Contents

Chapter 11

Systems of Differential Equations

Contents

11.1 Examples of Systems

Linear systems

A **linear system** is a system of differential equations of the form

$$
\begin{array}{rclcccccc}
x_1' & = & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & + & f_1, \\
x_2' & = & a_{21}x_1 & + & \cdots & + & a_{2n}x_n & + & f_2, \\
\vdots & & & \vdots & \cdots & \vdots & & \vdots & \\
x_m' & = & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & + & f_m,
\end{array}
\tag{1}
$$

where $' = d/dt$. Given are the functions $a_{ij}(t)$ and $f_j(t)$ on some interval $a < t < b$. The unknowns are the functions $x_1(t), \ldots, x_n(t)$.

The system is called **homogeneous** if all $f_j = 0$, otherwise it is called **non-homogeneous**.

Matrix Notation for Systems

A non-homogeneous system of linear equations (1) is written as the equivalent vector-matrix system

$$\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{f}}(t),$$

where

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{\mathbf{f}} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Brine Tank Cascade

Let brine tanks A, B, C be given of volumes 20, 40, 60, respectively, as in Figure 1.

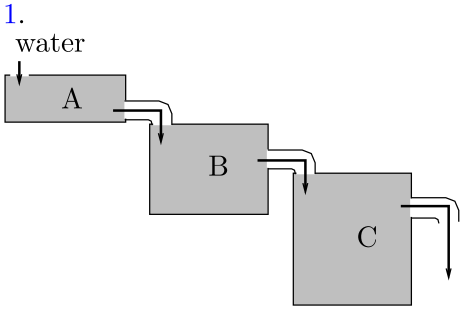

Figure 1. Three brine tanks in cascade.

It is supposed that fluid enters tank A at rate r, drains from A to B at rate r, drains from B to C at rate r, then drains from tank C at rate r. Hence the volumes of the tanks remain constant. Let $r = 10$, to illustrate the ideas.

Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.

Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t in each tank. We suppose **water containing no salt** is added to tank A . Therefore, the salt in all the tanks is eventually lost from the drains. The cascade is modeled by the **chemical balance law**

$$\text{rate of change} \quad = \quad \text{input rate} \quad - \quad \text{output rate}.$$

Application of the balance law, justified below in *compartment analysis*, results in the triangular differential system

$$\begin{aligned} x_1' &= -\frac{1}{2}x_1, \\ x_2' &= \frac{1}{2}x_1 - \frac{1}{4}x_2, \\ x_3' &= \frac{1}{4}x_2 - \frac{1}{6}x_3. \end{aligned}$$

The solution, to be justified later in this chapter, is given by the equations

$$\begin{aligned}
x_1(t) &= x_1(0)e^{-t/2}, \\
x_2(t) &= -2x_1(0)e^{-t/2} + (x_2(0) + 2x_1(0))e^{-t/4}, \\
x_3(t) &= \frac{3}{2}\,x_1(0)e^{-t/2} - 3(x_2(0) + 2x_1(0))e^{-t/4} \\
&\quad + (x_3(0) - \frac{3}{2}x_1(0) + 3(x_2(0) + 2x_1(0)))e^{-t/6}.
\end{aligned}$$

Cascades and Compartment Analysis

A **Linear Cascade** is a diagram of **compartments** in which input and output rates have been assigned from one or more different compartments. The diagram is a succinct way to summarize and document the various rates.

The method of **compartment analysis** translates the diagram into a system of linear differential equations. The method has been used to derive applied models in diverse topics like ecology, chemistry, heating and cooling, kinetics, mechanics and electricity.

The method. Refer to Figure 2. A compartment diagram consists of the following components.

Variable Names	Each **compartment** is labelled with a variable X.
Arrows	Each arrow is labelled with a **Flow Rate** R.
Input Rate	An arrowhead pointing at compartment X documents **Input Rate** R.
Output Rate	An arrowhead pointing away from compartment X documents **Output Rate** R.

