harmonic Motion. Shown is $x(t) = A\cos(\omega t - \alpha)$, period $2\pi/\omega$, phase shift α/ω and amplitude A.

Visualization of Harmonic Motion

A simple harmonic motion can be obtained graphically by means of the experiment shown in Figure 4, in which an undamped spring-mass system has an attached pen that writes on a moving paper chart. The chart produces the simple harmonic motion $x(t) = c_1\cos\omega t + c_2\sin\omega t$ or equivalently $x(t) = A\cos(\omega t - \alpha)$.

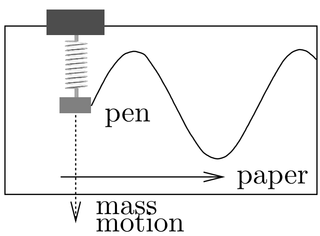

Figure 4. A Chart from Harmonic Motion. A moving paper chart records the vertical motion of a mass on a spring using an attached pen.

Phase-Amplitude Conversion

Given a simple harmonic motion $x(t) = c_1\cos\omega t + c_2\sin\omega t$, as in Figure 3, define **Amplitude** A and **Phase angle** α by the formulas

$$A = \sqrt{c_1^2 + c_2^2}, \quad c_1 = A\cos\alpha \quad \text{and} \quad c_2 = A\sin\alpha.$$

Then the simple harmonic motion has the **Phase-Amplitude form**

$$x(t) = A\cos(\omega t - \alpha). \tag{1}$$

Details. Equation (1) is derived from the cosine difference identity page 491 and basic triangle definitions of sine and cosine.

$x(t) = c_1\cos\omega t + c_2\sin\omega t$	Harmonic oscillator $x'' + \omega^2 x = 0$, general solution.

$x(t) = A\cos\alpha\cos\omega t + A\sin\alpha\sin\omega t$	Insert identities $c_1 = A\cos\alpha$ and $c_2 = A\sin\alpha$.
$x(t) = A\cos(\omega t - \alpha)$	Use $a = \omega t$ and $b = \alpha$ in the cosine difference identity.

Phase Shift Calculations. The phase shift is the amount of horizontal translation required to shift the cosine curve $\cos(\omega t - \alpha)$ so that its graph is atop $\cos(\omega t)$. To find the phase shift from equation (1), set the argument of the cosine term to zero, then solve for t.

To solve for $\alpha \geq 0$ and less than 2π, the expected range, form equations $c_1 = A\cos\alpha$, $c_2 = A\sin\alpha$, then compute numerically by calculator the radian angle $\phi = \arctan(c_2/c_1)$, $|\phi| < \pi/2$. Quadrantial angle rules are applied when $c_1 = 0$ or $c_2 = 0$. Calculators return a division by zero error for $c_1 = 0$ and maybe $\phi = 0$ for $c_2 = 0$, the latter incorrect if $c_1 < 0$. Computers should have **atan2**, a C library function that accepts c_1, c_2 and returns angle $|\phi| < \pi/2$. A calculator or computer answer that is negative requires correction by adding 2π to the radian answer. The corrected answer would give $\cos(\omega t - \alpha - 2\pi)$ instead of $\cos(\omega t - \alpha)$, however the cosine has period 2π: the phase-amplitude answers are equal.

Applications

Considered below are a variety of models with pendulum-like motion. The illustrations start with the simple pendulum and end with applications to auto suspension systems and rolling wheels.

Simple Pendulum

A pendulum is constructed from a thin massless wire or rod of length L and a body of mass m, as in Figure 5.

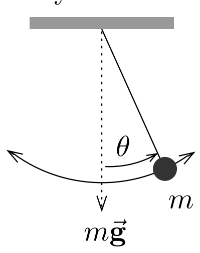

Figure 5. A Simple Pendulum

Derived below is the **Pendulum Equation**

$$\theta''(t) + \frac{g}{L}\sin\theta(t) = 0. \tag{2}$$

Details: Along the circular arc traveled by the mass, the velocity is ds/dt where $s = L\theta(t)$ is arclength. The acceleration is $L\theta''(t)$. Newton's second law for the force along this arc is $F = mL\theta''(t)$. Another relation for the force can be found by resolving the vector gravitational force $m\vec{\mathbf{g}}$ into its normal and tangential components. By trigonometry, the tangential component gives a second force equation $F = -mg\sin\theta(t)$. Equate competing forces and cancel m to obtain (2).

Because the mass m cancels from the equation, the pendulum oscillation depends only upon the length of the string and not upon the mass!

The **Linearized pendulum** equation is

$$\Theta''(t) + \frac{g}{L}\Theta(t) = 0. \tag{3}$$

Details: Approximation $\sin u \approx u$ is valid for small angles u. Apply the approximation to (2). The result is the **linearized pendulum** (3).

Equation (2) is indistinguishable from the classical harmonic oscillator, except for variable names. The solution of (3):

$$\Theta(t) = A\cos(\omega t - \alpha), \quad \omega^2 = g/L$$

Gymnast Swinging about a Horizontal Bar

The mass m of the gymnast is assumed concentrated at the center of the gymnast's physical height H. The problem is then simplified to a pendulum motion with $L = H/2$. The resulting equation of motion for the angle θ between the gravity vector and the gymnast is by equation (2) the **Gymnast's Equation**

$$\theta''(t) + \frac{2g}{H}\sin\theta(t) = 0. \tag{4}$$

The linearized version of this equation is not interesting, because the angle θ is never small. Commonly, $\theta(t)$ goes through many multiples of 2π radians, during an exercise.

Physical Pendulum

The **Compound Pendulum** or **Physical Pendulum** is a rigid body of total mass m having center of mass C which is suspended from a fixed origin O – see Figure 6.

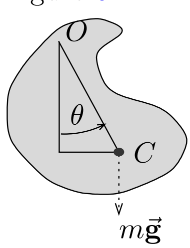

Figure 6. A Physical Pendulum

Derived by force competition is the **Compound Pendulum** equation

$$\theta''(t) + \frac{mgd}{I}\sin\theta(t) = 0. \tag{5}$$

Details: The vector $\vec{r}$ from O to C has magnitude $d = \|\vec{r}\| > 0$. The gravity force vector $\vec{\mathbf{G}} = m\vec{\mathbf{g}}$ (mass $\times$ acceleration due to gravity) makes angle θ with vector $\vec{r}$. The restoring torque $\vec{r} \times \vec{\mathbf{G}}$ has magnitude $F = -\|\vec{r} \times \vec{\mathbf{G}}\| = -\|\vec{r}\|\|\vec{\mathbf{G}}\| \sin\theta = -mgd \sin\theta$. Newton's second law gives a second force equation $F = I\theta''(t)$ where I is the torque of the rigid body about O. Force competition results in equation (5).

Approximation $\sin u \approx u$ applied to equation (5) gives a harmonic oscillator known as the **linearized compound pendulum**:

$$\Theta''(t) + \omega^2\Theta(t) = 0, \quad \omega = \sqrt{\frac{mgd}{I}}. \tag{6}$$

Swinging Rod

As depicted in Figure 7, a swinging rod is a special case of the compound pendulum. Assumed for the modeling is a rod of length L and mass m, with uniform mass density.

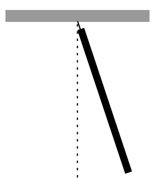

Figure 7. A Swinging Rod

The **Swinging Rod** equation

$$\theta''(t) + \frac{3g}{2L}\sin\theta(t) = 0 \tag{7}$$

will be derived from the compound pendulum equation (5).

Details: The center of mass distance $d = L/2$ appears in the calculus torque relation $I = mL^2/3$. Then:

$$\frac{mgd}{I} = \frac{3mgL}{2mL^2} = \frac{3g}{2L}$$

Insert this relation into the compound pendulum equation (5). The result is the **swinging rod** equation (7).

If equation (6) is used instead (5), then the result is the **linearized swinging rod** equation

$$\Theta''(t) + \omega^2\Theta(t) = 0, \quad \omega = \sqrt{\frac{3g}{2L}}. \tag{8}$$

Torsional Pendulum

A model for a balance wheel in a watch, a gavanometer or a Cavendish torsional balance is the torsional pendulum, which is a rigid body suspended by a solid wire – see Figure 8.

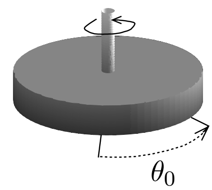

Figure 8. A Torsional Pendulum.
An example is a balance wheel in a watch. The wheel rotates angle θ_0 about the vertical axis, which acts as a spring, exerting torque I against the rotation.

The **Torsional Pendulum** equation:

$$\theta_0''(t) + \omega^2 \theta_0(t) = 0, \quad \omega = \sqrt{\frac{\kappa}{I}}. \tag{9}$$

Details: The wire undergoes twisting, which exerts a restoring force $F = -\kappa\theta_0$ when the body is rotated through angle θ_0. There is no small angle restriction on this restoring force, because it acts in the spirit of Hooke's law like a linear spring restoring force. The model uses Newton's second law force relation $F = I\theta_0''(t)$, as in the physical pendulum. Force competition against the restoring force $F = -\kappa\theta_0$ gives the **torsional pendulum** equation (9).

Shockless Auto

An automobile loaded with passengers is supported by four coil springs, as in Figure 9, but all of the shock absorbers are worn out. The simplistic linear model $mx''(t) + kx(t) = 0$ will be applied. The plan is to estimate the number of seconds it takes for one complete oscillation. This is the time between two consecutive *bottom–outs* of the automobile.[5]

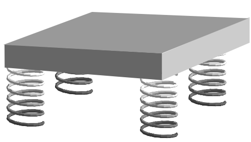

Figure 9. Car on Four Springs: Linear Model

Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter. The load is divided among the four springs equally, so each spring supports $m = 1350/4$ Kg. Let ω be the natural frequency of vibration. Then the number of seconds for one complete oscillation is the period $T = 2\pi/\omega$ seconds. The free vibration model for one spring is

$$\frac{1350}{4} x''(t) + 20000x(t) = 0.$$

The harmonic oscillator form is $x'' + \omega^2 x = 0$, where $\omega^2 = \frac{20000(4)}{1350} = 59.26$. Therefore, $\omega = 7.70$. Then the period is $T = 2\pi/\omega = 0.82$ seconds. The interpretation: the auto bottoms-out every 0.82 seconds.

[5]Teenagers popularized late-night cruising of Los Angeles boulevards in shockless 4-door sedans. They disabled the shock absorbers and modified the suspension to give a completely undamped ride.

Rolling Wheel on a Spring

A wheel of total mass m and radius R is attached at its center to a spring of Hooke's constant k, as in Figure 10. The wheel rolls without slipping. The spring is assumed to have negligible mass and zero kinetic energy. Let k be the Hooke's constant for the spring. Let $x(t)$ be the elongation of the spring from equilibrium, $x > 0$ corresponding to the wheel rolling to the right and $x < 0$ corresponding to the wheel rolling to the left.

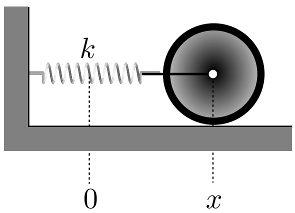

Figure 10. A Rolling Wheel on a Spring.

Derived below is the **Rolling Wheel Equation**

$$mx''(t) + \frac{2}{3}\,kx(t) = 0. \tag{10}$$

Details: The spring does not react only to tension, but it reacts like a coil spring with spacing that restores bi-directionally to equilibrium.

Figure 11. Restoring Force $F = kx$.
By Hooke's law, the spring restores to equilibrium for both compression and elongation.

If the wheel slides frictionless, then the model is the harmonic oscillator equation $mx''(t) + kx(t) = 0$. A wheel that rolls without slipping has inertia, and consideration of this physical difference will be shown to give equation (10).

A curious consequence is that $x(t)$ is identical to the frictionless sliding wheel with spring constant reduced from k to $2k/3$. This makes sense physically, because rolling wheel inertia is observed to reduce the apparent stiffness of the spring.

The derivation begins with the energy conservation law

$$\text{Kinetic} + \text{Potential} = \text{constant}.$$

The kinetic energy T is the sum of two energies, $T_1 = \frac{1}{2}mv^2$ for translation and $T_2 = \frac{1}{2}I\omega^2$ for the rolling wheel, whose inertia is $I = \frac{1}{2}mR^2$. The velocity is $v = R\omega = x'(t)$. Algebra gives $T = T_1 + T_2 = \frac{3}{4}mv^2$. The potential energy is $K = \frac{1}{2}kx^2$ for a spring of Hooke's constant k. Application of the energy conservation law $T + K = c$ gives the equation $\frac{3}{4}m(x'(t))^2 + \frac{1}{2}k(x(t))^2 = c$. Differentiate this equation on t to obtain $\frac{3}{2}mx'(t)x''(t) + kx(t)x'(t) = 0$, then cancel $x'(t)$ to give equation (10).

Examples and Methods

Example 6.35 (Harmonic Vibration)
A mass of $m = 250$ grams attached to a spring of Hooke's constant k undergoes free undamped vibration. At equilibrium, the spring is stretched 25 cm by a force of 8 Newtons. At time $t = 0$, the spring is stretched 0.5 m and the mass is set in motion with initial velocity 5 m/s directed downward from equilibrium. Find:

(a) The numerical value of Hooke's constant k.

(b) The initial value problem for vibration $x(t)$.

Solution:
(a): Hooke's law Force=k(elongation) is applied with force 8 Newtons and elongation $25/100 = 1/4$ meter. Equation $8 = k(1/4)$ implies $k = 32$ N/m.

(b): Given $m = 250/1000$ kg and $k = 32$ N/m from part (a), then the free vibration model $mx'' + kx = 0$ becomes $\frac{1}{4}x'' + 32x = 0$. Initial conditions are $x(0) = 0.5$ m and $x'(0) = 5$ m/s. The initial value problem is

$$\begin{cases} \dfrac{d^2x}{dt^2} + 128x &= 0, \\ x(0) &= 0.5, \\ x'(0) &= 5. \end{cases}$$

Example 6.36 (Phase-Amplitude Conversion)
Write the vibration equation

$$x(t) = 2\cos(3t) + 5\sin(3t)$$

in phase-amplitude form $x = A\cos(\omega t - \alpha)$. Create a graphic of $x(t)$ with labels for period, amplitude and phase shift.

Solution:
The answer and the graphic appear below.

$$x(t) = \sqrt{29}\cos(3t - 1.190289950) = \sqrt{29}\cos(3(t - 0.3967633167)).$$

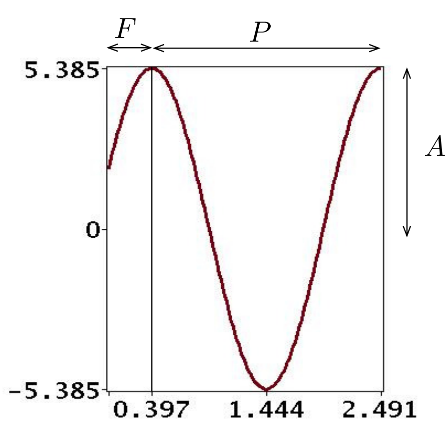

Figure 12. Harmonic Oscillation.

The graph of $2\cos(3t) + 5\sin(3t)$. It has amplitude $A = \sqrt{29} = 5.385$, period $P = 2\pi/3$ and phase shift $F = 0.3967633167$. The graph is on $0 \le t \le P + F$.

Algebra Details. The plan is to re-write $x(t)$ in the form $x(t) = A\cos(\omega t - \alpha)$, called the phase-amplitude form of the harmonic oscillation. The main tools from trigonometry appear on page 491.

Start with $x(t) = 2\cos(3t) + 5\sin(3t)$. Compare the expression for $x(t)$ with Trig identity $x(t) = A\cos(\omega t - \alpha) = A\cos(\alpha)\cos(\omega t) + A\sin(\alpha)\sin(\omega t)$. Then define accordingly

$$\omega = 3, \quad A\cos(\alpha) = 2, \quad A\sin(\alpha) = 5.$$

The Pythagorean identity $\cos^2\alpha + \sin^2\alpha = 1$ implies $A^2 = 2^2 + 5^2 = 29$ and then the amplitude is $A = \sqrt{29}$. Because $\cos\alpha = 2/A$, $\sin\alpha = 5/A$, then both the sine and cosine are positive, placing angle α in quadrant I. Divide equations $\cos\alpha = 2/A$, $\sin\alpha = 5/A$ to obtain $\tan(\alpha) = 5/2$, which by calculator implies $\alpha = \arctan(5/2) = 1.190289950$ radians or 68.19859051 degrees. Then $x(t) = A\cos(\omega t - \alpha) = \sqrt{29}\cos(3t - 1.190289950)$.

Computer Details. Either equation for $x(t)$ can be used to produce a computer graphic. A hand-drawn graphic would use only the phase-amplitude form. The period is $P = 2\pi/\omega = 2\pi/3$. The amplitude is $A = \sqrt{29} = 5.385164807$ and the phase shift is $F = \alpha/\omega = 0.3967633167$. The graph is on $0 \le t \le P + F$.

```
# Maple
F:=evalf(arctan(5/2)/3); P:=2*Pi/3;A:=sqrt(29);
X:=t->2*cos(3*t)+5*sin(3*t);
opts:=xtickmarks=[0,F,P/2+F,P+F],ytickmarks=[-A,0,A],
      axes=boxed,thickness=3,labels=["",""];
plot(X(t),t=0..P+F,opts);
```

Example 6.37 (Undamped Spring-Mass System)

A mass of 6 Kg is attached to a spring that elongates 20 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity -5 m/s. Find an equation for $x(t)$ using the free undamped vibration model $mx'' + kx = 0$.

Solution: The answer is $x(t) = -\sqrt{\frac{5}{2}}\sin(\sqrt{10}t)$.