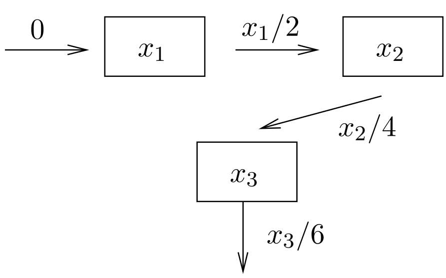

Figure 2. Compartment analysis diagram.
The diagram represents the classical brine tank problem of Figure 1.

Assembly of the single linear differential equation for a diagram compartment X is done by writing dX/dt for the left side of the differential equation and then algebraically adding the input and output rates to obtain the right side of the differential equation, according to the **balance law**

$$\frac{dX}{dt} = \text{sum of input rates} - \text{sum of output rates}$$

By convention, a compartment with no arriving arrowhead has input zero, and a compartment with no exiting arrowhead has output zero. Applying the balance law to Figure 2 gives one differential equation for each of the three compartments $\boxed{x_1}$, $\boxed{x_2}$, $\boxed{x_3}$.

$$
\begin{aligned}
x_1' &= 0 - \frac{1}{2}x_1, \\
x_2' &= \frac{1}{2}x_1 - \frac{1}{4}x_2, \\
x_3' &= \frac{1}{4}x_2 - \frac{1}{6}x_3.
\end{aligned}
$$

Recycled Brine Tank Cascade

Let brine tanks A, B, C be given of volumes 60, 30, 60, respectively, as in Figure 3.

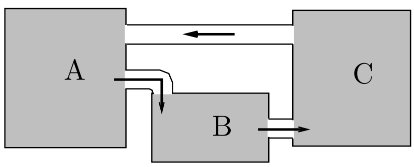

Figure 3. Three brine tanks in cascade with recycling.

Suppose that fluid drains from tank A to B at rate r, drains from tank B to C at rate r, then drains from tank C to A at rate r. The tank volumes remain constant due to constant recycling of fluid. For purposes of illustration, let $r = 10$.

Uniform stirring of each tank is assumed, which implies **uniform salt concentration** throughout each tank.

Let $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount of salt at time t in each tank. No salt is lost from the system, due to recycling. Using compartment analysis, the recycled cascade is modeled by the non-triangular system

$$
\begin{array}{lclcl}
x_1' & = & -\frac{1}{6}x_1 & & + \frac{1}{6}x_3, \\
x_2' & = & \frac{1}{6}x_1 & - \frac{1}{3}x_2, & \\
x_3' & = & & \frac{1}{3}x_2 & - \frac{1}{6}x_3.
\end{array}
$$

The solution is given by the equations

$$
\begin{aligned}
x_1(t) &= c_1 + (c_2 - 2c_3)e^{-t/3}\cos(t/6) + (2c_2 + c_3)e^{-t/3}\sin(t/6), \\
x_2(t) &= \frac{1}{2}c_1 + (-2c_2 - c_3)e^{-t/3}\cos(t/6) + (c_2 - 2c_3)e^{-t/3}\sin(t/6), \\
x_3(t) &= c_1 + (c_2 + 3c_3)e^{-t/3}\cos(t/6) + (-3c_2 + c_3)e^{-t/3}\sin(t/6).
\end{aligned}
$$

At infinity, $x_1 = x_3 = c_1$, $x_2 = c_1/2$. The meaning is that the total amount of salt is uniformly distributed in the tanks, in the ratio $2:1:2$.

Pond Pollution

Consider three ponds connected by streams, as in Figure 4. The first pond has a pollution source, which spreads via the connecting streams to the other ponds. The plan is to determine the amount of pollutant in each pond.

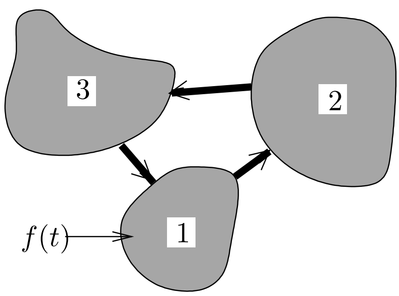

Figure 4. Three ponds 1, 2, 3 of volumes V_1, V_2, V_3 connected by streams. The pollution source $f(t)$ is in pond 1.