The mass is $m = 6$ kg. Hooke's law $F = kx$ is applied with $F = 12$ N and $x = 20/100$ m. Then Hooke's constant is $k = 60$ N/m. Initial conditions are $x(0) = 0$ m (equilibrium) and $x'(0) = -5$ m/s. The model is

$$\begin{cases} 6\dfrac{d^2x}{dt^2} + 60x &= 0, \\ x(0) &= 0, \\ x'(0) &= -5. \end{cases}$$

Solve the Initial Value Problem. The characteristic equation $6r^2 + 60 = 0$ is solved for $r = \pm i\sqrt{10}$, then the Euler solution atoms are $\cos(\sqrt{10}t)$, $\sin(\sqrt{10}t)$. The general solution is a linear combination of Euler atoms:

$$x(t) = c_1\cos(\sqrt{10}t) + c_2\sin(\sqrt{10}t).$$

The task remaining is determination of constants c_1, c_2 subject to initial conditions $x(0) = 0$, $x'(0) = -5$. The linear algebra problem uses the derivative formula

$$x'(t) = -\sqrt{10}c_1\sin(\sqrt{10}t) + \sqrt{10}c_2\cos(\sqrt{10}t).$$

The 2×2 system of linear algebraic equations for c_1, c_2 is obtained from the two equations $x(0) = 0$, $x'(0) = -5$ as follows.

$$\begin{cases} \cos(0)c_1 & + & \sin(0)c_2 & = & 0, & \text{Equation } x(0) = 0 \\ -\sqrt{10}\sin(0)c_1 & + & \sqrt{10}\cos(0)c_2 & = & -5, & \text{Equation } x'(0) = -5 \end{cases}$$

Because $\cos(0) = 1$, $\sin(0) = 0$, then $c_1 = 0$ and $c_2 = -5/\sqrt{10} = -\sqrt{5/2}$. Insert answers c_1, c_2 into the general solution to find the answer to the initial value problem:

$$x(t) = -\sqrt{\frac{5}{2}}\sin(\sqrt{10}t).$$

Example 6.38 (Pendulum)

A simple linearized pendulum of length 2.5 m oscillates with angle variable $\theta(t)$ satisfying $\theta(0) = 0$ (equilibrium position) and $\theta'(0) = 3$ (radial velocity). Find $\theta(t)$ in phase-amplitude form and report the period, amplitude and phase shift.

Solution: The answer is $\theta(t) = 3\sqrt{\frac{25}{98}}\sin\left(\sqrt{\frac{98}{25}}\,t\right)$, which has amplitude $3\sqrt{\dfrac{25}{98}}$, period $2\pi\sqrt{\dfrac{25}{98}}$, phase shift zero.

The mass is not given, because we use model equation (3), $\theta''(t) + \frac{g}{L}\theta(t) = 0$, in which $g = 9.8$ and $L = 2.5$. Then the initial value problem is

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{98}{25}\theta & = & 0, \\ \theta(0) & = & 0, \\ \theta'(0) & = & 3. \end{cases}$$

Solve the Initial Value Problem. The characteristic equation $r^2 + \frac{98}{25} = 0$ is solved for $r = \pm i\omega$ where $\omega = \sqrt{\frac{98}{25}}$. The Euler solution atoms are $\cos(\omega t)$, $\sin(\omega t)$. The general solution:

$$\theta(t) = c_1\cos(\omega t) + c_2\sin(\omega t).$$

The task remaining is determination of constants c_1, c_2 subject to initial conditions $\theta(0) = 0$, $\theta'(0) = 3$. The linear algebra problem uses the derivative formula

$$\theta'(t) = -\omega c_1\sin(\omega t) + \omega c_2\cos(\omega t).$$

The 2×2 system of equations for c_1, c_2 is obtained from equations $\theta(0) = 0$, $\theta'(0) = 3$ as follows.

$$\begin{cases} \cos(0)c_1 & + & \sin(0)c_2 & = & 0, & \text{Equation } \theta(0) = 0 \\ -\omega\sin(0)c_1 & + & \omega\cos(0)c_2 & = & 3, & \text{Equation } \theta'(0) = 3 \end{cases}$$

Because $\cos(0) = 1$, $\sin(0) = 0$, then $c_1 = 0$ and $c_2 = 3/\omega = 3\sqrt{\frac{25}{98}}$. The solution to the initial value problem is

$$\theta(t) = 3\sqrt{\frac{25}{98}}\sin\left(\sqrt{\frac{98}{25}}\,t\right).$$

Example 6.39 (Gymnast)
Consider the change of variables $x(t) = \theta(t)$, $y(t) = \theta'(t)$, called the **position-velocity substitution**. Re-write the gymnast equation (4), $\theta'' + \frac{2g}{H}\sin\theta = 0$, in the form

$$\begin{aligned} \frac{dx}{dt} &= y(t), \\ \frac{dy}{dt} &= -\frac{2g}{H}\sin(x(t)). \end{aligned} \tag{11}$$

Apply the method of quadrature to develop the equation for the **total mechanical energy**

$$\frac{1}{2}y^2 + \frac{2g}{H}(1 - \cos x) = E. \tag{12}$$

Solution: The terms in the energy equation (12) are $\frac{1}{2}y^2$, called the **Kinetic Energy**, and $\omega^2(1 - \cos x)$, called the **Potential Energy**. We will show that $E = \frac{1}{2}y(0)^2$.

Details for (11): Define $x(t) = \theta(t)$ and $y(t) = \theta'(t)$. Then

$$\begin{aligned} x' &= \theta' \\ &= y & &\text{Used } x(t) = \theta(t) \text{ and } y(t) = \theta'(t). \\ y' &= \theta'' \\ &= -\frac{2g}{H}\sin(\theta) & &\text{Used } x(t) = \theta(t) \text{ and } \theta'' + \tfrac{2g}{H}\sin\theta = 0. \\ &= -\frac{2g}{H}\sin(x) \end{aligned}$$

Details for (12): Because $y = x'$, we multiply the second equation in (11) by y and then re-write the resulting equation as

$$yy' = -\frac{2g}{H}x'\sin(x).$$

This is a quadrature equation. Integrate on variable t across the equation to obtain for some constant C the identity

$$\frac{1}{2}y^2 = \frac{2g}{H}\cos(x) + C.$$

Let $t = 0$ in this equation to evaluate $C = \frac{1}{2}(y(0))^2 - \frac{2g}{H}$. Then rearrange terms to obtain the equation

$$\frac{1}{2}y^2 + \frac{2g}{H}(1 - \cos(x)) = \frac{1}{2}(y(0))^2.$$

This is equation (12) with $E = \frac{1}{2}(y(0))^2$.

Example 6.40 (Swinging Rod)
A uniform rod of length 16 cm swings from a support at origin $\mathcal{O}$. The motion started at angle $\theta(0) = \pi/12$ radians with radial velocity zero. Find approximate equations for the motion at the extreme end of the rod in rectangular coordinates.

Solution: The answer is

$$\begin{aligned} x(t) &= \frac{16}{100}\cos(\theta(t)), \\ y(t) &= \frac{16}{100}\sin(\theta(t)), \\ \theta(t) &= \frac{\pi}{12}\cos\left(\frac{t}{2}\sqrt{735}\right). \end{aligned}$$

The mass is not given, because we use model equation (8), $\theta''(t) + \frac{3g}{2L}\sin(\theta(t)) = 0$, in which $g = 9.8$ m/s^2 and $L = 16/100$ m. Then the initial value problem is

$$\begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{735}{4}\sin(\theta) &= 0, \\ \theta(0) &= \pi/2, \\ \theta'(0) &= 0. \end{cases}$$

The linearized equation will be used to find an approximate formula for the motion. The initial value problem is

$$(13) \qquad \begin{cases} \dfrac{d^2\theta}{dt^2} + \dfrac{735}{4}\theta &= 0, \\ \theta(0) &= \pi/12, \\ \theta'(0) &= 0. \end{cases}$$

The rectangular coordinates for the end of the rod are

$$x(t) = L\cos(\theta(t)), \quad y(t) = L\sin(\theta(t)).$$

Solve the Initial Value Problem. As in two previous examples, system (8) is readily solved with general solution

$$\theta(t) = c_1\cos(\omega t) + c_2\sin(\omega t), \quad \omega = \frac{\sqrt{735}}{2}.$$

Initial conditions imply $c_1 = \frac{\pi}{12}, c_2 = 0$. Details not supplied. Then

$$\theta(t) = \frac{\pi}{12}\cos\left(\frac{\sqrt{735}}{2}t\right).$$

Final Answer. The formula for $\theta(t)$ is inserted into polar coordinate equations $x = r\cos\theta, y = r\sin\theta$ with $r = L$ to obtain the reported answers.

Example 6.41 (Torsional Pendulum)

The balance wheel of a classical watch oscillates with angular amplitude π radians and period 0.5 seconds. Find the following values.

(a) The maximum angular speed of the balance wheel.

(b) The angular speed when the angle equals $\pi/2$ radians.

(c) The angular acceleration when the angle equals $\pi/4$ radians.

Solution: The answers are (a) $4\pi^2$, (b) $-2\pi^2\sqrt{3}$, (c) $-4\pi^3$.

The model is equation (9), $\theta_0''(t) + \omega^2\theta_0(t) = 0$, where $\omega = \sqrt{\frac{\kappa}{I}}$. The general solution in phase-amplitude form is $\theta_0(t) = A\cos(\omega t - \alpha)$, with constants A, α replacing the constants c_1, c_2 in a general solution. We are given that $A = \pi$. The period $\frac{2\pi}{\omega}$ equals 0.5, which implies $\omega = 4\pi$. Then

$$\theta_0(t) = \pi\cos(4\pi t - \alpha).$$

The constant α is undetermined by the information supplied.

(a): The angular speed is $\theta_0'(t) = -4\pi^2\sin(4\pi t - \alpha)$. It is a maximum when the sine factor equals -1. Then $\theta_0'(t) = 4\pi^2$ is the maximum angular speed of the balance wheel.

(b): The angle $\theta_0(t) = \pi/2$ is valid only when the cosine factor in $\theta_0(t) = \pi\cos(4\pi t - \alpha)$ is equal to $1/2$. Then $\sin(4\pi t - \alpha) = \sqrt{3}/2$, from trigonometry. The angular speed at this moment is $\theta_0'(t) = -4\pi^2\sin(\omega t - \alpha) = -2\pi^2\sqrt{3}$.

(c): Apply the equation $\theta_0''(t) + \omega^2\theta_0(t) = 0$ to obtain the acceleration relation $\theta_0''(t) = -16\pi^2\theta_0(t)$. When $\theta_0(t) = \pi/4$, then the acceleration equals $-4\pi^3$.

Example 6.42 (Shockless Auto)

A shockless auto of total mass 1400 kg bounces on a level street, making 8 **bottom-outs** in 10 seconds. Estimate the Hooke's constant k for each of the four coil springs.

Solution: The answer is $k = 8750\pi^2 \approx 86596$.

The model equation $mx'' + kx = 0$ is used. Then $x(t) = A\cos(\omega t - \alpha)$ is a general solution, with A and α constant and $\omega^2 = \frac{k}{m}$. The mass is not 1400 kg, but 1/4 of that, because each of the four springs carries an equal load. Let $m = 1400/4$. The period of oscillation is $2\pi/\omega$, which has to equal $\frac{1}{2}\frac{8}{10}$, because two bottom-outs mark one complete cycle. Then $\frac{2\pi}{\omega} = \frac{4}{10}$ implies $\omega = 5\pi$. Finally, $k = m\omega^2 = \frac{1400}{4}(5\pi)^2 = 8750\pi^2$.

Example 6.43 (Rolling Wheel)

A wheel of mass 10 kg and radius 0.35 m rolls frictionless with attached coil spring as in Figure 23. The observed frequency of oscillation is 8 full cycles every 3 seconds. Estimate the Hooke's constant k of the spring.

Solution: The answer is $k = \frac{135\pi^2}{16}$.

The rolling wheel model (10) will be used, equation $mx''(t) + \frac{2}{3}\,kx(t) = 0$. Known is the mass $m = 10$ kg and the general solution $x(t) = A\cos(\omega t - \alpha)$ with A and α constant and natural frequency $\omega = \sqrt{\dfrac{2}{3}\dfrac{k}{m}}$. Then the period of oscillation is $\frac{2\pi}{\omega} = \frac{8}{3}$, because the cosine factor passes through 8 periods in 3 seconds. The equation determines $\omega = \frac{3}{8}(2\pi)$. Then $k = m\frac{3}{2}\omega^2 = 10\frac{3}{2}\left(\frac{6\pi}{8}\right)^2 = \frac{135\pi^2}{16}$.

Exercises 6.6

Simple Harmonic Motion

Determine the model equation $mx''(t) + kx(t) = 0$, the natural frequency $\omega = \sqrt{k/m}$, the period $2\pi/\omega$ and the solution $x(t)$ for the following spring–mass systems.

1. A mass of 4 Kg attached to a spring of Hooke's constant 20 Newtons per meter starts from equilibrium plus 0.05 meters with velocity 0.

2. A mass of 2 Kg attached to a spring of Hooke's constant 20 Newtons per meter starts from equilibrium plus 0.07 meters with velocity 0.

3. A mass of 2 Kg is attached to a spring that elongates 20 centimeters due to a force of 10 Newtons. The motion starts at equilibrium with velocity −5 meters per second.

4. A mass of 4 Kg is attached to a spring that elongates 20 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity −8 meters per second.

5. A mass of 3 Kg is attached to a coil spring that compresses 2 centimeters when 1 Kg rests on the top coil. The motion starts at equilibrium plus 3 centimeters with velocity 0.

6. A mass of 4 Kg is attached to a coil spring that compresses 2 centimeters when 2 Kg rests on the top coil. The motion starts at equilibrium plus 4 centimeters with velocity 0.

7. A mass of 5 Kg is attached to a coil spring that compresses 1.5 centimeters when 1 Kg rests on the top coil. The motion starts at equilibrium plus 3 centimeters with velocity −5 meters per second.

8. A mass of 4 Kg is attached to a coil spring that compresses 2.2 centimeters when 2 Kg rests on the top coil. The motion starts at equilibrium plus 4 centimeters with velocity −8 meters per second.

9. A mass of 5 Kg is attached to a spring that elongates 25 centimeters due to a force of 10 Newtons. The motion starts at equilibrium with velocity 6 meters per second.

10. A mass of 5 Kg is attached to a spring that elongates 30 centimeters due to a force of 15 Newtons. The motion starts at equilibrium with velocity 4 meters per second.

Phase–amplitude Form

Solve the given differential equation and report the general solution. Solve for the constants c_1, c_2. Report the solution in phase–amplitude form

$$x(t) = A\cos(\omega t - \alpha)$$

with $A > 0$ and $0 \le \alpha < 2\pi$.

11. $x'' + 4x = 0$,
$x(0) = 1$, $x'(0) = -1$

12. $x'' + 4x = 0$,
$x(0) = 1$, $x'(0) = 1$

13. $x'' + 16x = 0$,
$x(0) = 2$, $x'(0) = -1$

14. $x'' + 16x = 0$,
$x(0) = -2$, $x'(0) = -1$

15. $5x'' + 11x = 0$,
$x(0) = -4$, $x'(0) = 1$

16. $5x'' + 11x = 0$,
$x(0) = -4$, $x'(0) = -1$

17. $x'' + x = 0$,
$x(0) = 1$, $x'(0) = -2$

18. $x'' + x = 0$,
$x(0) = -1$, $x'(0) = 2$

19. $x'' + 36x = 0$,
$x(0) = 1$, $x'(0) = -4$

20. $x'' + 64x = 0$,
$x(0) = -1$, $x'(0) = 4$

Pendulum

The formula

$$\frac{P_1}{P_2} = \frac{R_1}{R_2}\sqrt{\frac{L_1}{L_2}}$$

is valid for the periods P_1, P_2 of two pendulums of lengths L_1, L_2 located at distances R_1, R_2 from the center of the earth. The formula implies that a pendulum can be used to find the radius of the earth at a location. It is also useful for designing a pendulum clock adjustment screw.

21. Derive the formula, using $\omega = \sqrt{g/L}$, period $P = 2\pi/\omega$ and the gravitational relation $g = GM/R^2$.

22. A pendulum clock taken on a voyage loses 2 minutes a day compared to its exact timing at home. Determine the altitude change at the destination.

23. A pendulum clock with adjustable length L loses 3 minutes per day when $L = 30$ inches. What length L adjusts the clock to perfect time?

24. A pendulum clock with adjustable length L loses 4 minutes per day when $L = 30$ inches. What fineness length F is required for a 1/4–turn of the adjustment screw, in order to have 1/4–turns of the screw set the clock to perfect time plus or minus one second per day?

Torsional Pendulum

Solve for $\theta_0(t)$.

25. $\theta_0''(t) + \theta_0(t) = 0$

26. $\theta_0''(t) + 4\theta_0(t) = 0$

27. $\theta_0''(t) + 16\theta_0(t) = 0$

28. $\theta_0''(t) + 36\theta_0(t) = 0$

Shockless Auto

Find the period and frequency of oscillation of the car on four springs. Use model $mx''(t) + kx(t) = 0$.

29. Assume the car plus occupants has mass 1650 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter.

30. Assume the car plus occupants has mass 1850 Kg. Let each coil spring have Hooke's constant $k = 20000$ Newtons per meter.

31. Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 18000$ Newtons per meter.

32. Assume the car plus occupants has mass 1350 Kg. Let each coil spring have Hooke's constant $k = 16000$ Newtons per meter.

Rolling Wheel on a Spring

Solve the rolling wheel model $mx''(t) + \frac{2}{3}kx(t) = 0$ and also the frictionless model $mx''(t) + kx(t) = 0$, each with the given initial conditions. Graph the two solutions $x_1(t), x_2(t)$ on one set of axes.

33. $m = 1$, $k = 4$,
$x(0) = 1$, $x'(0) = 0$

34. $m = 5$, $k = 18$,
$x(0) = 1$, $x'(0) = 0$

35. $m = 11$, $k = 18$,
$x(0) = 0$, $x'(0) = 1$

36. $m = 7$, $k = 18$,
$x(0) = 0$, $x'(0) = 1$

6.7 Forced and Damped Vibrations

The study of vibrating mechanical systems continues. The main example is a system consisting of an externally forced mass on a spring with damping.[6] Both undamped and damped systems are studied. A few physical examples are included: clothes dryer, cafe door, pet door, bicycle trailer.

Forced Undamped Motion

The equation for study is a forced spring–mass system

$$mx''(t) + kx(t) = f(t).$$

The model originates by equating the Newton's second law force $mx''(t)$ to the sum of the Hooke's force $-kx(t)$ and the external force $f(t)$. The physical model is a laboratory box containing an undamped spring–mass system, transported on a truck as in Figure 13, with external force $f(t) = F_0 \cos \omega t$ induced by the speed bumps.

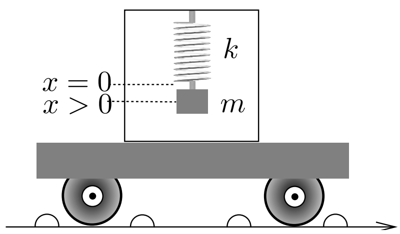

Figure 13. An undamped, forced spring-mass system.
A box containing a spring-mass system is transported on a truck. Speed bumps on the shoulder of the road transfer periodic vertical oscillations to the box.