Assume the following.

- Symbol $f(t)$ is the pollutant flow rate into pond 1 (lb/min).
- Symbols f_1, f_2, f_3 denote the pollutant flow rates out of ponds 1, 2, 3, respectively (gal/min). It is assumed that the pollutant is well-mixed in each pond.
- The three ponds have volumes V_1, V_2, V_3 (gal), which remain constant.
- Symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount (lbs) of pollutant in ponds 1, 2, 3, respectively.

The pollutant flux is the flow rate times the pollutant concentration, e.g., pond 1 is emptied with flux f_1 times $x_1(t)/V_1$. A compartment analysis is summarized in the following diagram.

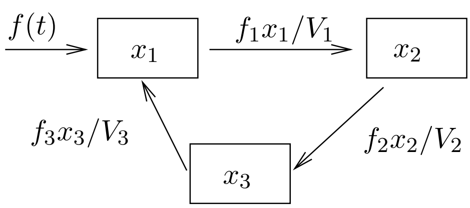

Figure 5. Pond diagram.
The compartment diagram represents the three-pond pollution problem of Figure 4.

The diagram plus compartment analysis gives the following differential equations.

$$\begin{aligned}
x_1'(t) &= \frac{f_3}{V_3}x_3(t) - \frac{f_1}{V_1}x_1(t) + f(t), \\
x_2'(t) &= \frac{f_1}{V_1}x_1(t) - \frac{f_2}{V_2}x_2(t), \\
x_3'(t) &= \frac{f_2}{V_2}x_2(t) - \frac{f_3}{V_3}x_3(t).
\end{aligned}$$

For a specific numerical example, take $f_i/V_i = 0.001$, $1 \le i \le 3$, and let $f(t) = 0.125$ lb/min for the first 48 hours (2880 minutes), thereafter $f(t) = 0$. We expect due to uniform mixing that after a long time there will be $(0.125)(2880) = 360$ pounds of pollutant uniformly deposited, which is 120 pounds per pond.

Initially, $x_1(0) = x_2(0) = x_3(0) = 0$, if the ponds were pristine. The specialized problem for the first 48 hours is

$$\begin{aligned} x_1'(t) &= 0.001\,x_3(t) - 0.001\,x_1(t) + 0.125, \\ x_2'(t) &= 0.001\,x_1(t) - 0.001\,x_2(t), \\ x_3'(t) &= 0.001\,x_2(t) - 0.001\,x_3(t), \\ x_1(0) &= x_2(0) = x_3(0) = 0. \end{aligned}$$

The solution to this system is

$$\begin{aligned} x_1(t) &= e^{-\frac{3t}{2000}}\left(\frac{125\sqrt{3}}{9}\sin\left(\frac{\sqrt{3}t}{2000}\right) - \frac{125}{3}\cos\left(\frac{\sqrt{3}t}{2000}\right)\right) + \frac{125}{3} + \frac{t}{24}, \\ x_2(t) &= -\frac{250\sqrt{3}}{9}e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}t}{2000}\right) + \frac{t}{24}, \\ x_3(t) &= e^{-\frac{3t}{2000}}\left(\frac{125}{3}\cos\left(\frac{\sqrt{3}t}{2000}\right) + \frac{125\sqrt{3}}{9}\sin\left(\frac{\sqrt{3}t}{2000}\right)\right) + \frac{t}{24} - \frac{125}{3}. \end{aligned}$$

After 48 hours elapse, the approximate pollutant amounts in pounds are

$$x_1(2880) = 162.30, \quad x_2(2880) = 119.61, \quad x_3(2880) = 78.08.$$

It should be remarked that the system above and its solution will require a change in order to predict the state of the ponds after 48 hours/ The equations change by replacing constant 0.125 by zero. The corresponding homogeneous system has an equilibrium solution $x_1(t) = x_2(t) = x_3(t) = 120$. This constant solution, called the **steady-state**, is the limit at infinity of the solution to the homogeneous system using the initial values $x_1(0) \approx 162.30$, $x_2(0) \approx 119.61$, $x_3(0) \approx 78.08$, which are values from the forced system at $t = 48$ hours.