The **forced spring-mass system** takes the form $x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m} \cos \omega t$. Symbol $\omega_0 = \sqrt{k/m}$ is called the **Natural Frequency**. It is the number of full periods of free oscillation per second for the **unforced spring–mass system** $x''(t) + \omega_0^2\, x(t) = 0$. The **External Frequency** ω is the number of full periods of oscillation per second of the external force $f(t) = F_0 \cos \omega t$. In the case of Figure 13, $f(t)$ is the vertical force applied to the box containing the spring–mass system, due to the speed bumps. The general solution $x(t)$ always presents itself in two pieces, as the sum of the homogeneous solution x_h and a particular solution x_p. For $\omega \ne \omega_0$, the solution formulas are (full details on page 522)

$$(1) \qquad \begin{array}{lcl} x''(t) + \omega_0^2\, x(t) = \dfrac{F_0}{m} \cos \omega t, & & \omega_0 = \sqrt{\dfrac{k}{m}}, \\ x(t) & = & x_h(t) + x_p(t), \\ x_h(t) & = & c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \quad c_1,\, c_2 \text{ constants}, \\ x_p(t) & = & \dfrac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t. \end{array}$$

A general statement can be made about the solution decomposition:

[6] Damping is energy dissipation and dampening is making something wet.

The solution is a sum of two harmonic oscillations, one of natural frequency ω_0 due to the spring and the other of natural frequency ω due to the external force $F_0 \cos \omega t$.

Beats

The physical phenomenon of **beats** refers to the periodic interference of two sound waves of slightly different frequencies. Human heartbeat uses the same terminology. Our pulse rate is $40 - 100$ **beats** per minute at rest. The phenomenon of **beats** will be explained mathematically *infra*. An illustration of the graphical meaning is in Figure 14.

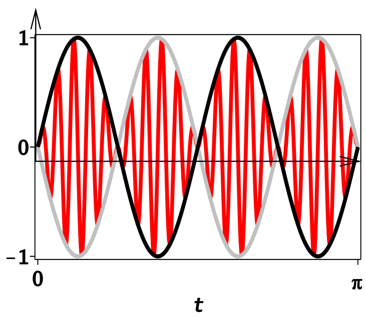

Figure 14. Beats.
Shown is a periodic oscillation

$$x(t) = 2 \sin 4t \sin 40t$$

with rapidly–varying factor $\sin 40t$ and the two slowly–varying envelope curves

$$x_1(t) = 2 \sin 4t, \quad x_2(t) = -2 \sin 4t.$$

A key example is piano tuning. A tuning fork is struck, then the piano string is tuned until the beats are not heard. The number of beats per second (unit Hz) is approximately the frequency difference between the two sources, e.g., two tuning forks of frequencies 440 Hz and 437 Hz would produce 3 beats per second.

The average human ear can detect beats only if the two interfering sound waves have a frequency difference of about 7 Hz or less. Ear-tuned pianos are subject to the same human ear limitations. Two piano keys are more than 7 Hz apart, even for a badly tuned piano, which is why simultaneously struck piano keys are heard as just one sound (no beats).

A **destructive interference** occurs during a very brief interval, so our impression is that the sound periodically stops, only briefly, and then starts again with a *beat*, a section of sound that is instantaneously loud again. The beat we hear corresponds to maxima in Figure 14.

In Figure 14, we see not the two individual sound waves, but their **superposition**, because $2\sin(4t)\sin(40t) = \cos(36t) - \cos(44t) =$ sum of two harmonic oscillations of different frequencies. See equation 2 below for details. When the tuning fork and the piano string have the same exact frequency ω, then Figure 14 would show a simple harmonic wave, because the two sounds would **superimpose** to a graph that looks like $\cos(\omega t - \alpha)$.

The origin of the phenomenon of **beats** can be seen from the formula

$$x(t) = 2\sin at \sin bt.$$

There is no sound when $x(t) \approx 0$: this is when destructive interference occurs. When a is small compared to b, e.g., $a = 4$ and $b = 40$, then there are long intervals between the zeros of $A(t) = 2\sin at$, at which destructive interference occurs. Otherwise, the amplitude of the sound wave is the average value of $A(t)$, which is 1. The sound stops at a zero of $A(t)$ and then it is rapidly loud again, causing the beat.

Black Box in the Trunk

Return to the forced harmonic oscillator

$$x''(t) + \omega_0^2\, x(t) = \frac{F_0}{m}\cos\omega t, \quad \omega_0 = \sqrt{\frac{k}{m}},$$

whose solution $x(t)$ appears in equation (1). The expression for $x(t)$ will show the phenomenon of **beats** for certain choices of frequencies ω_0, ω and initial position and velocity $x(0)$, $x'(0)$.

For instance, consider one possible expression $x(t) = \cos(\omega_0 t) - \cos(\omega t)$. Use the trigonometric identity $2\sin c \sin d = \cos(c-d) - \cos(c+d)$, derived from identities on page 491, to write

$$x(t) = \cos(\omega_0 t) - \cos(\omega t) = 2\sin\frac{1}{2}(\omega - \omega_0)t \sin\frac{1}{2}(\omega_0 + \omega)t. \tag{2}$$

If $\omega \approx \omega_0$, then the first factor $2\sin\frac{1}{2}(\omega-\omega_0)t$ has natural frequency $a = \frac{1}{2}(\omega-\omega_0)$ near zero. The natural frequency $b = \frac{1}{2}(\omega_0 + \omega)$ of the other factor can be relatively large and therefore $x(t)$ is a product of a **Slowly Varying** oscillation $2\sin at$ and a **Rapidly Varying** oscillation $\sin bt$. A graphic of $x(t)$ looks like Figure 14.

Rotating Drum on a Cart

Figure 15 shows a model for a rotating machine, like a front–loading clothes dryer.

For modeling purposes, the rotating drum with load is replaced by an idealized model: a mass $\mathcal{M}$ on a string of radius R rotating with angular speed ω. The center of rotation is located along the center–line of the cart. The total mass m of the cart includes the rotating mass $\mathcal{M}$, which we imagine to be an off–center lump of wet laundry inside the dryer drum. Vibrations cause the cart to skid left or right.

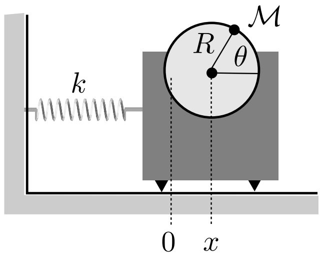

Figure 15. Rotating Vertical Drum.

Like a front-loading clothes dryer, or a washing machine, the drum is installed on a cart with skids. An internal spring restores the cart to equilibrium $x = 0$.

A spring of Hooke's constant k restores the cart to its equilibrium position $x = 0$. The cart has position $x > 0$ corresponding to skidding distance x to the right of the equilibrium position, due to the off-center load. Similarly, $x < 0$ means the cart skidded distance $|x|$ to the left.

Modeling. Friction ignored, Newton's second law gives force $F = m\overline{x}''(t)$, where $\overline{x}$ locates the cart's center of mass. Hooke's law gives force $F = -kx(t)$. The centroid $\overline{x}$ can be expanded in terms of $x(t)$ by using calculus moment of inertia formulas. Let $m_1 = m - \mathcal{M}$ be the cart mass, $m_2 = \mathcal{M}$ the drum mass, $x_1 = x(t)$ the moment arm for m_1 and $x_2 = x(t) + R\cos\theta$ the moment arm for m_2. Then $\theta = \omega t$ in Figure 15 gives

$$\begin{aligned} \overline{x}(t) &= \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \\ &= \frac{(m - \mathcal{M})x(t) + \mathcal{M}(x(t) + R\cos\theta)}{m} \\ &= x(t) + \frac{R\mathcal{M}}{m}\cos\omega t. \end{aligned} \tag{3}$$

Force competition $m\overline{x}'' = -kx$ and derivative expansion results in the forced harmonic oscillator

$$mx''(t) + kx(t) = R\mathcal{M}\omega^2 \cos\omega t. \tag{4}$$

Forced Damped Motion

Real systems do not exhibit idealized harmonic motion, because **damping** occurs. A watch balance wheel submerged in oil is a key example: frictional forces due to the viscosity of the oil will cause the wheel to stop after a short time. The same wheel submerged in air will appear to display harmonic motion, but indeed there is friction present, however small, which slows the motion.

Consider a spring–mass system consisting of a mass m and a spring with Hooke's constant k, with an added **dashpot** or **damper**, depicted in Figure 16 as a piston inside a cylinder attached to the mass. A useful physical model, for purposes of intuition, is a screen door with USA hardware: the door is equipped with a spring to restore the door to the jamb position and an adjustable piston–cylinder style dashpot.

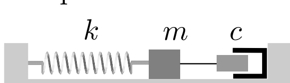

Figure 16. A spring-mass system with dashpot

The dashpot is assumed to operate in the **viscous domain**, which means that the force due to the damper device is proportional to the speed that the mass is moving: $F = cx'(t)$. The number $c \geq 0$ is called the **dashpot constant**. Three forces act: (1) Newton's second law $F_1 = mx''(t)$, (2) viscous damping $F_2 = cx'(t)$ and (3) the spring restoring force $F_3 = kx(t)$. The sum of the forces $F_1 + F_2 + F_3$ acting on the system must equal the **External Force** $f(t)$, which gives the equation for a **Forced Damped Spring–Mass System**

$$mx''(t) + cx'(t) + kx(t) = f(t). \tag{5}$$

If there is no external force, $f(t) = 0$, then the vibration is called **free** or **unforced** and otherwise it is called **forced**. Equation (5) is called **damped** if $c > 0$ and **undamped** if $c = 0$.

A useful visualization for a forced system is a vertical laboratory spring–mass system with dashpot placed inside a box, which is transported down a washboard road inside an auto trunk. The function $f(t)$ is the vertical oscillation of the auto trunk. The function $x(t)$ is the signed excursion of the mass in response to the washboard road. See Figure 17.

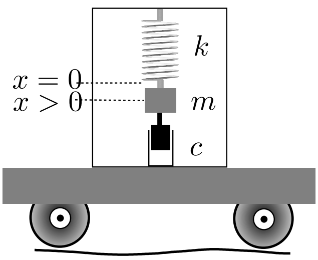

Figure 17. A Damped Spring-Mass System with External Forcing.
The apparatus is placed in a box, then transported in an auto trunk along a washboard road. Vertical excursion $x(t)$ of the mass is measured from equilibrium.

Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.

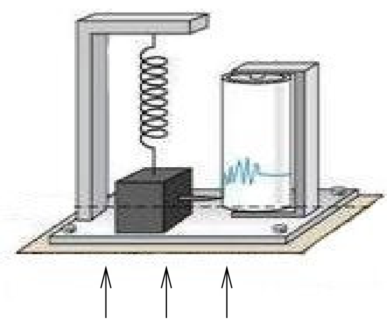

Figure 18. A Simplistic Vertical Motion Seismoscope.

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up.

The motion of the heavy mass m in Figure 18 can be modeled by a forced spring-mass system with damping. The first model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, Newton's second law gives the force equation $f(t) = -mu''(t)$. The second model for the motion of the mass is then

$$(6)\quad \begin{aligned} &x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ &\tfrac{c}{m} = 2\beta\Omega_0, \quad \tfrac{k}{m} = \Omega_0^2, \\ &x(t) = \text{mass position measured from equilibrium}, \\ &u(t) = \text{vertical ground motion due to the earthquake}. \end{aligned}$$

Some observations about equation (6):

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (6) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (6) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

A **release test** will find β, Ω_0 experimentally. See the exercises for details.

The point of (6) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

Free damped motion

Consider the special case of no external force, $f(t) = 0$. The vibration $x(t)$ satisfies the homogeneous differential equation

$$(7)\qquad mx''(t) + cx'(t) + kx(t) = 0.$$

Cafe Door

Restaurant waiters and waitresses are accustomed to the cafe door, which partially blocks the view of onlookers, but allows rapid, collision-free trips to the kitchen – see Figure 19. The door is equipped with a spring which tries to restore the door to the equilibrium position $x = 0$, which is the plane of the door frame. There is a dashpot attached, to keep the number of oscillations low.

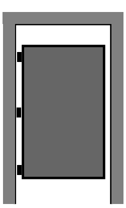

Figure 19. A Cafe Door.
There are three hinges with dashpot in the lower hinge.
The equilibrium position is the plane of the door frame.

The top view of the door, Figure 20, shows how the angle $x(t)$ from equilibrium $x = 0$ is measured from different door positions.

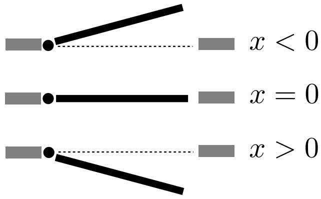

Figure 20. Top View of a Cafe Door.
The three possible door positions.

$x < 0$ kitchen
$x = 0$ door frame
$x > 0$ restaurant

The figure shows that, for modeling purposes, the cafe door can be reduced to a torsional pendulum with viscous damping. This results in the **cafe door** equation

$$Ix''(t) + cx'(t) + \kappa x(t) = 0. \tag{8}$$

The removal of the spring ($\kappa = 0$) causes the vibration $x(t)$ to be monotonic, which is a reasonable fit to a springless cafe door.

Pet Door

Designed for dogs and cats, the small door in Figure 21 permits free entry and exit.

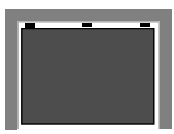

Figure 21. A Pet Door.
The equilibrium position is the plane of the door frame.
The door swings from hinges on the top edge.
One hinge is spring-loaded with dashpot.

Like the cafe door, the spring restores the door to the equilibrium position while the dashpot acts to eventually stop the oscillations. However, there is one fundamental difference: if the spring–dashpot system is removed, then the door continues to oscillate! The cafe door model will not describe the pet door.

For modeling purposes, the door can be compressed to a linearized swinging rod of length L (the door height). The torque $I = mL^2/3$ of the door assembly becomes important, as well as the linear restoring force kx of the spring and the viscous damping force cx' of the dashpot. All considered, a suitable model is the **pet door** equation

$$I\,x''(t) + cx'(t) + \left(k + \frac{mgL}{2}\right) x(t) = 0. \tag{9}$$

Derivation of (9) is by equating to zero the algebraic sum of the forces.

Removing the dashpot and spring ($c = k = 0$) gives a harmonic oscillator $x''(t) + \omega^2 x(t) = 0$ with $\omega^2 = \dfrac{mgL}{2I}$, which matches physical intuition. Equation (9) is *formally* the cafe door equation with an added linearization term $\dfrac{mgL}{2}x(t)$ obtained from $\dfrac{mgL}{2}\sin x(t)$.

Modeling Unforced Damped Vibration

The cafe door (8) and the pet door (9) have equations in the same form as a damped spring–mass system (7), and all equations can be reduced, for suitable definitions of constants p and q, to the simplified second order differential equation

(10) $$x''(t) + p\,x'(t) + q\,x(t) = 0.$$

The solution $x(t)$ of this equation is a linear combination of two Euler atoms determined by the roots of the characteristic equation

$$r^2 + pr + q = 0.$$

There are three types of solutions possible, organized by the sign of the discriminant

$$p^2 - 4q.$$

Positive Discriminant	Distinct real roots $r_1 \neq r_2$ $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$
Zero Discriminant	Double real root $r_1 = r_2$ $x = c_1 e^{r_1 t} + c_2\, t\, e^{r_1 t}$
Negative Discriminant	Complex conjugate roots $a \pm i\,b$ $x = e^{at}(c_1 \cos bt + c_2 \sin bt)$

Tuning a dashpot

The pet door and the cafe door have dashpots with an adjustment screw. The screw changes the dashpot coefficient c which in turn changes the size of coefficient p in (10). More damping c means p is larger.

There is a *critical damping effect* for a certain screw setting: if the setting is decreased more, then the door *oscillates*, whereas if the setting is increased, then the door has a *monotone non-oscillatory behavior*. The monotonic behavior can result in the door opening in one direction followed by slowly settling to exactly the door jamb position. If p is too large, then it could take 10 minutes for the door to close!

The critical case corresponds to the least $p > 0$ (the smallest damping constant $c > 0$) required to close the door with this kind of monotonic behavior. The same can be said about decreasing the damping: the more p is decreased, the more the door oscillations approach those of no dashpot at all, which is a pure harmonic oscillation.

As viewed from the characteristic equation $r^2 + pr + q = 0$, the change is due to a change in character of the roots from real to complex, which is measured by a sign change from positive to negative for the **Discriminant** $p^2 - 4q$. The physical response and the three cases of the constant–coefficient theorem, page 430, lead to the following terminology.

Classification	Defining properties
Overdamped	Distinct real roots $r_1 \neq r_2$ Positive discriminant $x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ $\quad$ = exponential × monotonic function

Critically damped	Double real root $r_1 = r_2$ Zero discriminant $x = c_1 e^{r_1 t} + c_2\, t\, e^{r_1 t}$ $\quad =$ exponential $\times$ monotonic function
Underdamped	Complex conjugate roots $a \pm i\, b$ Negative discriminant $x = e^{at}(c_1 \cos bt + c_2 \sin bt)$ $\quad =$ exponential $\times$ harmonic oscillation

Envelope Curves and Pseudo-Period

In the under-damped case the solution $x(t)$ of $mx''+cx'+kx = 0$ can be expressed in phase-amplitude form

$$\begin{aligned} x(t) &= e^{at}(c_1 \cos bt + c_2 \sin bt) \\ &= e^{at} C \cos(bt - \alpha). \end{aligned}$$

In this formula, $c_1 = C \cos\alpha$, $c_2 = C \sin\alpha$ and $C = \sqrt{c_1^2 + c_2^2}$. The **Pseudo-Period** is $T = \dfrac{2\pi}{b}$, so named because the harmonic factor $\cos(bt - \alpha)$ has period $2\pi/b$. The factor Ce^{at} generates the two **envelope curves**

$$y = Ce^{at}, \quad y = -Ce^{at}.$$

The solution $x(t)$ oscillates entirely inside the region defined by the envelope curves. Crossings of the t-axis happen at $bt = n\pi + \alpha$, $n = 0, \pm 1, \pm 2, \ldots$. Contact with the envelope curves happens at $bt = n\pi + \pi/2 + \alpha$, $n = 0, \pm 1, \pm 2, \ldots$.

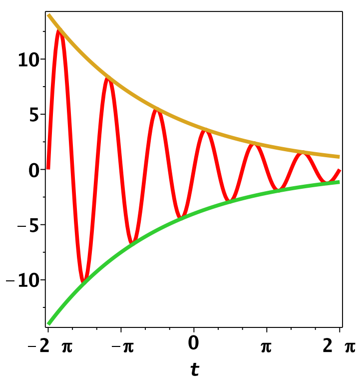

Figure 22. Envelope Curves.
A particular solution of the differential equation $25x'' + 10x' + 226x = 0$ is
$x(t) = 4e^{-t/5} \sin 3t$ red,
which has pseudo-period $T = \dfrac{2\pi}{3}$.
The envelope curves are
$x_1(t) = 4e^{-t/5}$ yellow,
$x_2(t) = -4e^{-t/5}$ green.

Bicycle trailer

An auto tows a one–wheel trailer over a washboard road. Shown in Figure 23 is the trailer strut, which has a single coil spring and two dampers. The mass m includes the trailer and the bicycles.

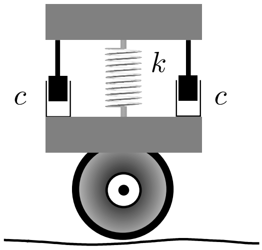

Figure 23. A trailer strut with dampers on a washboard road

Suppose a washboard dirt road has about 2 full oscillations (2 bumps and 2 valleys) every 3 meters and a full oscillation has amplitude 6 centimeters. Let s denote the horizontal distance along the road and let ω be the number of full oscillations of the roadway per unit length. The oscillation period is $2\pi/\omega$, therefore $2\pi/\omega = 3/2$ or $\omega = 4\pi/3$. A model for the road surface is

$$y = \frac{5}{100}\cos \omega s.$$

Let $x(t)$ denote the vertical elongation of the spring, measured from equilibrium. Newton's second law gives a force $F_1 = mx''(t)$ and the viscous damping force is $F_2 = 2cx'(t)$. The trailer elongates the spring by $x - y$, therefore the Hooke's force is $F_3 = k(x - y)$. The sum of the forces $F_1 + F_2 + F_3$ must be zero, which implies

$$mx''(t) + 2cx'(t) + k(x(t) - y(t)) = 0.$$

Write $s = vt$ where v is the speedometer reading of the car in meters per second. The expanded differential equation is the forced damped spring-mass system equation

$$mx''(t) + 2cx'(t) + kx(t) = \frac{k}{20}\cos(4\pi vt/3).$$

The solution $x(t)$ of this model, with $x(0)$ and $x'(0)$ given, describes the vertical excursion of the trailer bed from the roadway. The **observed oscillations** of the trailer are modeled by the steady-state solution

$$x_{\text{SS}}(t) = A\cos(4\pi vt/3) + B\sin(4\pi vt/3),$$

where A, B are constants determined by the method of undetermined coefficients. From physical data, the amplitude $C = \sqrt{A^2 + B^2}$ of this oscillation might be 6cm or larger. The maximum amplitude C over all speedometer readings v can be found by calculus. The computation uses the formula

$$C(v) = \frac{k/20}{\sqrt{(k - m\omega^2)^2 + (2c\omega)^2}}, \quad \omega = \frac{4\pi v}{3}. \tag{11}$$

Set $\frac{dC}{dv} = 0$ and then solve for the speed v^* which maximizes $C(v)$. The maximum excursion of the trailer is then

$$C(v^*) = \frac{km}{40c\sqrt{km - c^2}}.$$

The values of k, m, c can be found from an experiment: record $C(v)$ at three different speeds $v = v_1, v_2, v_3$. Then solve the system of three equations in three unknowns m, k, c, arising from (11).

Examples and Methods

Example 6.44 (Forced Undamped Vibration)
Solve the vibration equation

$$x'' + 225x = 209\cos(4t).$$

Solution: The answer is $x(t) = c_1\cos(15t) + c_2\sin(15t) + \cos(4t)$. The vibration is an example of **beats** for certain values of c_1, c_2. The solution is a superposition of two harmonic oscillations of frequencies 15 and 4. There are two ways to solve the problem, detailed below.

First Solution Details. A shortcut is to use equations (1), page 506. The given equation $x'' + 225x = 209\cos(4t)$ provides symbols $m = 1$, $k = 225$, $F_0 = 209$, $\omega = 4$. Then $\omega_0 = \sqrt{225} = 15$ is the unforced natural frequency of vibration. Substitution of the symbols into equations (1) gives $x_h = c_1\cos(15t) + c_2\sin(15t)$ and $x_p = F_1\cos(4t)$ with $F_1 = (209/1)/(225 - 4^2) = 1$. By superposition $x = x_h + x_p$. The reported solution is verified.

Second Solution Details. The characteristic equation $r^2 + 225 = 0$ of the homogeneous problem $x'' + 225x = 0$ has complex conjugate roots $\pm 15i$ and Euler solution atoms $\cos(15t), \sin(15t)$. Then $x_h(t) = c_1\cos(15t) + c_2\sin(15t)$.

A particular solution by Rule I of the method of undetermined coefficients is $x(t) = A\cos(4t) + B\sin(4t)$. Substitution into the non-homogeneous equation $x'' + 225x = 209\cos(4t)$ gives the relation

$$-16(A\cos(4t) + B\sin(4t)) + 225(A\cos(4t) + B\sin(4t)) = 209\cos(4t).$$

It reduces to the equation

$$209A\cos(4t) + 226B\sin(4t) = 209\cos(4t).$$

Independence of Euler atoms $\cos(4t)$, $\sin(4t)$ implies matching coefficients. Then $B = 0$ and $A = 1$. The trial solution $x(t) = A\cos(4t) + B\sin(4t)$ upon substitution of $A = 1, B = 0$ becomes particular solution $x_p(t) = \cos(4t)$.

Superposition gives general solution $x(t) = x_h(t) + x_p(t)$, therefore the answer reported has been verified.

Example 6.45 (Beats)
Write the linear combination $x(t) = \cos 10t - \cos 20t$ in the form $x(t) = C\sin at \sin bt$. Then graph the slowly-varying envelope curves and the curve $x(t)$.

Solution: The answer is $x(t) = 2\sin(5t)\sin(15t)$, which implies $C = 2, a = 5, b = 15$ with envelope curves $\pm 2\sin 5t$ (sine factor with longer period appears first). The graphic in Figure 24 is made from these formulas using a computer graphics program.

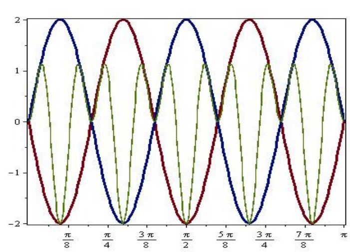

Figure 24. Beats Oscillation.
Plot of slowly-varying envelopes $\pm 2\sin(5t)$ and the oscillation $x(t) = 2\sin(5t)\sin(15t)$.

Details. The basic tool is the cosine sum formula from page 491. Let's assemble the formulas

$$\begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B,\\ \cos(A+B) &= \cos A\cos B - \sin A\sin B.\end{aligned}$$

Because $x(t) = \cos 10t - \cos 20t = \cos(A-B) - \cos(A+B) = 2\sin A\sin B$, then choose $A - B = 10t$ and $A + B = 20t$. Then the unique solution is $A = 15t, B = 5t$, which implies the formula

$$x(t) = 2\sin A\sin B = 2\sin(15t)\sin(5t)\,.$$

The slowly-varying envelope curves are $\pm 2\sin(5t)$, because the sine factor periods are $2\pi/15$ and $2\pi/5$, the second being the longer period.

Example 6.46 (Rotating Drum)

An unloaded European-style washing machine weighs 156 lbs. When loaded with an off-center wet mass of 4 kg, it has horizontal excursions x from equilibrium satisfying approximately the rotating drum equation (4):

$$mx''(t) + kx(t) = R\mathcal{M}\omega^2 \cos\omega t.$$

Assume Hooke's spring constant $k = 10$ slugs per foot. The drum has diameter 30 in and during a water extraction cycle it rotates at 600 rpm. Discuss assumptions and computations for the values $\mathcal{M} = 0.275$, $m = 5.15$, $R = 1.25$ and $\omega = 20\pi$. Then compute the approximate expression

$$x(t) = c_1\cos\left(\frac{20t}{\sqrt{206}}\right) + c_2\sin\left(\frac{20t}{\sqrt{206}}\right) - 55\,\frac{\pi^2\cos(20\,\pi\,t)}{824\,\pi^2 - 4}. \tag{12}$$

Solution:

Details. Central to the mathematical formulation is Newton's formula $W = mg$, which in words is *weight W (a force) equals mass m times gravitational acceleration g*. Use $g = 32$ ft/sec per second, for simplicity of discussion. Using $g = 32.2$ changes constants in a minor way.

Basic plan. Use model (4). After, we will be tormented and humiliated by closer analysis of the physical problem. Let's assume the centroid of the wet load is approximately on the edge of the rotating drum, in order to simplify the formulas and use model (4). The rotating machine in the absence of the wet load is assumed to operate at equilibrium $x = 0$. Issues like additional internal damping and frictional forces on the mounting surface will be patently ignored with no apologies.

Wet load mass $\mathcal{M}$: A unit conversion is required for the wet load mass: 4 kg represents 4(2.2) lbs. Then $W = 8.8$ lbs is the wet load weight and its mass is $\mathcal{M} = W/g = 8.8/32 = 0.275$ slugs.

Total machine mass m: Total machine weight is $W = 156 + 8.8 = 164.8$ lbs, then formula $W = mg$ implies the total mass is $m = 164.8/32 = 5.15$ slugs.

Drum radius R: A conversion to feet is required, giving $R = \frac{1}{2}(30)$ in $= \frac{15}{12}$ in $= 1.25$ ft.

Natural frequency of rotation ω: Supplied is the rotational period $2\pi/\omega$, which is equal to $1/10$ second (600 revolutions in 60 seconds). Solve $2\pi/\omega = 1/10$ for $\omega = 20\pi$.

Solution $x(t)$: We'll use equation (4) with the constants inserted:

$$5.15x''(t) + 10x(t) = 1.25(0.275)(20\pi)^2 \cos(20\pi t).$$

Without machine assist, the homogeneous equation $5.15x''(t) + 10x(t) = 0$ is solved as $x_h = c_1 \cos(bt) + c_2 \sin(bt)$ where $b = \sqrt{k/m} = 20/\sqrt{206}$. Then undetermined coefficients is applied with (shortcut) trial solution $x = A\cos(20\pi t)$ to the non-homogeneous problem, giving

$$A = \frac{-55\,\pi^2}{824\,\pi^2 - 4}, \quad x_p = \frac{-55\,\pi^2}{824\,\pi^2 - 4}\cos(20\pi t).$$

The reported answer in equation (12) is $x = x_h + x_p$.

Answer check: Computer algebra system `maple` solves the equation using this code:

```
f:=t->1.25*(0.275)*(20*Pi)^2*cos (20*Pi* t);
de:=5.15*diff(x(t),t,t)+10*x(t)=f(t);
dsolve(de,x(t));
```

Vibrations of x_p have amplitude about 0.13 cm and period 0.1. The harmonic vibrations of x_h have a longer period of about 4.5. For example, if the spin cycle starts from rest, then $x(t)$ will have amplitude of about 0.13 and its graphic on $0 < t < 4.5$ will look like a beats figure, with slow oscillation envelope of approximate period 4.5.

Example 6.47 (Damped Spring-Mass System)
Let $x(t)$ be the defected distance from equilibrium in a damped spring-mass system with free oscillation equation

$$4x''(t) + 3x'(t) + 17x(t) = 0.$$

Find an expression for $x(t)$.

Solution: The answer is

$$x(t) = c_1 e^{-3t/8}\cos(\sqrt{263}t/8) + c_2 e^{-3t/8}\sin(\sqrt{263}t/8).$$

Details. The homogeneous solution $x(t)$ is a linear combination of two Euler solution atoms found from the characteristic equation $4r^2 + 3r + 17 = 0$. The roots according to the quadratic formula are $-\frac{3}{8} \pm \frac{i}{8}\sqrt{263}$. Then the two Euler solution atoms are

$$e^{-3t/8}\cos(\sqrt{263}t/8), \quad e^{-3t/8}\sin(\sqrt{263}t/8),$$

from which the solution formula follows.

Remarks. The oscillation is classified as **under-damped**, because of the presence of sine and cosine oscillatory factors in the Euler solution atoms. Any solution is the product of an exponential factor and a harmonic oscillation, therefore the solution is **pseudo-periodic** with **pseudo-period** $16\pi/\sqrt{263}$.

Example 6.48 (Seismoscope)
Consider the seismoscope equation

$$x''(t) + 12x'(t) + 100x(t) = -u''(t).$$

Find an expression for the seismoscope stylus record $x(t)$ in terms of the ground motion $u(t)$.

Solution: In terms of particular solution $x_p(t)$, defined below in integral equation (14) or (15), the answer is

$$x(t) = c_2 e^{-6t}\cos(8t) + c_2 e^{-6t}\sin(8t) + x_p(t). \tag{13}$$

Details. The solution method is superposition $x(t) = x_h(t) + x_p(t)$ where x_h is the solution of the homogeneous equation $x''(t) + 12x'(t) + 100x(t) = 0$ and x_p is a variation of parameters solution of the non-homogeneous equation $x''(t)+12x'(t)+100x(t) = f(t)$, where $f(t) = -u''(t)$.

Homogeneous solution x_h. The characteristic equation $r^2 + 12r + 100 = 0$ has factorization $(r + 6)^2 + 64 = 0$, hence complex conjugate roots $r = -6 \pm 8i$. The Euler solution atoms are $e^{-6t}\cos(8t)$, $e^{-6t}\sin(8t)$, from which we construct the general solution

$$x_h(t) = c_2 e^{-6t}\cos(8t) + c_2 e^{-6t}\sin(8t).$$

Non-homogeneous solution x_p. Let's start by writing the variation of parameters formula in the different form

$$\begin{aligned} x_p(t) &= y_1(t)\left(\int_0^t -\frac{y_2(x)f(x)}{W(x)}dx\right) + y_2(t)\left(\int_0^t \frac{y_1(x)f(x)}{W(x)}dx\right) \\ &= \int_0^t \frac{W_1(t,x)}{W(x)} f(x)dx \end{aligned}$$

where

$$\begin{aligned} f(x) &= -u''(x), \\ y_1(t) &= e^{-6t}\cos(8t), \\ y_2(t) &= e^{-6t}\sin(8t), \\ W(x) &= 8e^{-12x}, && \text{Details below in } \boxed{\mathbf{1}}. \\ W_1(t,x) &= -y_1(t)y_2(x) + y_2(t)y_1(x) \\ &= e^{-6t-6x}(\sin 8t\cos 8x - \cos 8t \sin 8x) \\ &= e^{-6t-6x}\sin(8t-8x). && \text{Trig identity.} \end{aligned}$$

Condensing the definitions gives the final formula

$$x_p(t) = -\int_0^t e^{-6t+6x}\sin(8t-8x)u''(x)dx. \tag{14}$$

It is possible to integrate this equation by parts and express the answer entirely in terms of $u(t)$. Some integration by parts free terms are collected into $x_h(t)$ to produce the replacement formula

$$\begin{aligned} x_p^*(t) &= -u(t) + \int_0^t K(t-x)u(x)dx, \\ K(w) &= 12\,e^{-6w}\cos(8w) + \frac{7}{2}\,e^{-6w}\sin(8w). \end{aligned} \tag{15}$$

Laplace theory can derive formula (15) using the convolution theorem. Generally, (14) and (15) are different answers.

$\boxed{\mathbf{1}}$ **Wronskian determinant details**.

A shortcut is to use Theorem 6.17, page 464. The answer is $W(x) = W(0)e^{-12x}$ where $W(0) = 8$ is computed from the first line of the determinant expansion below. Details below compute $W(x)$ directly from the definition.

$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$	Variation of parameters definition of the Wronskian of y_1, y_2.
$= \begin{vmatrix} y_1 & y_2 \\ -6y_1 - 8y_2 & -6y_2 + 8y_1 \end{vmatrix}$	Because $y_1' = -6y_1 - 8y_2$ and $y_2' = -6y_2 + 8y_1$.
$= \begin{vmatrix} y_1 & y_2 \\ -8y_2 & 8y_1 \end{vmatrix}$	Combination rule `combo(1,2,6)`.
$= 8(y_1^2 + y_2^2)$	Sarrus' Rule.
$= 8e^{-12x}(\cos^2(8x) + \sin^2(8x))$	Expand $y_1(x) = e^{-6x}\cos(8x)$ and $y_2(x) = e^{-6x}\sin(8x)$.
$= 8e^{-12x}.$	Pythagorean identity.

Example 6.49 (Cafe Door)
Consider the cafe door equation (8):

$$Ix''(t) + cx'(t) + \kappa x(t) = 0.$$

Find an expression for $x(t)$. Then show details for why the motion $x(t)$ is eventually monotonic when the spring is removed.

Solution:

First, divide by torque $I > 0$ to obtain equation $x'' + 2ax' + bx = 0$ with new symbols $2a = c/I$, $b = \kappa/I$. The characteristic equation is $(r + a)^2 + b - a^2 = 0$. There are three cases determined by the sign of $b - a^2$ for the form of the solution. Because $b - a^2 = \frac{4I\kappa - c^2}{4I^2}$, then $b - a^2$ has sign determined by $4I\kappa - c^2$.

Case $4I\kappa - c^2 > 0$.
Then the characteristic equation roots are complex conjugates $-a \pm i\sqrt{b - a^2}$. The solution is **under-damped**, oscillatory and given by

$$\begin{aligned} x(t) &= c_1 e^{-at}\cos(\sqrt{b - a^2}\,t) + c_2 e^{-at}\sin(\sqrt{b - a^2}\,t) \\ &= c_1 e^{\frac{ct}{2I}}\cos\left(\sqrt{4I\kappa - c^2}\,\frac{t}{2I}\right) + c_2 e^{\frac{ct}{2I}}\sin\left(\sqrt{4I\kappa - c^2}\,\frac{t}{2I}\right). \end{aligned}$$

Case $4I\kappa - c^2 = 0$.
Then the characteristic equation roots are equal, $-a, -a$. The solution is **critically damped**, non-oscillatory and given by

$$x(t) = c_1 e^{-at} + c_2\,t\,e^{-at} = c_1 e^{\frac{ct}{2I}} + c_2\,t\,e^{\frac{ct}{2I}}.$$

Case $4I\kappa - c^2 < 0$.
Then the characteristic equation roots are real and unequal, $-a \pm \sqrt{a^2 - b}$. The solution is **over-damped**, non-oscillatory and given by

$$\begin{aligned} x(t) &= c_1 e^{-at - \sqrt{a^2 - b}\,t} + c_2\,t\,e^{-at + \sqrt{a^2 - b}\,t} \\ &= c_1\,e^{\left(c - \sqrt{c^2 - 4I\kappa}\right)\frac{t}{2I}} + c_2\,e^{\left(c + \sqrt{c^2 - 4I\kappa}\right)\frac{t}{2I}}. \end{aligned}$$

Cafe door with no spring. This event is defined by $\kappa = 0$, which eliminates the under-damped case $4I\kappa - c^2 > 0$. Suppose hereafter that $x(t)$ is a nonzero solution. The critically damped case is $a = 0$. Then the solution can be written as $x(t) = c_1 + c_2\, t$, which crosses the axis $x = 0$ at most once. The over-damped case $4I\kappa - c^2 < 0$ can be written $x(t) = \left(c_1 + c_2\, e^{B}\right) e^{At}$ where $B > 0$. Similarly, it crosses the axis $x = 0$ at most once, due to the factor $c_1 + c_2\, e^{Bt}$.