Home Heating

Consider a typical home with attic, basement and insulated main floor.

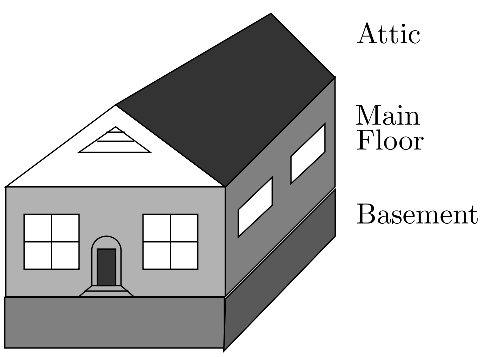

Figure 6. Typical home with attic and basement. The below-grade basement and the attic are uninsulated. Only the main living area is insulated.

It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation. The walls and floor in the basement are insulated by earth. The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement. We will analyze the changing temperatures in the three levels using Newton's cooling law and the variables

$$\begin{aligned} z(t) &= \text{Temperature in the attic,} \\ y(t) &= \text{Temperature in the main living area,} \\ x(t) &= \text{Temperature in the basement,} \\ t &= \text{Time in hours.} \end{aligned}$$

Initial data. Assume it is winter time and the outside temperature in constantly 35°F during the day. Also assumed is a basement earth temperature of 45°F. Initially, the heat is off for several days. The initial values at noon ($t = 0$) are then $x(0) = 45$, $y(0) = z(0) = 35$.

Portable heater. A small electric heater is turned on at noon, with thermostat set for 100°F. When the heater is running, it provides a 20°F rise per hour, therefore it takes some time to reach 100°F (probably never!). Newton's cooling law

$$\text{Temperature rate} = \text{k(Temperature difference)}$$

will be applied to five boundary surfaces: (0) the basement walls and floor, (1) the basement ceiling, (2) the main floor walls, (3) the main floor ceiling, and (4) the attic walls and ceiling. Newton's cooling law gives positive cooling constants k_0, k_1, k_2, k_3, k_4 and the equations

$$\begin{aligned} x' &= k_0(45 - x) + k_1(y - x), \\ y' &= k_1(x - y) + k_2(35 - y) + k_3(z - y) + 20, \\ z' &= k_3(y - z) + k_4(35 - z). \end{aligned}$$

The insulation constants will be defined as $k_0 = 1/2$, $k_1 = 1/2$, $k_2 = 1/4$, $k_3 = 1/4$, $k_4 = 3/4$ to reflect insulation quality. The reciprocal $1/k$ is approximately the amount of time in hours required for 63% of the temperature difference to be exchanged. For instance, 4 hours elapse for the main floor. The model:

$$\begin{aligned} x' &= \frac{1}{2}(45 - x) + \frac{1}{2}(y - x), \\ y' &= \frac{1}{2}(x - y) + \frac{1}{4}(35 - y) + \frac{1}{4}(z - y) + 20, \\ z' &= \frac{1}{4}(y - z) + \frac{3}{4}(35 - z). \end{aligned}$$

The homogeneous solution in vector form is given in terms of constants $a = 1 + \sqrt{5}/4$, $b = 1 - \sqrt{5}/4$, and arbitrary constants c_1, c_2, c_3 by the formula

$$\begin{pmatrix} x_h(t) \\ y_h(t) \\ z_h(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + c_2 e^{-at} \begin{pmatrix} 2 \\ \sqrt{5} \\ 1 \end{pmatrix} + c_3 e^{-bt} \begin{pmatrix} 2 \\ -\sqrt{5} \\ 1 \end{pmatrix}.$$