Example 6.50 (Pet Door)

A pet door of height $L = 1.5$ feet and weight 8 pounds oscillates freely because the dashpot has been removed. Assume Hooke's spring constant $k = 10$. Find an expression for the angular motion $x(t)$ using equation (9) with torque $I = mL^2/3$:

$$I\, x''(t) + cx'(t) + \left(k + \frac{mgL}{2}\right) x(t) = 0.$$

Solution:

Removal of the dashpot corresponds to $c = 0$. The mass m satisfies $W = mg$, which from $W = 8$ and $g = 32$ gives $m = 0.25$ slugs. Then the torque is $I = mL^2/3 = L^2/12 = 3/16$ and $mgL/2 = 3g/16 = 6$. Equation (9) becomes

$$\frac{3}{16}\, x''(t) + 16x(t) = 0.$$

This is the classical harmonic oscillator $x'' + \omega^2 x = 0$ with $\omega^2 = 16^2/3$. Then $\omega = 16/\sqrt{3}$ and

$$x(t) = c_1 \cos\left(\frac{16\, t}{\sqrt{3}}\right) + c_2 \sin\left(\frac{16\, t}{\sqrt{3}}\right).$$

Example 6.51 (Tuning a Dashpot)

Classify the following equations as over-damped, critically damped or under-damped free vibrations.

(a) $x'' + 2x' + 3x = 0$

(b) $x'' + 4x' + 3x = 0$

(c) $x'' + 2x' + x = 0$

Solution: The answers: (a) Under-damped, (b) Over-damped, (c) Critically damped. Definitions on page 513.

Details (a). The characteristic equation $r^2 + 2r + 3 = 0$ factors into $(r + 1)^2 + 2 = 0$ with complex conjugate roots $-1 \pm i\sqrt{2}$. The Euler solution atoms contain sines and cosines, therefore (a) is oscillatory, classified as under-damped.

Details (b). The characteristic equation $r^2+4r+3 = 0$ factors into $(r+3)(r+1) = 0$ with distinct real roots $-3, -1$. Therefore, (b) is non-oscillatory, classified as over-damped because of distinct roots.

Details (c). The characteristic equation $r^2 + 2r + 1 = 0$ factors into $(r + 1)(r + 1) = 0$ with equal real roots $-1, -1$. Therefore, (b) is non-oscillatory, classified as critically damped because of equal roots.

Summary of Methods. Classification requires only the roots of the characteristic equation.

Over-damped means *too much damping*. In the screen door example, the tuning screw has made the dashpot constant c large, which means an overly-aggressive dashpot that halts motion. This means the screen door hangs open. Then the screen door has no oscillations, equivalently, $x(t)$ has no sines or cosines.

Critically damped is an unstable state. In the screen door example, it is the impossible to achieve the ideal dashpot tuning screw setting on a screen door: the door opens and then slowly closes to the jamb position, the door hardware making a single click as it locks the door on the jamb. A turn of the tuning screw in either direction jumps between oscillation and non-oscillation of the screen door.

Under-damped means *not enough damping effect*. Physically, the dashpot is not effective. In the screen door example this means the screen door oscillates and bangs repeatedly on the door jamb. Detection in $x(t)$ is the presence of oscillating sines and cosines. Solution $x(t)$ is called *oscillatory*.

Example 6.52 (Pseudo-Period)

Find the pseudo-period and time-varying amplitude for the free damped vibration

$$4x'' + 2x' + 3x = 0, \quad x(0) = 1, \quad x'(0) = -1.$$

Solution: The answers: Pseudo period $8\pi/\sqrt{11}$ and amplitude $4e^{-t/4}$ are obtained from the solution $x(t) = 4e^{-t/4}\cos\left(\sqrt{11}\,\frac{t}{4}\right)$.

Details. The characteristic equation $4r^2 + 2r + 3 = 0$ has complex conjugate roots $-\frac{1}{4} \pm i\,\frac{\sqrt{11}}{4}$, obtained from the quadratic formula. Then the general solution is

$$x(t) = c_1\, e^{-t/4}\cos\left(\sqrt{11}\,\frac{t}{4}\right) + c_2\, e^{-t/4}\sin\left(\sqrt{11}\,\frac{t}{4}\right).$$

Initial conditions $x(0) = 4, x'(0) = -1$ give the two equations

$$\begin{array}{rcrcr} (1)c_1 & + & (0)c_2 & = & 4, \\ \left(\frac{-1}{4}\right)c_1 & + & \left(\frac{\sqrt{11}}{4}\right)c_2 & = & -1, \end{array}$$

with unique solution $c_1 = 4, c_2 = 0$. The pseudo-period is the period $2\pi/\omega$ of the trig factor $\cos(\omega t)$, where $\omega = \frac{1}{4}\sqrt{11}$. The time-varying amplitude is the factor in front of the cosine factor, namely $4e^{-t/4}$.

Remark on Method. If both c_1, c_2 are nonzero, then a trig identity is applied first to write $x(t) = Ae^{-t/4}\cos(\omega t - \alpha)$. The amplitude is then $Ae^{-t/4}$. The period is unchanged.

Proofs and Details

Details for equation (1), page 506:

Homogeneous solution x_h. The characteristic equation for $x'' + \omega_0^2 x = 0$ is $r^2 + \omega_0^2 = 0$ with complex conjugate roots $r = \pi i\omega_0$. Then the Euler solution atoms are $\cos(\omega_0 t), \sin(\omega_0 t)$. The general solution is a linear combination of the Euler solution atoms, as displayed in equation (1).

Particular solution x_p. The method of undetermined coefficients applies, because the equation has constant coefficients and the forcing term $f(t) = (F_0/m)\cos(\omega t)$ is a linear combination of Euler solution atoms. Derivatives of $f(t)$ are linear combinations of the two atoms $\cos(\omega t), \sin(\omega t)$ and therefore the initial trial solution in the method of undetermined coefficients is $x(t) = d_1 \cos(\omega t) + d_2 \sin(\omega t)$. Neither of the two atoms appearing in the trial solution are solutions of the unforced equation $x'' + \omega_0^2 x = 0$, because that would require the false equation $\omega_0 = \omega$). Therefore, the initial trial solution is the final trial solution, no changes made, no Rule II applied.

The trial solution $x(t) = d_1 \cos(\omega t) + d_2 \sin(\omega t)$ is substituted into $x'' + \omega_0^2 x = \frac{F_0}{m}\cos \omega t$ in order to determine d_1, d_2. The calculation uses the equation $x'' + \omega^2 x = 0$, satisfied by $\cos \omega t, \sin \omega t$ and the trial solution $x(t)$. Then

$$\begin{array}{lcl} x'' + \omega_0^2 x & = & \frac{F_0}{m}\cos(\omega t), \\ -\omega^2 x + \omega_0^2 x & = & \frac{F_0}{m}\cos(\omega t), \\ \left(\omega_0^2 - \omega^2\right) x & = & \frac{F_0}{m}\cos(\omega t), \\ C d_1 \cos(\omega t) + C d_2 \sin(\omega t) & = & \frac{F_0}{m}\cos(\omega t), \end{array}$$

where $C = \left(\omega_0^2 - \omega^2\right)$. Matching coefficients of the Euler atoms $\cos(\omega t), \sin(\omega t)$ then implies

$$\begin{array}{lcl} C d_1 & = & \frac{F_0}{m}, \\ C d_2 & = & 0. \end{array}$$

Division by C gives $d_1 = \frac{F_0}{mC}$ and $d_2 = 0$, which implies $x(t) = \frac{F_0}{mC}\cos(\omega t)$. This is the answer for x_p reported in equation (1).

Exercises 6.7

Forced Undamped Vibration

Solve the given equation.

1. $x'' + 100x = 20\cos(5t)$

2. $x'' + 16x = 100\cos(10t)$

3. $x'' + \omega_0^2 x = 100\cos(\omega t)$, when the internal frequency ω_0 is twice the external frequency ω.

4. $x'' + \omega_0^2 x = 5\cos(\omega t)$, when the internal frequency ω_0 is half the external frequency ω.

Black Box in the Trunk

5. Construct an example $x'' + \omega_0^2 x = F_1 \cos(\omega t)$ with a solution $x(t)$ having beats every two seconds.

6. A solution $x(t)$ of $x'' + 25x = 100\cos(\omega t)$ has beats every two seconds. Find ω.

Rotating Drum

Solve the given equation.

7. $x'' + 100x = 500\,\omega^2 \cos(\omega t)$, $\omega \ne 10$.

8. $x'' + \omega_0^2 x = 5\,\omega^2 \cos(\omega t)$, $\omega \ne \omega_0$.

Harmonic Oscillations

Express the general solution as a sum of two harmonic oscillations of different frequencies, each oscillation written in phase-amplitude form.

9. $x'' + 9x = \sin 4t$

10. $x'' + 100x = \sin 5t$

11. $x'' + 4x = \cos 4t$

12. $x'' + 4x = \sin t$

Beats: Convert and Graph

Write each linear combination as $x(t) = C \sin at \sin bt$. Then graph the slowly-varying envelope curves and the curve $x(t)$.

13. $x(t) = \cos 4t - \cos t$

14. $x(t) = \cos 10t - \cos t$

15. $x(t) = \cos 16t - \cos 12t$

16. $x(t) = \cos 25t - \cos 23t$

Beats: Solve, find Envelopes

Solve each differential equation with $x(0) = x'(0) = 0$ and determine the slowly-varying envelope curves.

17. $x'' + x = 99 \cos 10t$.

18. $x'' + 4x = 252 \cos 10t$.

19. $x'' + x = 143 \cos 12t$.

20. $x'' + 256x = 252 \cos 2t$.

Waves and Superposition

Graph the individual waves x_1, x_2 and then the superposition $x = x_1 + x_2$. Report the apparent period of the superimposed waves.

21. $x_1(t) = \sin 22t$, $x_2(t) = 2 \sin 20t$

22. $x_1(t) = \cos 16t$, $x_2(t) = 4 \cos 20t$

23. $x_1(t) = \cos 16t$, $x_2(t) = 4 \sin 16t$

24. $x_1(t) = \cos 25t$, $x_2(t) = 4 \cos 27t$

Periodicity

25. Let $x_1(t) = \cos 25t$, $x_2(t) = 4 \cos 27t$. Their sum has period $T = m\frac{2\pi}{25} = n\frac{2\pi}{27}$ for some integers m, n. Find all m, n and the least period T.

26. Let $x_1(t) = \cos \omega_1 t$, $x_2(t) = \cos \omega_2 t$. Find a condition on ω_1, ω_2 which implies that the sum $x_1 + x_2$ is periodic.

27. Let $x(t) = \cos(t) - \cos(\sqrt{2}t)$. Explain without proof, from a graphic, why $x(t)$ is not periodic.

28. Let $x(t) = \cos(5t) + \cos(5\sqrt{2}t)$. Is $x(t)$ is periodic? Explain without proof.

Rotating Drum

Let $x(t)$ and $x_p(t)$ be defined as in Example 4, page 509. Replace Hooke's constant $k = 10$ by $k = 1$, all other constants unchanged.

29. Re-compute the amplitude $A(t)$ of solution $x_p(t)$. Find the decimal value for the maximum of $|A(t)|$.

30. Find $x(t)$ when $x(0) = x'(0) = 0$. It is known that $x(t)$ fails to be periodic. Let $t_1 = 0, \ldots, t_{29}$ be the consecutive extrema on $0 \le t \le 1.4$. Verify graphically or by computation that $|x(t_{i+1}) - x(t_i)| \approx 0.133$ for $i = 1, \ldots, 28$.

Musical Instruments

Melodious tones are superpositions of harmonics $\sin(n\omega t)$, with $n =$ an integer, $\omega =$ fundamental frequency.

In 1885 Alexander J. Ellis introduced a measurement unit **Cent** by the equation **one cent** $= 2^{\frac{1}{12}} \approx 1.0005777895$. On most pianos, the frequency ratio between two adjacent keys equals 100 **cents**, called an **equally tempered semitone**. Two piano keys of frequencies 480 Hz and 960 Hz span 1200 **cents** and have tones $\sin(\omega t)$ and $\sin(2\omega t)$ with $\omega = 480$. A span of 1200 **cents** between two piano key frequencies is called an **Octave**.

31. **(Equal Temperament)** Find the 12 frequencies of equal temperament for octave 480 Hz to 960 Hz. The first two frequencies are 480, 508.5422851.

32. **(Flute or Noise)** Equation $x(t) = \sin 220\pi t + 2 \sin 330\pi t$ could represent a tone from a flute or just a dissonant, unpleasing sound. Discuss the impossibility of answering the question with a simple yes or no.

33. **(Guitar)** Air inside a guitar vibrates a little like air in a bottle when you blow across the top. Consider a flask of volume $V = 1$ liter, neck length $L = 5$ cm and neck cross-section $S = 3$ cm^2. The vibration has model $x'' + f^2 x = 0$ with $f = c\sqrt{\frac{S}{VL}}$, where $c = 343$ m/s is the speed of sound in air. Compute $\frac{f}{2\pi}$ and $\lambda = \frac{2\pi c}{f}$, the frequency and wavelength. The answers are about 130 Hz and $\lambda = 2.6$ meters, a low sound.

34. **(Helmholtz Resonance)** Repeat the previous exercise calculations, using a flask with neck diameter 2.0 cm and neck length 3 cm. The tone should be lower, about 100 Hz, and the wavelength λ should be longer.

Seismoscope

35. Verify that x_p given in (14) and x_p^* given by (15), page 519, have the same initial conditions when $u(0) = u'(0) = 0$, that is, the ground does not move at $t = 0$. Conclude that $x_p = x_p^*$ in this situation.

36. A **release test** begins by starting a vibration with $u = 0$. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. Explain how to find β, Ω_0 in the equation $x'' + 2\beta\Omega_0 x' + \Omega_0^2 x = 0$, using Exercises 69 and 70, *infra*.

Free Damped Motion

Classify the homogeneous equation $mx'' + cx' + kx = 0$ as **over-damped**, **critically damped** or **under-damped**. Then solve the equation for the general solution $x(t)$.

37. $m = 1, c = 2, k = 1$

38. $m = 1, c = 4, k = 4$

39. $m = 1, c = 2, k = 3$

40. $m = 1, c = 5, k = 6$

41. $m = 1, c = 2, k = 5$

42. $m = 1, c = 12, k = 37$

43. $m = 6, c = 17, k = 7$

44. $m = 10, c = 31, k = 15$

45. $m = 25, c = 30, k = 9$

46. $m = 9, c = 30, k = 25$

47. $m = 9, c = 24, k = 41$

48. $m = 4, c = 12, k = 34$

Cafe and Pet Door

Classify as a cafe door model and/or a pet door model. Solve the equation for the general solution and identify as oscillatory or non-oscillatory.

49. $x'' + x' = 0$

50. $x'' + 2x' + x = 0$

51. $x'' + 2x' + 5x = 0$

52. $x'' + x' + 3x = 0$

53. $9x'' + 24x' + 41x = 0$

54. $6x'' + 17x' = 0$

55. $9x'' + 24x' = 0$

56. $6x'' + 17x' + 7x = 0$

Classification

Classify $mx'' + cx' + kx = 0$ as **over-damped**, **critically damped** or **under-damped** without solving the differential equation.

57. $m = 5, c = 12, k = 34$

58. $m = 7, c = 12, k = 19$

59. $m = 5, c = 10, k = 3$

60. $m = 7, c = 12, k = 3$

61. $m = 9, c = 30, k = 25$

62. $m = 25, c = 80, k = 64$

Critically Damped

The equation $mx'' + cx' + kx = 0$ is critically damped when $c^2 - 4mk = 0$. Establish the following results for $c > 0$.

63. The mass undergoes no oscillations, because

$$x(t) = (c_1 + c_2 t)e^{-\frac{ct}{2m}}.$$

64. The mass passes through $x = 0$ at most once.

Over-Damped

Equation $mx'' + cx' + kx = 0$ is defined to be over-damped when $c^2 - 4mk > 0$. Establish the following results for $c > 0$.

65. The mass undergoes no oscillations, because if r_1, r_2 are the roots of $mr^2 + cr + c = 0$, then

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

66. The mass passes through equilibrium position $x = 0$ at most once.

Under-Damped

Equation $mx'' + cx' + kx = 0$ is defined to be under-damped when $c^2 - 4mk < 0$. Establish the following results.

67. The mass undergoes infinitely many oscillations. If $c = 0$, then the oscillations are harmonic.

68. The solution $x(t)$ can be factored as an exponential function $e^{-\frac{ct}{2m}}$ times a harmonic oscillation. In symbols:

$$x(t) = e^{-\frac{ct}{2m}} \left(A\cos(\omega t - \alpha)\right).$$

Experimental Methods

Assume model $mx''+cx'+kx = 0$ is oscillatory. The results apply to find nonnegative constants m, c, k from one experimentally known solution $x(t)$. Provide details.

69. Let $x(t)$ have consecutive maxima at $t = t_1$ and $t = t_2 > t_1$. Then $t_2 - t_1 = T = \frac{2\pi}{\omega}$ = pseudo period of $x(t)$.

70. Let (t_1, x_1) and (t_2, x_2) be two consecutive maximum points of the graph of a solution $x(t) = Ce^{-ct/(2m)}\cos(\omega t - \alpha)$ of $mx'' + cx' + kx = 0$. Let $a \pm \omega i$ be the two complex roots of $mr^2 + cr + k = 0$ where $a = -c/(2m)$ and $\omega = \frac{1}{2m}\sqrt{4mk - c^2}$. Then

$$\ln\frac{x_1}{x_2} = \frac{c\pi}{m\omega},$$

71. **(Bike Trailer)** Assume *fps* units. A trailer equipped with one spring and one shock has mass $m = 100$ in the model $mx'' + cx' + kx = 0$. Find c and k from this experimental data: two consecutive maxima of $x(t)$ are $(0.35, 10/12)$ and $(1.15, 8/12)$.
Hint: Use exercises 69 and 70.

72. **(Auto)** Assume *fps* units. An auto weighing 2.4 tons is equipped with four identical springs and shocks. Each spring-shock module has damped oscillations satisfying $mx'' + cx' + kx = 0$. Find m. Then find c and k from this experimental data: two consecutive maxima of $x(t)$ are $(0.3, 3/12)$ and $(0.7, 2/12)$.
Hint: Use exercises 69 and 70.

Structure of Solutions

Establish these results for the damped spring-mass system $mx'' + cx' + kx = 0$. Assume $m > 0$, $c > 0$, $k > 0$.

73. **(Monotonic Factor)** Let the equation be critically damped or over-damped. Prove that

$$x(t) = e^{-pt} f(t)$$

where $p \ge 0$ and $f(t)$ is monotonic (f' one-signed).

74. **(Harmonic Factor)** Let the equation be under-damped. Prove that

$$x(t) = e^{-at} f(t)$$

where $a > 0$ and $f(t) = c_1\cos\omega t + c_2\sin\omega t = A\cos(\omega t - \alpha)$ is a harmonic oscillation.

75. **(Limit Zero and Transients)** A term appearing in a solution is called **transient** if it has limit zero at $t = \infty$. Prove that positive damping $c > 0$ implies that the homogeneous solution satisfies $\lim_{t\to\infty} x(t) = 0$.

76. **(Steady-State)** An **observable** or **steady-state** is expression obtained from a solution by excluding all terms with limit zero at $t = \infty$. The **Transient** is the expression excluded to obtain the steady state. Assume $mx'' + cx' + kx = 25\cos 2t$ has a solution

$$x(t) = 2te^{-t} - \cos 2t + \sin 2t.$$

Find the transient and steady-state terms.

Damping Effects

Construct a figure on $0 \le t \le 2$ with two curves, to illustrate the effect of removing the dashpot. Curve 1 is the solution of $mx'' + cx' + kx = 0$, $x(0) = x_0$, $x'(0) = v_0$. Curve 2 is the solution of $my'' + ky = 0$, $y(0) = x_0$, $y'(0) = v_0$.

77. $m = 2, c = 12, k = 50,$
$x_0 = 0, v_0 = -20$

78. $m = 1, c = 6, k = 25,$
$x_0 = 0, v_0 = 20$

79. $m = 1, c = 8, k = 25,$
$x_0 = 0, v_0 = 60$

80. $m = 1, c = 4, k = 20,$
$x_0 = 0, v_0 = 4$

Envelope and Pseudo-period

Plot on one graphic the envelope curves and the solution $x(t)$, over two pseudo-periods. Use initial conditions $x(0) = 0$, $x'(0) = 4$.

81. $x'' + 2x' + 5x = 0$

82. $x'' + 2x' + 26x = 0$

83. $2x'' + 12x' + 50x = 0$

84. $4x'' + 8x' + 20x = 0$

6.8 Resonance

A highlight in the study of vibrating mechanical systems is the theory of pure and practical resonance.

Pure Resonance and Beats

The notion of **pure resonance** in the differential equation

$$x''(t) + \omega_0^2\, x(t) = F_0 \cos(\omega t) \tag{1}$$

is the existence of a solution that is unbounded as $t \to \infty$. Unbounded means *not bounded*. Bounded means a constant M exists such that $|x(t)| \le M$ for all values of t. Already known, The theory of **Beats** page 507 solves (1) for $\omega \ne \omega_0$. The solution is the sum of two harmonic oscillations, hence it is bounded. Equation (1) for $\omega = \omega_0$ has by the method of undetermined coefficients the unbounded oscillatory solution $x(t) = \dfrac{F_0}{2\omega_0}\, t\, \sin(\omega_0\, t)$. Technical details are similar to Example 6.53, *infra*.

> Pure resonance occurs exactly when the natural internal frequency ω_0 matches the natural external frequency ω, in which case all solutions of the differential equation are unbounded.

Figure 25 illustrates pure resonance for $x''(t)+16x(t) = 8\cos 4t$, which in equation (1) corresponds to $\omega = \omega_0 = 4$ and $F_0 = 8$.

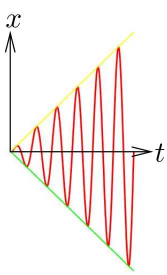

Figure 25. Pure resonance.
Equation $x''(t) + 16x(t) = 8\cos\omega t$, $\omega = 4$.
Graphs:
envelope curve $x = t$ yellow
envelope curve $x = -t$ green
solution $x(t) = t \sin 4t$ red

Resonance and Undetermined Coefficients

An explanation of resonance can be based upon the theory of undetermined coefficients. An initial trial solution for

$$x''(t) + 16x(t) = 8\cos\omega t$$

is $x = d_1 \cos\omega t + d_2 \sin\omega t$. The homogeneous solution is $x_h = c_1 \cos 4t + c_2 \sin 4t$. Euler atoms in $x_h(t)$ match Euler atoms in the trial solution $x = d_1 \cos\omega t +$

$d_2 \sin \omega t$ exactly when $\omega = 4$. RULE II in undetermined coefficients applies exactly for $\omega = 4$. The two cases $\omega \neq 4$ and $\omega = 4$ give final trial solution

$$x(t) = \begin{cases} d_1 \cos \omega t + d_2 \sin \omega t & \omega \neq 4, \\ t(d_1 \cos \omega t + d_2 \sin \omega t) & \omega = 4. \end{cases} \tag{2}$$

Even before the undetermined coefficients d_1, d_2 are evaluated, it is decided that unbounded solutions occur exactly when frequency matching $\omega = 4$ occurs, because of the amplitude factor t. If $\omega \neq 4$, then $x_p(t)$ is a pure harmonic oscillation, hence bounded. If $\omega = 4$, then amplitude factor t times a pure harmonic oscillation makes x_p unbounded.

Practical Resonance

The notion of pure resonance is easy to understand both mathematically and physically, because frequency matching characterizes the event. This ideal situation never happens in the physical world, because *damping is always present.* In the presence of damping $c > 0$, it will be established below that *only bounded solutions exist* for the forced spring-mass system

$$mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t. \tag{3}$$

Our intuition about resonance seems to vaporize in the presence of damping effects. But not completely. Most would agree that the undamped intuition is correct when the damping effects are nearly zero.

Practical resonance is said to occur when the external frequency ω has been tuned to produce the largest possible solution (a more precise definition appears below). It will be shown that the steady-state solution $x_{\text{SS}}(t)$ has maximum amplitude, over all possible input frequencies ω, at the precise tuned frequency $\omega = \Omega$ given by the equation

$$\Omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}. \tag{4}$$

The equation only makes sense when $\frac{k}{m} - \frac{c^2}{2m^2} > 0$. Pure resonance $\omega = \sqrt{k/m}$ is the limiting case obtained by setting the damping constant c to zero in condition (4). This strange but predictable interaction exists between the damping constant c and the magnitude of a solution, relative to the external frequency ω, even though all solutions remain bounded.

The decomposition of $x(t)$ into homogeneous solution $x_h(t)$ and particular solution $x_p(t)$ gives some intuition into the complex relationship between the input frequency ω and the size of the solution $x(t)$.

Homogeneous Solution $x_h(t)$

Solution $x_h(t)$ for homogeneous equation $mx''(t) + cx'(t) + kx(t) = 0$ for positive constants m, c, k will be shown to have limit zero at $t = \infty$, which means the graph of $x_h(t)$ follows the t-axis to $t = \infty$. An inequality of the form $|x_h(t)| \le e^{-qt}$ holds as $t \to \infty$, for some $q > 0$: see the proof of Theorem 6.21. Figure 26 shows that the graph of $x_h(t)$ can cross the t-axis infinitely often, even though it is trapped between envelope curves $x = \pm e^{-qt}$ near $t = \infty$.[7]

Theorem 6.21 (Transient Solution)
Assume positive values for m, c, k. The solution $x_h(t)$ of the homogeneous equation $mx''(t) + cx'(t) + kx(t) = 0$ has limit zero at $t = \infty$:

$$\lim_{t\to\infty} x_h(t) = 0 \qquad \text{for positive } m,\ c,\ k$$

Proof on page 542.

Definition 6.3 (Transient Solution)
A solution $x(t)$ of a differential equation is called a **transient solution** provided it satisfies the relation $\lim_{t\to\infty} x(t) = 0$.

A transient solution $x(t)$ for large t has its graph atop the axis $x = 0$, as in Figure 26.

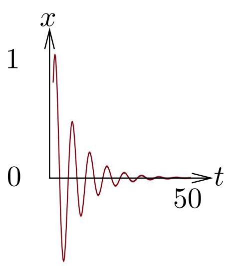

Figure 26. Transient Oscillatory Solution. Shown is solution $x = e^{-t/8}(\cos t + \sin t)$ of differential equation $64x'' + 16x' + 65x = 0$.

Particular Solution $x_p(t)$

Let's find $x_p(t)$ for $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$ by the method of undetermined coefficients. It will be found that $x_p(t)$ equals $x_{\rm SS}(t)$ defined in Definition 6.4 and explicitly given in equation (5) *infra*.

Definition 6.4 (Steady-State Solution)
Assume for non-homogeneous equation $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$ that m, c, k are all positive values. The **steady–state** solution $x_{\rm SS}(t)$ is a particular solution $x_p(t)$ in superposition $x(t) = x_p(t) + x_h(t)$, found from any general solution $x(t)$ by removing all terms containing negative exponentials. The terms removed add to some homogeneous solution $x_h(t)$.

[7] A funnel in first order theory for $y' = f(y)$ may also have limit $y = 0$ at infinity, but the funnel graph cannot cross $y = 0$.

Steady-state solution $x_{SS}(t)$ is **observable**, because it is visible as the graph of $x(t)$ for t large enough for the negative exponential terms become zero to pixel resolution. Uniqueness of $x_{SS}(t)$ implies Definition 6.4 is sensible, details in the proof of Theorem 6.22.

Theorem 6.22 (Steady-State Solution)
Assume positive values for m, c, k. The unique **steady-state solution** $x_{SS}(t)$ of the non-homogeneous equation $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$ with period $2\pi/\omega$ is given by

$$\begin{aligned} x_{SS}(t) &= \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2} \left((k - m\omega^2) \cos \omega t + (c\omega) \sin \omega t \right) \\ &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \alpha), \end{aligned} \tag{5}$$

where α is defined by the phase–amplitude relations (see page 492)

$$\begin{aligned} &C \cos \alpha = k - m\omega^2, \quad C \sin \alpha = c\omega, \\ &C = F_0 / \sqrt{(k - m\omega^2)^2 + (c\omega)^2}. \end{aligned} \tag{6}$$

Proof on page 543.

It is possible to be mislead by the method of undetermined coefficients, in which it turns out that $x_p(t)$ and $x_{SS}(t)$ are the same. Alternatively, a particular solution $x_p(t)$ can be calculated by variation of parameters, a method which produces in $x_p(t)$ extra terms containing negative exponentials. These extra terms come from the homogeneous solution – their appearance cannot always be avoided. This justifies the careful definition of steady–state solution, in which the transient terms are removed from a general solution $x(t)$ to produce $x_{SS}(t)$.

Definition 6.5 (Practical Resonance)
Assume positive values for m, c, k in non-homogeneous equation $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$. **Practical resonance** occurs if there is a value of external frequency $\omega > 0$ in which produces the largest possible steady-state amplitude $C(\omega)$ in the steady-state periodic solution x_{SS} defined by equation (5) in Theorem 6.22.

Theorem 6.23 (Practical Resonance Identity)
Assume positive values for m, c, k in non-homogeneous equation $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$. Practical resonance for $mx''(t) + cx'(t) + kx(t) = F_0 \cos \omega t$ occurs precisely when the external frequency ω is tuned to

$$\Omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}$$

and the square root argument $\frac{k}{m} - \frac{c^2}{2m^2}$ is positive.

Proof on page 543.

Theorem 6.24 (Pure Resonance Identity)
Assume m and k are positive in non-homogeneous equation $mx''(t) + kx(t) = F_0 \cos \omega t$. **Pure resonance** results from tuned external frequency value

$$\omega = \sqrt{\frac{k}{m}} = \left(\sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} \right) \Bigg|_{c=0}$$

This value is the limiting case $c = 0$ in Theorem 6.23. If $\omega = \frac{k}{m}$ is inserted into $mx''(t) + kx(t) = F_0 \cos \omega t$, then $x_p(t) = \dfrac{F_0}{2m\omega} t \sin(\omega t)$ is an unbounded solution, causing all solutions $x(t)$ to be unbounded. Proof on page 543.

An Illustration. Figure 27 illustrates practical resonance for $x'' + cx' + 26x = 10 \cos \omega t$. The amplitude $C(\omega)$ of the steady–state periodic solution is graphed against the external natural frequency ω for damping constants $c = 1, 2, 3$. The practical resonance condition is $\Omega = \sqrt{26 - c^2/2}$. As c increases from 1 to 3, the maximum point $(\Omega, C(\Omega))$ satisfies a monotonicity condition: both Ω and $C(\Omega)$ decrease as c increases. The maxima for the three curves in the figure occur at $\omega = \sqrt{25.5}$, $\sqrt{24}$, $\sqrt{21.5}$. Pure resonance occurs when $c = 0$ and $\omega = \sqrt{26}$.

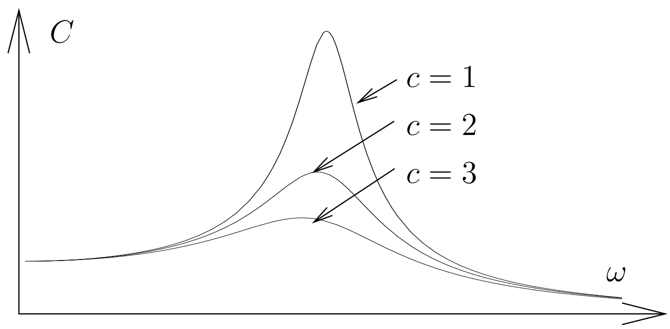

Figure 27. Practical resonance for $x'' + cx' + 26x = 10 \cos \omega t$.
The amplitude $C(\omega) = 10/\sqrt{(26 - \omega^2)^2 + (c\omega)^2}$ is plotted versus external frequency ω for $c = 1, 2, 3$.

Uniqueness of the Steady–State Periodic Solution

Any two solutions of the nonhomogeneous differential equation (3) which are periodic of period $2\pi/\omega$ must be identical by Theorem 6.22. A more general statement is true:

Theorem 6.25 (Uniqueness of a T-Periodic Solution)
Assume m, c, k positive. Consider the equation $mx''(t) + cx'(t) + kx(t) = f(t)$ with f continuous and T-periodic: $f(t + T) = f(t)$. Then a T–periodic solution is unique. Proof on page 544.

An Illustration. In Figure 28, the unique steady–state periodic solution is graphed for the differential equation $x'' + 2x' + 2x = \sin t + 2 \cos t$. The transient solution of the homogeneous equation and the steady–state solution appear in Figure 29. In Figure 30, several solutions are shown for the differential equation $x'' + 2x' + 2x = \sin t + 2\cos t$, all of which reproduce at $t = \infty$ the steady–state solution $x = \sin t$.

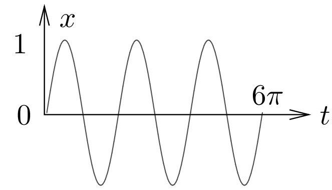

Figure 28. Steady-state solution.
Differential equation $x'' + 2x' + 2x = \sin t + 2\cos t$.
Periodic steady-state solution $x_{SS} = \sin t$.

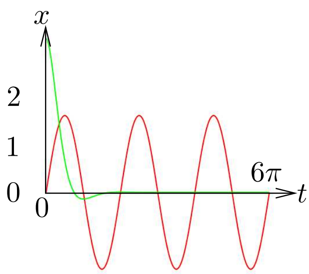

Figure 29. Transient and Steady-state.
General solution $x(t)$ is the graphical sum of x_h (green) and x_{SS} (red):
Transient Green $x_h = e^{-t}(2 * \cos t + 2\sin t)$
Steady-state Red $x_{SS} = \sin t$

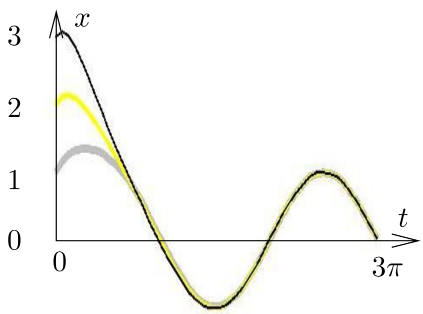

Figure 30. Steady-state.
Initial value problem solutions of $x'' + 2x' + 2x = \sin t + 2\cos t$ with $x'(0) = 1$ and $x(0) = 1, 2, 3$.
All graphically coincide with the steady-state solution $x = \sin t$ for $t \geq \pi$.

Pseudo–Periodic Solution

Resonance gives rise to solutions of the form $x(t) = A(t)\sin(\omega t - \alpha)$ where $A(t)$ is a time–varying amplitude. Figure 31 shows such a solution, which is called a **pseudo–periodic solution** because it has a natural period $2\pi/\omega$ arising from the trigonometric factor $\sin(\omega t - \alpha)$. The only requirement on $A(t)$ is that it be non–vanishing, so that it acts like an amplitude. The **pseudo–period** of a pseudo–periodic solution can be determined graphically, by computing the length of time it takes for $x(t)$ to vanish three times.

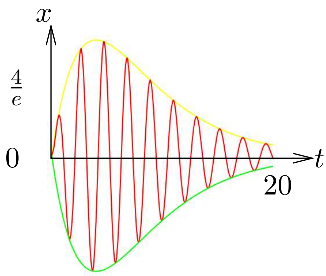

Figure 31. Pseudo-periodic solution.
Equation $16x'' + 8x' + 145x = 96e^{-t/4}\cos 3t$.
Legend for the graphic:
Envelope $x = te^{-t/4}$ Yellow
Envelope $x = -te^{-t/4}$ Green
Solution $x = te^{-t/4}\sin(3t)$ Red

The pseudo-period $2\pi/3$ of $x = te^{-t/4}\sin(3t)$ is found by solving for t in $x(t) = 0$, equivalently $te^{-t/4}\sin(3t) = 0$. Then $3t = 0, \pi, 2\pi$ are the first three crossings of $x(t)$ with the t-axis. The pseudo-period is $2\pi/3$. The terminology does not mean that $x(t)$ is periodic, but pseudo-periodic, which is a periodic function multiplied by a nonzero amplitude function.

Resonance History

Soldiers Breaking Cadence, 1831

Figure 32. The Rebuilt Broughton Suspension Bridge.
On 12 April 1831, the original bridge collapsed, blamed on mechanical resonance from troops marching in cadence. The bridge spans the River Irwell between Broughton and Pendleton near Manchester, England. Photo from 1883.

The collapse of the Broughton suspension bridge in 1831 reportedly caused the now–standard military rule of breaking cadence when soldiers cross a bridge. Bridges like the Broughton bridge have many natural low frequencies of vibration, so it is possible for a column of soldiers to vibrate the bridge at one of the bridge's natural frequencies. The bridge locks onto the frequency while the soldiers continue to add to the excursions with every step, causing larger and larger bridge oscillations.