A particular solution is an equilibrium solution

$$\begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix} = \begin{pmatrix} \frac{620}{11} \\ \frac{745}{11} \\ \frac{475}{11} \end{pmatrix}.$$

The homogeneous solution has limit zero at infinity, hence the temperatures of the three spaces hover around $x = 56.4$, $y = 67.7$, $z = 43.2$ degrees Fahrenheit. Specific information can be gathered by solving for c_1, c_2, c_3 according to the initial data $x(0) = 45$, $y(0) = z(0) = 35$. The answers are

$$c_1 = 5, \quad c_2 = \frac{25}{2} + \frac{7}{2}\sqrt{5}, \quad c_3 = \frac{25}{2} - \frac{7}{2}\sqrt{5}.$$

Underpowered heater. To the main floor each hour is added 20°F, but the heat escapes at a substantial rate, so that after one hour $y \approx 68$°F. After five hours, $y \approx 68$°F. The heater in this example is so inadequate that even after many hours, the main living area is still under 69°F.

Forced air furnace. Replacing the space heater by a normal furnace adds the difficulty of **switches** in the input, namely, the thermostat turns off the furnace when the main floor temperature reaches 70°F, and it turns it on again after a 4°F temperature drop. We will suppose that the furnace has four times the BTU rating of the space heater, which translates to an 80°F temperature rise per hour. The study of the forced air furnace requires two differential equations, one with 20 replaced by 80 (DE 1, furnace on) and the other with 20 replaced by 0 (DE 2, furnace off). The plan is to use the first differential equation on time interval $0 \le t \le t_1$, then switch to the second differential equation for time interval $t_1 \le t \le t_2$. The time intervals are selected so that $y(t_1) = 70$ (the thermostat setting) and $y(t_2) = 66$ (thermostat setting less 4 degrees). Numerical work gives the following results.

Time in minutes	Main floor temperature	Model	Furnace
31.6	70	DE 1	on
40.9	66	DE 2	off
45.3	70	DE 1	on
54.6	66	DE 2	off

The reason for the non-uniform times between furnace cycles can be seen from the model. Each time the furnace cycles, heat enters the main floor, then escapes through the other two levels. Consequently, the initial conditions on each floor applied to models 1 and 2 are changing, resulting in different solutions to the models on each switch.

Chemostats and Microorganism Culturing

A vessel into which nutrients are pumped, to feed a microorganism, is called a **Chemostat**[1]. Uniform distributions of microorganisms and nutrients are assumed, for example, due to stirring effects. The pumping is matched by draining to keep the volume constant.

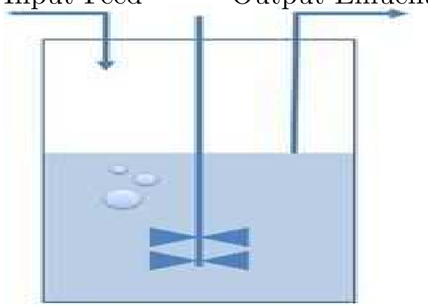

Figure 7. A Basic Chemostat. A stirred bio-reactor operated as a chemostat, with continuous inflow and outflow. The flow rates are controlled to maintain a constant culture volume.

In a typical chemostat, one nutrient is kept in short supply while all others are abundant. We consider here the question of **survival** of the organism subject to the limited resource. The problem is quantified as follows:

$$x(t) = \text{the concentration of the limited nutrient in the vessel},$$
$$y(t) = \text{the concentration of organisms in the vessel}.$$

A special case of the derivation in J.M. Cushing's text [?] for the organism *E. Coli*[2] is the set of **nonlinear** differential equations[3]

$$\begin{aligned} x' &= -0.075x + (0.075)(0.005) - \frac{1}{63}g(x)y, \\ y' &= -0.075y + g(x)y, \end{aligned} \tag{2}$$

where $g(x) = 0.68x(0.0016+x)^{-1}$. Of special interest to the study of this equation are two linearized equations at equilibria, given by

$$\begin{aligned} u_1' &= -0.075\,u_1 - 0.008177008175\,u_2, \\ u_2' &= 0.4401515152\,u_2, \end{aligned} \tag{3}$$

[1]The October 14, 2004 issue of the journal *Nature* featured a study of the co-evolution of a common type of bacteria, Escherichia coli, and a virus that infects it, called bacteriophage T7. Postdoctoral researcher Samantha Forde set up "microbial communities of bacteria and viruses with different nutrient levels in a series of chemostats – glass culture tubes that provide nutrients and oxygen and siphon off wastes."