Figure 33. The London Albert Bridge.
A sign added in 1973 warns marching ranks of soldiers to break cadence.

The Tacoma Narrows Bridge, 1940

The literature is rich with accounts of the November 7, 1940 Tacoma bridge disaster, the date when the bridge fell into the Tacoma Narrows.

Figure 34. The Tacoma Narrows Bridge, 1940.
Historically, the disaster has been presented as an instance of **resonance**, a technical term which requires a periodic input of energy. No observer witnessed a **periodic** input of energy, and this is the source of the controversy over the cause of the bridge failure.

The bridge disaster has been blamed on **Aeroelastic Flutter**, a term used for aircraft:

> If energy input by aerodynamic excitation is larger than what is dissipated by system damping, then the amplitude of vibration will increase, resulting in self-exciting oscillation.

The Tacoma bridge was injected with energy from a 40 mph wind. The energy did not dissipate through the damping properties of the bridge structure. The energy was dissipated by the formation of longitudinal and transverse vibrations of the roadway, which eventually lead to failure.

There have been other explanations, none of which are more popular than aeroelastic flutter.

1940 Theodore von Karman proposed that **vortex shedding** had created a (periodic) force in its wake that excited the bridge into resonant oscillations. This resonance theory requires a periodic input caused by a 40 mph wind acting on the bridge structure. Wind tunnel experiments seemed to verify the explanation. The final *Federal Works Administration* report rejected the explanation.

2000 The resonance model was re-visited, because the hanging bridge suspension cables produce a force only in one direction. Using a modification of the classical linear resonance model, simulations reproduced oscillation magnitudes seen in the 1940 film of the bridge failure.

The Wine Glass Experiment, 1985

The equation $mx'' + cx' + kx = F_0 \cos(\omega t)$ with c replaced by zero is advertised as the basis for a physics experiment to break a wine glass with resonant sound waves.

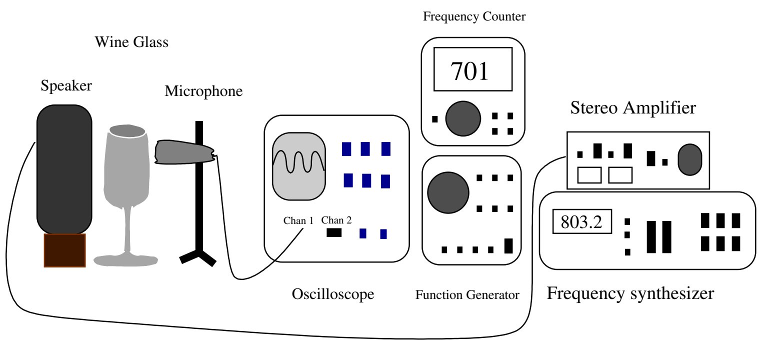

Figure 35. The Wine Glass Experiment Lab Table.
Equipment: A wine glass, a stereo amplifier, a speaker for sound waves, a frequency generator and a microphone connected to an oscilloscope.

The *wine glass experiment* is a portion of a film produced in 1985 by the Annenberg/CPB Project in Episode 17, **Resonance**, which is one of 52 episodes in **The Mechanical Universe** series. A **synopsis** appears below for a portion of episode 17, with parenthetical remarks inserted for the model equation $mx'' + kx = F_0 \cos \omega t$.

> A physicist in front of an audience of physics students equips a lab table with a frequency generator, an amplifier and an audio speaker. The *valuable* wine glass is replaced by a glass beaker. The frequency generator is tuned to the natural frequency of the glass beaker ($\omega \approx \omega_0$), then the volume knob on the amplifier is suddenly turned up (F_0 adjusted larger), whereupon the sound waves emitted from the speaker break the glass beaker.

The glass itself will vibrate at a certain frequency, as can be determined experimentally by *pinging* the glass rim. This vibration operates within elastic limits of the glass and the glass will not break under these circumstances. A physical explanation for the breakage is that an incoming sound wave from the speaker is timed to add to the glass rim excursion. After enough amplitude additions, the glass rim moves beyond the elastic limit and the glass breaks. The explanation implies that the external frequency from the speaker has to match the natural frequency of the glass. But there is more to it: the glass has some natural damping that nullifies feeble attempts to increase the glass rim amplitude. The physicist uses to great advantage this natural damping to *tune* the external frequency to the glass. The reason for turning up the volume on the amplifier is to nullify the damping effects of the glass. The amplitude additions then build rapidly and the glass breaks.

The London Millennium Foot-Bridge, 2000

Figure 36. The London Millennium Foot-Bridge.
Opened June 10, 2000 and closed two days later, London visitors nicknamed it the **Wobbly Bridge**. The reconstruction finished in 2002 added 5M pounds to the initial cost of 18M.

The opening of the bridge brought crowds of 90,000 people per day. The natural swaying motion of people walking across the span caused small sideways bridge oscillations, which in turn caused people on the bridge to sway in step, adding to the amplitude of the bridge oscillations.

Engineers fixed the vibration problem by retrofitting 37 energy dissipating viscous fluid dashpots to control horizontal movement and 52 tuned inertial mass dampers to control vertical movement.

Examples and Methods

Example 6.53 (Beats and Pure Resonance)

Solve by undetermined coefficients for a particular solution of the equation $x''(t) + 16x(t) = 8\cos\omega t$ for all values of $\omega > 0$, verifying that

$$x_p(t) = \begin{cases} \dfrac{8}{16-\omega^2}\cos(\omega t) & \omega \neq 4, \\ t\sin(4t) & \omega = 4. \end{cases}$$

Solution:

Trial solution details. Rule I of undetermined coefficients requires derivatives of $f(t) = 8\cos(\omega t)$, which are linear combinations of Euler atoms $\cos(\omega t)$, $\sin(\omega t)$. Then the Rule I trial solution is $x = d_1\cos(\omega t) + d_2\sin(\omega t)$.

The homogeneous solution solves $x'' + 16x = 0$, then $x_h = c_1\cos(4t) + c_2\sin(4t)$. Euler atoms $\cos(\omega t), \sin(\omega t)$ will be homogeneous solutions if and only if $\omega = 4$. Rule II applies only in the case $\omega = 4$, in which case the trial solution is $x = d_1 t\cos(4t) + d_2 t\sin(4t)$ (ωt equals $4t$).

Details for Beats, $\omega \neq 4$: Write $u = \cos(\omega t), v = \sin(\omega t)$ and $x(t) = d_1u + d_2v$. Then $x(t) = d_1u + d_2v$. Because $u'' + \omega^2 u = 0$ and $v'' + \omega^2 v = 0$, then $x'' + \omega^2 x = 0$.

$x'' + 16x = 8u$	Original equation, $u = \cos(\omega t)$.
$-\omega^2 x + 16x = 8u$	Substitute from $x'' + \omega^2 x = 0$.
$(16-\omega^2)(d_1u + d_2v) = 8u$	Collect on x. Substitute $x = d_1u + d_2v$.
$\left\lvert \begin{array}{l} (16-\omega^2)d_1 = 8, \\ (16-\omega^2)d_2 = 0. \end{array} \right\rvert$	Independence. Match coefficients of u, v.
$d_1 = \dfrac{8}{16-\omega^2},\ d_2 = 0$	Solve for d_1, d_2.

Details for Pure Resonance, $\omega = 4$: Define $u = \cos(4t), v = \sin(4t)$. The modified trial solution $x(t)$ then satisfies

$$\begin{array}{rcl} x(t) & = & d_1tu + d_2tv, \\ x'(t) & = & d_1u + d_2v - 4d_1tv + 4d_2tu, \\ x''(t) & = & -8d_1v + 8d_2u - 16x(t). \end{array} \tag{7}$$

Then

$x'' + 16x = 8u$	Original equation, $u = \cos(\omega t)$.
$-8d_1v + 8d_2u = 8u$	Use equation (7), then cancel $16x(t)$.
$\left\lvert \begin{array}{rcl} -8d_1 & = & 0, \\ 8d_2 & = & 8. \end{array} \right\rvert$	Independence of u, v implies matching coefficients.
$\begin{aligned} x(t) &= d_1tu + d_2tv \\ &= tv, \\ &= t\sin(4t). \end{aligned}$	Insert answers $d_1 = 0, d_2 = 1$. Answer found.

Example 6.54 (Damped Forced Spring-Mass System Trial Solution)

To equation $mx'' + cx' + kx = F_0 \cos(\omega t)$ with all coefficients positive apply undetermined coefficients to obtain trial solution

$$x(t) = A\cos\omega t + B\sin\omega t.$$

Solution: The derivatives of $f(t) = F_0\cos(\omega t)$ are linear combinations of Euler solution atoms $\cos(\omega t), \sin(\omega t)$. Rule I of the method of undetermined coefficients gives trial solution $x(t) = d_1\cos(\omega t) + d_2\sin(\omega t)$.

For characteristic equation $mr^2 + cr + k = 0$ with positive m, c, k, there are 3 cases to consider, based on the sign of the discriminant. In all 3 cases, equation $mr^2 + cr + k = 0$ has roots with nonzero real part. For instance, the real part is $-\frac{c}{2m}$ for a negative discriminant. Then the trial solution is not a solution of the homogeneous differential equation $mx'' + cx' + kx = 0$. Rule I in the method of undetermined coefficients does not fail and Rule II is not applied.

The reported trial solution is the final trial solution. To agree with notation, replace symbols d_1, d_2 by symbols A, B and report trial solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$.

Example 6.55 (Undetermined Coefficients Calculation)

Substitute the trial solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$ into the equation $mx'' + cx' + kx = F_0\cos(\omega t)$ to obtain the system of equations

$$\begin{array}{rcrcl} (k - m\omega^2)A & + & (c\omega)B & = & F_0, \\ (-c\omega)A & + & (k - m\omega^2)B & = & 0. \end{array} \tag{8}$$

Solution: Define $u = \cos(\omega t), v = \sin(\omega t)$, to simplify the displays. Equations $u'' + \omega^2 u = 0$ and $v'' + \omega^2 v = 0$ are valid. By superposition, $x'' + \omega^2 x = 0$ holds for the trial solution $x(t) = A\cos(\omega t) + B\sin(\omega t)$.

$mx'' + cx' + kx = F_0 u$	Original differential equation.
$-m\omega^2 x + cx' + kx = F_0 u$	Use $x'' + \omega^2 x = 0$.
$(k - m\omega^2)x + cx' = F_0 u$	Collect on x and x'.
$\begin{aligned}&(k - m\omega^2)(Au + Bv) + \\ &\quad c(-A\omega v + B\omega u) = F_0 u\end{aligned}$	Expand with $x = Au + Bv$ and $x' = -A\omega v + B\omega u$.
$\begin{aligned}&\left((k - m\omega^2)A + c\omega B\right) u + \\ &\left(-c\omega A + (k - m\omega^2)B\right) v = F_0 u\end{aligned}$	Collect on u, v.
$\begin{aligned}\left((k - m\omega^2)A + c\omega B\right) &= F_0, \\ \left(-c\omega A + (k - m\omega^2)B\right) &= 0.\end{aligned}$	Independence of u, v implies their coefficients match.
$\begin{array}{rrl}(k - m\omega^2)A & +c\omega B & = F_0, \\ -c\omega A & +(k - m\omega^2)B & = 0.\end{array}$	Linear equations in unknowns A, B. System (8) found.

Example 6.56 (Cramer's Rule Solution for A, B)

Verify using Cramer's determinant rule the formulas

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}$$

for the answers A, B to the system of equations (8).

Solution: Cramer's 2×2 rule for system $a_{11}x_1 + a_{12}x_2 = b_1$, $a_{21}x_1 + a_{22}x_2 = b_2$ is the set of equations

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta},\ \Delta = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},\ \Delta_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix},\ \Delta_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}.$$

Apply these formulas to system (8). Then

$$\Delta = \begin{vmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{vmatrix},\ \Delta_1 = \begin{vmatrix} F_0 & c\omega \\ 0 & k - m\omega^2 \end{vmatrix},\ \Delta_2 = \begin{vmatrix} k - m\omega^2 & F_0 \\ -c\omega & 0 \end{vmatrix}.$$

Sarrus' 2×2 rule is applied to evaluate the determinants. Then

$$\Delta = (k - m\omega^2)^2 + (c\omega)^2,\ \Delta_1 = (k - m\omega^2)F_0,\ \Delta_2 = c\omega F_0.$$

Cramer's rule formulas $A = \frac{\Delta_1}{\Delta}$, $B = \frac{\Delta_2}{\Delta}$ give the reported answers.

Example 6.57 (Transient and Steady-State Solutions)

Compute the transient and steady-state solutions x_{tr} and x_{ss} for the equation $2x'' + 3x' + 2x = 174\cos(4t)$, verifying the formulas

$$\begin{aligned} x_{\text{tr}} &= e^{-3t/4}\left(c_1\cos(kt) + c_2\sin(kt)\right),\ k = \tfrac{\sqrt{7}}{4}, \\ x_{\text{ss}} &= -5\cos(4t) + 2\sin(4t). \end{aligned}$$

Solution:
Homogeneous Solution: The characteristic equation $2r^2 + 3r + 2 = 0$ has complex conjugate roots $-\frac{3}{4} \pm \frac{\sqrt{7}}{4}i$. Then the Euler solution atoms are $e^{-3t/4}\cos(kt)$, $e^{-3t/4}\sin(kt)$ where $k = \frac{\sqrt{7}}{4}$. The homogeneous solution is then

$$x_h = e^{-3t/4}\left(c_1\cos(kt) + c_2\sin(kt)\right).$$

Particular Solution: The method of undetermined coefficients applies with Rule I trial solution $x = A\cos(4t) + B\sin(4t)$. Let's justify this statement. The right side $f(t) = 174\cos(4t)$ has derivatives a linear combination of the Euler solution atoms $\cos(4t), \sin(4t)$. Rule I does not fail, because these Euler atoms are not solutions of the homogeneous equation. Rule II does not apply, and the final trial solution is $x = A\cos(4t) + B\sin(4t)$.

Let $u = \cos(4t), v = \sin(4t)$. Then $u'' + 16u = 0, v'' + 16v = 0$. Superposition implies $x'' + 16x = 0$. The following steps find the undetermined coefficients $A = -5, B = 2$.

$2x'' + 3x' + 2x = 174u$	Original differential equation, $u = \cos 4t$.
$-32x + 3x' + 2x = 174u$	Substitute $x'' + 16x = 0$, where $x = Au + Bv$.
$-30(Au + Bv) +$ $3(-4Av + 4Bu) = 174u$	Substitute $x = Au + Bv$, $x' = -4Av + 4Bu$.
$(-30A + 12B)u +$ $(-12A - 30B)v = 174u$	Collect on u, v.
$\begin{vmatrix} -30A + 12B = & 174, \\ -12A - 30B = & 0. \end{vmatrix}$	Independence of u, v implies matching coefficients (independent Euler atoms).

$$\left| \begin{array}{l} A = -5, \\ B = 2. \end{array} \right| \qquad \text{Solve for } A, B \text{ by elimination or Cramer's rule.}$$

The particular solution is $x_p = -5\cos 4t + 2\sin 4t$.

General Solution: Superposition gives general solution

$$x = x_h + x_p = e^{-3t/4}\left(c_1\cos(kt) + c_2\sin(kt)\right) - 5\cos 4t + 2\sin 4t.$$

Transient Solution: This is the part of the general solution with negative exponential terms (terms that limit to zero at infinity). Then

$$x_{\text{tr}} = e^{-3t/4}\left(c_1\cos(kt) + c_2\sin(kt)\right).$$

Steady-State Solution: This is the part of the solution left over after the transients are removed. Then

$$x_{\text{SS}} = -5\cos 4t + 2\sin 4t.$$

Example 6.58 (Pseudo-periodic solution)
Derive the pseudo-periodic solution $x = te^{-t/4}\sin(3t)$ and its envelope curves $x = \pm te^{-t/4}$ for the equation $16x'' + 8x' + 145x = 96e^{-t/4}\cos 3t$.

Solution:
Envelope Curves. For damped oscillations, a solution of the form $x(t) = e^{at}(c_1\cos(bt) + c_2\sin(bt))$ has to be re-written in phase-amplitude form, using the formulas from page 492. Then $x(t) = Ce^{at}\cos(bt - \alpha)$ and by definition the envelope curves are $x = \pm Ce^{at}$, because the cosine factor has extreme values ± 1.

In the present example, the pseudo-periodic solution is $x(t) = te^{-t/4}\sin(3t)$. The same logic applies. The sine factor has extreme values ± 1, then the envelope curves are $x = \pm te^{-t/4}$.

Pseudo-periodic Solution. Undetermined coefficients will be applied to find a particular solution x_p of $16x'' + 8x' + 145x = 96e^{-t/4}\cos 3t$. It turns out that the desired pseudo-periodic solution is the undetermined coefficients answer $x = te^{-t/4}\sin 3t$. This is because the method subtracts all homogeneous terms from the particular solution. Superposition $x = x_h + x_p$ was invisibly used here. If x_p was found from another method, then homogeneous terms should be removed from the answer, before reporting the pseudo-periodic solution.

Homogeneous Solution. It is found from $16x'' + 8x' + 145x = 0$. The Euler solution atoms are $e^{-t/4}\cos(3t), e^{-t/4}\sin(3t)$, found from the characteristic equation $16r^2 + 8r + 145 = 0$, which has complex conjugate roots $r = -\frac{1}{4} \pm 3i$. Then

$$x_h(t) = c_1 e^{-t/4}\cos(3t) + c_2 e^{-t/4}\sin(3t).$$

Particular solution x_p. It is found by undetermined coefficients. The answer to be justified below is

$$x_p(t) = te^{-t/4}\sin(3t).$$

Differentiate the right side $f(t) = 96e^{-t/4}\cos 3t$ of the non-homogeneous equation to identify the Euler atoms $e^{-t/4}\cos 3t, e^{-t/4}\sin 3t$. Rule I of undetermined coefficients fails, because these atoms are solutions of the homogeneous equation. Then Rule II is applied to find the final trial solution

$$\begin{aligned} x &= t\left(d_1 e^{-t/4}\cos 3t + d_2 e^{-t/4}\sin 3t\right) \\ &= t(d_1 u + d_2 v) \end{aligned}$$

where $u = e^{-t/4}\cos 3t$ and $v = e^{-t/4}\sin 3t$. Then u, v are solutions of $16x''+8x'+145x = 0$. Define $w = d_1 u + d_2 v$. Superposition implies w is also a solution of $16x''+8x'+145x = 0$.