[2]In a biology Master's thesis, two strains of Escherichia coli were grown in a glucose-limited chemostat coupled to a modified Robbins device containing plugs of silicone rubber urinary catheter material. Reference: Jennifer L. Adams and Robert J. C. McLean, Applied and Environmental Microbiology, September 1999, p. 4285-4287, Vol. 65, No. 9.

[3]More details can be found in *The Theory of the Chemostat Dynamics of Microbial Competition*, ISBN-13: 9780521067348, by Hal Smith and Paul Waltman, June 2008.

$$(4) \qquad \begin{aligned} v_1' &= -1.690372243\, v_1 - 0.001190476190\, v_2, \\ v_2' &= 101.7684513\, v_1. \end{aligned}$$

Although we cannot solve the nonlinear system explicitly, nevertheless there are explicit formulas for u_1, u_2, v_1, v_2 that complete the picture of how solutions $x(t)$, $y(t)$ behave at $t = \infty$. The result of the analysis is that *E. Coli* survives indefinitely in this vessel at concentration $y \approx 0.3$.

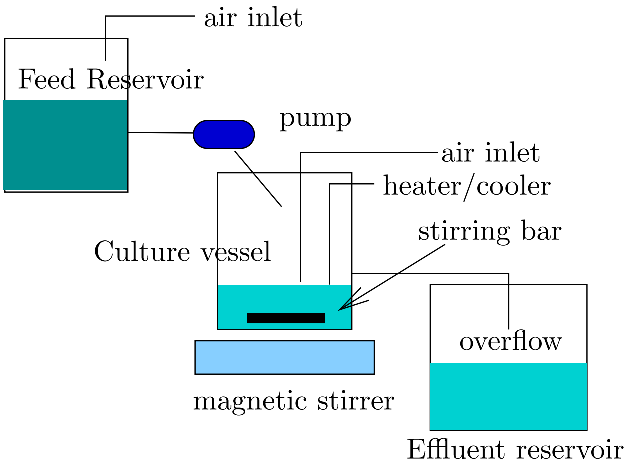

Figure 8. Laboratory Chemostat. The components are the **Feed reservoir**, which contains the nutrients, a stirred chemical reactor labeled the **Culture vessel**, and the **Effluent reservoir**, which holds the effluent overflow from the reactor.

Irregular Heartbeats and Lidocaine

The human malady of **Ventricular Arrhythmia** or irregular heartbeat is treated clinically using the drug **lidocaine**.

Figure 9. Xylocaine label, a brand name for the drug lidocaine.

To be effective, the drug has to be maintained at a bloodstream concentration of 1.5 milligrams per liter, but concentrations above 6 milligrams per liter are considered lethal in some patients. The actual dosage depends upon body weight. The adult dosage maximum for ventricular tachycardia is reported at 3 mg/kg.[4] The drug is supplied in 0.5%, 1% and 2% solutions, which are stored at room temperature.