Compute the derivatives of the trial solution:

$$\begin{array}{rcl} x &=& t\,(d_1 u + d_2 v) = tw, \\ x' &=& w + tw' \\ x'' &=& 2w' + tw''. \end{array} \tag{9}$$

$16x'' + 8x' + 145x = 96u$	Original equation, $u = e^{-t/4}\cos 3t$.
$16(2w' + tw'')+$ $8(w + tw') + 145tw = 96u$	Use equations (9).
$32w' + 8w+$ $t(16w'' + 8w' + 145w) = 96u$	Collect terms on factor t.
$32w' + 8w = 96u$	Use homogeneous equation $16w'' + 8w' + 145w = 0$.
$-96d_1 v + 96d_2 u = 96u$	Expand $w = d_1u+d_2v$, $w' = -\frac{1}{4}w - 3d_1v + 3d_2u$. Cancel $8w$.
$d_1 = 0, d_2 = 1$	Independence of u, v implies matching coefficients.

The trial solution $x = tw$ becomes $x = te^{-t/4}\sin(3t)$.

Other Methods to Find x_p. The possible methods are variation of parameters, Laplace theory and a computer algebra system. Below is sample `maple` code to check the answer given above.

```
de:=16*diff(x(t),t,t)+8*diff(x(t),t)+145*x(t)=
    96*exp(-t/4)*cos(3*t);
dsolve(de,x(t));
```

The answer involves homogeneous terms with arbitrary constants _C1, _C2. These terms must be removed to check the answer, $x_p = te^{-t/4}\sin(3t)$.

The example is complete.

Proofs and Technical Details

Proof of Theorem 6.21, Transient Solution:

For positive damping $c > 0$, equation (3) has homogeneous solution $x_h(t) = c_1x_1(t) + c_2x_2(t)$ where Euler atoms x_1 and x_2 are according to Theorem 6.1 page 430 given in terms of the roots of the characteristic equation $mr^2 + cr + k = 0$ as follows:

Let $D = c^2 - 4mk$.	The discriminant of $mr^2 + cr + k = 0$.
Case 1, $D > 0$	$x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$ with r_1 and r_2 negative.

Case 2, $D = 0$	$x_1 = e^{r_1 t}$, $x_2 = te^{r_1 t}$ with r_1 negative.
Case 3, $D < 0$	$x_1 = e^{at}\cos bt$, $x_2 = e^{at}\sin bt$ with $b > 0$ and a negative.

Let's verify that $x_h(t) = e^{-qt}$(bounded function) for some $q > 0$, regardless of the positive values of m, c, k. For instance, Case 2 implies $x_h = e^{r_1 t/2}(c_1 e^{r_1 t/2} + c_2 te^{r_1 t/2})$ and $(c_1 e^{r_1 t/2} + c_2 te^{r_1 t/2})$ is bounded by some number M. Let $-q = r_1/2 < 0$. Then $|x_h(t)| \le Me^{-qt}$, which proves $x_h(t)$ has limit zero at $t = \infty$. A similar analysis applied to cases 1,2,3 reveals that $|x_h(t)| \le Me^{-qt}$ holds if q is smaller than $|\mathcal{R}\mathrm{e}(\lambda)|$ for all roots λ of the characteristic equation. ■

Proof of Theorem 6.22, Steady-State Solution

Uniqueness. Assume (5) has been proved. Suppose $x(t)$ is a periodic solution of period $2\pi/\omega$. Superposition implies $x(t) = x_{\mathrm{SS}}(t) + x_h(t)$ for some homogeneous solution $x_h(t)$. Then $x(t) - x_{\mathrm{SS}}(t)$ has period $2\pi/\omega$ and equals some $x_h(t)$, which has limit zero at $t = \infty$ by Theorem 6.21. Because a nonzero periodic function cannot have limit zero at $t = \infty$, then $x_h(t) = 0$, proving uniqueness $x(t) = x_{\mathrm{SS}}(t)$.

Details for (6). The method of undetermined coefficients applies to $mx''(t) + cx'(t) + kx(t) = F_0\cos\omega t$ with trial solution $x(t) = A\cos\omega t + B\sin\omega t$. The TEST succeeds, because by Theorem 6.21 the Euler atoms in $x_h(t)$ cannot match $\cos\omega t$ or $\sin\omega t$. More details in Example 6.54 page 539. Substitution of $x(t)$ into $mx''(t) + cx'(t) + kx(t) = F_0\cos\omega t$ produces a linear combination of Euler atoms on the left. Match the coefficients of the atoms left and right to verify the equations

$$\begin{array}{rcrcl} (k - m\omega^2)A & + & (c\omega)B & = & F_0, \\ (-c\omega)A & + & (k - m\omega^2)B & = & 0. \end{array} \tag{10}$$

Details in Example 6.55 page 539. Solve (10) for A, B with Cramer's rule or elimination. Then:

$$A = \frac{(k - m\omega^2)F_0}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2}. \tag{11}$$

Details in Example 6.56 page 539. Substitute the answers in (11) into trial solution $x(t) = A\cos\omega t + B\sin\omega t$. Convert this solution to phase-amplitude form using formulas on page 492. Then (6) holds. ■

Proof of Theorem 6.23, Practical Resonance Identity:

Mathematically, a maximum happens exactly when the amplitude function $C = C(\omega)$ defined in (6) has a maximum. If a maximum exists on $0 < \omega < \infty$, then $C'(\omega) = 0$ at the maximum. The derivative is computed by the power rule:

$$\begin{aligned} C'(\omega) &= \frac{-F_0}{2}\,\frac{2(k - m\omega^2)(-2m\omega) + 2c^2\omega}{((k - m\omega^2)^2 + (c\omega)^2)^{3/2}} \\ &= \omega\left(2mk - c^2 - 2m^2\omega^2\right)\frac{C(\omega)^3}{F_0^2} \end{aligned} \tag{12}$$

If $2km - c^2 \le 0$, then $C'(\omega)$ does not vanish for $0 < \omega < \infty$. Then $C'(\omega)$ is one-signed and there is no maximum. If $2km - c^2 > 0$, then $2km - c^2 - 2m^2\omega^2 = 0$ has exactly one root $\omega = \sqrt{k/m - c^2/(2m^2)}$ in $0 < \omega < \infty$. Because $C(\infty) = 0$, then $C(\omega)$ is a maximum. ■

Proof of Theorem 6.24, Pure Resonance Identity:

The details follow Example 6.53 page 538. Let $\omega = \frac{k}{m}$. The homogeneous equation

$mx''(t) + kx(t) = 0$ has general solution $x = c_1x_1 + c_2x_2$ given by Euler atoms $x_1 = \cos\omega t$, $x_2 = \sin\omega t$. Undetermined coefficients applies, RULE II giving modified trial solution $X = t(d_1\cos\omega t + d_2\sin\omega t)$. Like Example 6.53, the trial solution is inserted into $mx''(t) + kx(t) = F_0\cos\omega t$, then Euler atom coefficients are matched left and right to obtain a diagonal system of linear algebraic equations for d_1, d_2. The answer: $d_1 = 0$, $d_2 = \frac{F_0}{2m\omega}$. Insert the answers into the trial solution to find $x_p(t) = 0 + d_2 t\sin\omega t = \frac{F_0}{2m\omega}t\sin\omega t$. ■

Proof of Theorem 6.25, Uniqueness T-periodic Solution:
The vehicle of proof is to show that the difference $x(t)$ of two T-periodic solutions is zero. Difference $x(t)$ is a solution of the homogeneous equation, it is T–periodic and it has limit zero at infinity. A periodic function with limit zero must be zero, therefore $x(t) = 0$, which proves the two solutions are identical. ■

Exercises 6.8

Beats

Each equation satisfies the beats relation $\omega \ne \omega_0$. Find the general solution. See Example 6.53, page 538.

1. $x'' + 100x = 10\sin 9t$

2. $x'' + 100x = 5\sin 9t$

3. $x'' + 25x = 5\sin 4t$

4. $x'' + 25x = 5\cos 4t$

Pure Resonance

Each equation satisfies the pure resonance relation $\omega = \omega_0$. Find the general solution. See Example 6.53, page 538.

5. $x'' + 4x = 10\sin 2t$

6. $x'' + 4x = 5\sin 2t$

7. $x'' + 16x = 5\sin 4t$

8. $x'' + 16x = 10\sin 4t$

Practical Resonance

For each model, find the **tuned practical resonance** frequency Ω and the **resonant amplitude** C:

$$\Omega = \sqrt{k/m - c^2/(2m^2)},$$
$$C = F_0/\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}$$

9. $x'' + 2x' + 17x = 100\cos(4t)$

10. $x'' + 2x' + 10x = 100\cos(4t)$

11. $x'' + 4x' + 5x = 10\cos(2t)$

12. $x'' + 2x' + 6x = 10\cos(2t)$

Transient Solution

Identify from superposition $x = x_h + x_p$ a shortest particular solution, given one particular solution.

13. $x'' + 2x' + 10x = 26\cos(3t)$,
$x = 100e^{-t}\cos(3t) + 3\cos(2t) + 2\sin(2t)$

14. $x'' + 4x' + 13x = 920\cos(3t)$,
$x = 5e^{-2t}\cos(3t) + 23\cos(3t) + 69\sin(3t)$

15. $x'' + 2x' + 2x = 2\cos(t)$,
$x = 3e^{-t}\sin(t) + 5e^{-t}\cos(t) + \cos(t) + 2\sin(t)$

16. $x'' + 2x' + 17x = 65\cos(4t)$,
$x = -2e^{-t}\sin(4t) + 7e^{-t}\cos(4t) + \cos(4t) + 8\sin(4t)$

Steady-State Periodic Solution

Consider the model $mx'' + cx' + kx = F_0\cos(\omega t)$ of external frequency ω. Compute the unique steady-state solution $A\cos(\omega t) + B\sin(\omega t)$ and its amplitude $C(\omega) = \sqrt{A^2 + B^2}$. Graph the ratio $100C(\omega)/C(\Omega)$ on $0 < \omega < \infty$, where Ω is the tuned practical resonance frequency.

17. $x'' + 2x' + 17x = 100\cos(4t)$

18. $x'' + 2x' + 10x = 100\cos(4t)$

19. $x'' + 4x' + 5x = 10\cos(2t)$

20. $x'' + 2x' + 6x = 10\cos(2t)$

21. $x'' + 4x' + 5x = 5\cos(2t)$

22. $x'' + 2x' + 5x = 5\cos(1.5t)$

Phase-Amplitude

Solve for a particular solution in the form $x(t) = C\cos(\omega t - \alpha)$.

23. $x'' + 6x' + 13x = 174\sin(5t)$

24. $x'' + 8x' + 25x = 100\cos(t) + 260\sin(t)$

6.9 Kepler's laws

Kepler's empirical laws of planetary motion are:

1. All planets move in elliptical orbits with the sun at one focus.
2. The radius vector from the sun to any planet sweeps out equal areas in equal times.
3. The square of the orbital period is proportional to the cube of the major semi-axis of its elliptical orbit.

Precise observations over 20 years on the planets and 777 stars visible to the naked eye were made by the Danish astronomer Tycho Brahe (1546-1601), who was a teacher of the German astronomer Johannes Kepler (1571-1630). It is Kepler who is credited with analyzing his teacher's observations, from which he deduced the three laws of planetary motion, about 1605. The results were published in 1609 and 1618.

About 100 years after Kepler, Isaac Newton formulated his renowned universal gravitation law. Newton showed in his *Principia Mathematica* (1687) that Kepler's laws implied his universal gravitation law. Newton also showed that Kepler's first two laws were a consequence of the universal gravitation law.

The purpose of this section is to establish Kepler's first two laws from Newton's universal gravitation law. Modern calculus courses provide the differential equations background outlined below.

Background

The derivation of Kepler's first two laws from Newton's law requires diverse background from calculus, analytic geometry, physics and differential equations. Outlined here is the material required to understand the derivation.

Analytic Geometry

An ellipse or circle equation in standard form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The numbers $a > 0$, $b > 0$ are called the major and minor semi-axis lengths, respectively. They are related by $b = a\sqrt{1 - e^2}$, where $0 \le e < 1$ is called the eccentricity. The equation is a circle if and only if $e = 0$.

Polar Coordinates

A point (r, θ) in polar coordinates is related to its rectangular coordinates (x, y) defined by the equations

$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = 1, \quad \tan\theta = y/x.$$

Circles and ellipses have respectively the polar equations

$$r = 2a\cos(\theta - \theta_0), \quad r = \frac{ed}{1 + e\cos(\theta - \theta_0)}.$$

The number $a > 0$ is the radius of the circle. The number $d > 0$ is the distance to the directrix. The eccentricity satisfies $0 < e < 1$.

Calculus

The area of a sector in polar coordinates is given by

$$A = \frac{1}{2} r^2 \theta.$$

A polar equation $r = f(\theta)$ encloses on the interval $\theta_1 \le \theta \le \theta_2$ the area

$$A = \int_{\theta_1}^{\theta_2} |f(\theta)|^2 \, d\theta.$$

Physics

Newton's universal gravitation law is given by the formula

$$F = G\,\frac{m_1 m_2}{r^2},$$

where $G = 6.672 \times 10^{-11} \frac{\mathrm{N{\cdot}m^2}}{\mathrm{kg}^2}$ is the universal gravitation constant and r is the distance between the two masses m_1, m_2. This equation gives only the magnitude of the force. Implied by the formula is the value of the fundamental constant $g \approx 9.80$ meters per second, the acceleration due to gravity:

$$g = G\frac{M}{R},$$

where $M \approx 5.98 \times 10^{24}$ kilograms and $R \approx 6.38 \times 10^6$ meters are respectively the mass of the earth and the radius of the earth. A similar formula applies for any planet. While g is computed for sea level, it varies significantly with altitude, e.g., 7.33 to 0.13 at altitudes from 1000 to $50,000$ kilometers.

Differential Equations

The second order differential equation

$$u'' + u = 0$$

is called the **harmonic oscillator**. It's solution is $u = c_1 \cos x + c_2 \sin x$, by the classical constant-coefficient Theorem 6.1. The forced equation $u'' + u = c$, where c is a constant, has a particular solution $u = c$, obtained by the equilibrium method. Therefore, the forced equation has the general solution

$$u = c_1 \cos x + c_2 \sin x + c.$$

Derivation of Kepler's First Two Laws

The second law will be derived first, then the details are used to derive the first law. The third law is not discussed here.

Kepler's Second Law

Assumed is the sun at the origin in the plane of motion of the planet. The position of the planet is written in vector form in polar coordinates by the formula

$$\vec{\mathbf{r}}(t) = \begin{pmatrix} r(t)\cos\theta(t) \\ r(t)\sin\theta(t) \end{pmatrix}.$$

Newton's universal gravitation law implies that the acceleration vector $\vec{\mathbf{r}}''(t)$ satisfies

$$\vec{\mathbf{r}}''(t) = -\frac{k}{|\vec{\mathbf{r}}(t)|^3}\vec{\mathbf{r}}(t).$$

The planet's motion can be expanded by the product rule and chain rule of calculus to give the relation

$$\vec{\mathbf{r}}'(t) = \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} r'(t) + \begin{pmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{pmatrix} r(t)\theta'(t).$$

The column vectors in this formula are orthogonal hence independent. One more application of the product and chain rules gives

$$\begin{aligned} \vec{\mathbf{r}}''(t) &= \left(r''(t) - r(t)(\theta'(t))^2\right) \begin{pmatrix} \cos\theta(t) \\ \sin\theta(t) \end{pmatrix} \\ &\quad + \left(\frac{1}{r(t)}\left(r^2(t)\theta'(t)\right)'\right) \begin{pmatrix} -\sin\theta(t) \\ \cos\theta(t) \end{pmatrix}. \end{aligned}$$

The independent vectors appearing in the formula happen to be the normal and tangential components of the acceleration, although we don't use this fact.

Newton's law expansion of $\vec{\mathbf{r}}''(t)$ requires that corresponding vector components must match, giving the relations

$$\begin{aligned} r''(t) - r(t)(\theta'(t))^2 &= -\frac{k}{r^2(t)}, \\ \frac{1}{r(t)}\left(r^2(t)\theta'(t)\right)' &= 0. \end{aligned} \tag{1}$$

The second formula in (1) implies that $dA(t) = 0$, where $dA(t)$ is the polar area increment swept out by the planet. Kepler's second law is proved.

Kepler's First Law

Write the second equation in (1) in integrated form $r^2(t)\theta'(t) = h$. Combine the first formula in (1) with the second (in integrated form) to obtain the nonlinear second order differential equation

$$r''(t) - \frac{h^2}{r^3(t)} = -\frac{k}{r^2(t)}. \tag{2}$$

Because $\theta'(t) = h/r^2(t) \neq 0$, then a variable change $t = t(\theta)$ is possible: $r(t) = r(t(\theta))$ is a function of θ. Let $u(\theta) = 1/r(t(\theta))$, then by the chain rule

$$\begin{aligned} r'(t) &= -\frac{du/dt}{u^2(\theta)} \\ &= -r^2(t)u'(\theta)\theta'(t) \\ &= -hu'(\theta). \end{aligned}$$

Differentiate again on θ and use (2) to obtain

$$u''(\theta) + u(\theta) = c,$$

where $c = k/h^2$. Solving gives

$$u(\theta) = c_1 \cos\theta + c_2 \sin\theta + c\,.$$

Use $u = 1/r$ to re-write this formula in the new form

$$r(t(\theta)) = \frac{1}{c + c_1 \cos\theta + c_2 \sin\theta}\,.$$

Define angle θ_0 and amplitude R by the formulas $R\cos\theta_0 = c_1$, $R\sin\theta_0 = c_2$. The sum formula for the cosine implies

$$\begin{aligned} R\cos(\theta - \theta_0) &= R\cos\theta\cos\theta_0 + R\sin\theta\sin\theta_0 \\ &= c_1\cos\theta + c_2\sin\theta. \end{aligned}$$

Substitution gives the ellipse equation in polar coordinates

$$\begin{aligned} r(t(\theta)) &= \frac{1}{c + R\cos(\theta - \theta_0)} \\ &= \frac{\ell}{1 + e\,\cos(\theta - \theta_0)}\,. \end{aligned}$$

Here, $\ell = h^2/k$ is half the latus rectum and $e = R\ell$ is the eccentricity of the ellipse. Initially, we don't know that $0 \leq e < 1$, but the requirement that a planetary orbit be bounded discards the possibility $e \geq 1$ (parabola or hyperbola). This completes the proof of Kepler's first law.

PDF Sources

Text, Solutions and Corrections

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`https://github.com/ggustaf/github.io` and mirror
`https://gitlab.com/ggustaf/answers`.

Communication: To contribute a solution or correction, ask a question or request an answer, click the link below, then create a GitHub issue and post. Contributions and corrections are credited, privacy respected.

`https://github.com/ggustaf/github.io/issues`