A differential equation model for the dynamics of the drug therapy uses

[4]Source: **Family Practice Notebook**, `http://www.fpnotebook.com/`. The author is Scott Moses, MD, who practises in Lino Lakes, Minnesota.

$$x(t) = \text{amount of } \textit{lidocaine} \text{ in the bloodstream},$$
$$y(t) = \text{amount of } \textit{lidocaine} \text{ in body tissue}.$$

A typical set of equations, valid for a special body weight only, appears below; for more detail see J.M. Cushing's text [?].

$$(5) \qquad \begin{aligned} x'(t) &= -0.09x(t) + 0.038y(t), \\ y'(t) &= 0.066x(t) - 0.038y(t). \end{aligned}$$

The physically significant initial data is zero drug in the bloodstream $x(0) = 0$ and injection dosage $y(0) = y_0$. The answers:

$$\begin{aligned} x(t) &= -0.3367y_0e^{-0.1204t} + 0.3367y_0e^{-0.0076t}, \\ y(t) &= 0.2696y_0e^{-0.1204t} + 0.7304y_0e^{-0.0076t}. \end{aligned}$$

The answers can be used to estimate the maximum possible safe dosage y_0 and the duration of time that the drug *lidocaine* is effective.

Nutrient Flow in an Aquarium

Consider a vessel of water containing a radioactive isotope, to be used as a tracer for the food chain, which consists of aquatic plankton varieties A and B.

Plankton are aquatic organisms that drift with the currents, typically in an environment like Chesapeake Bay. Plankton can be divided into two groups, phytoplankton and zooplankton. The phytoplankton are *plant-like* drifters: diatoms and other alga. Zooplankton are *animal-like* drifters: copepods, larvae, and small crustaceans.

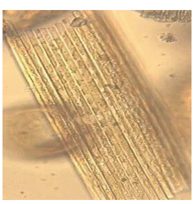
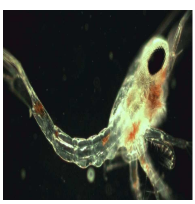

Figure 10. Left: Bacillaria paxillifera, phytoplankton. Right: Anomura Galathea zoea, zooplankton.

Let

$$x(t) = \text{isotope concentration in the water},$$
$$y(t) = \text{isotope concentration in } A,$$
$$z(t) = \text{isotope concentration in } B.$$

Typical differential equations are

$$\begin{aligned} x'(t) &= -3x(t) + 6y(t) + 5z(t), \\ y'(t) &= 2x(t) - 12y(t), \\ z'(t) &= x(t) + 6y(t) - 5z(t). \end{aligned}$$

The answers are

$$
\begin{aligned}
x(t) &= 6c_1 + (1+\sqrt{6})c_2 e^{(-10+\sqrt{6})t} + (1-\sqrt{6})c_3 e^{(-10-\sqrt{6})t},\\
y(t) &= c_1 + c_2 e^{(-10+\sqrt{6})t} + c_3 e^{(-10-\sqrt{6})t},\\
z(t) &= \frac{12}{5}c_1 - \left(2+\sqrt{1.5}\right) c_2 e^{(-10+\sqrt{6})t} + \left(-2+\sqrt{1.5}\right) c_3 e^{(-10-\sqrt{6})t}.
\end{aligned}
$$

The constants c_1, c_2, c_3 are related to the initial radioactive isotope concentrations $x(0) = x_0$, $y(0) = 0$, $z(0) = 0$, by the 3×3 system of linear algebraic equations

$$
\begin{array}{rcrcrcl}
6c_1 & + & (1+\sqrt{6})c_2 & + & (1-\sqrt{6})c_3 & = & x_0,\\
c_1 & + & c_2 & + & c_3 & = & 0,\\
\frac{12}{5}c_1 & - & \left(2+\sqrt{1.5}\right) c_2 & + & \left(-2+\sqrt{1.5}\right) c_3 & = & 0.
\end{array}
$$

Biomass Transfer

Consider a European forest having one or two varieties of trees. We select some of the oldest trees, those expected to die off in the next few years, then follow the cycle of living trees into dead trees. The dead trees eventually decay and fall from seasonal and biological events. Finally, the fallen trees become humus.

Figure 11. Forest Biomass. Total biomass is a parameter used to assess atmospheric carbon that is harvested by trees. Forest management uses biomass subclasses to classify fire risk.

Let variables x, y, z, t be defined by

$$
\begin{aligned}
x(t) &= \text{biomass decayed into humus},\\
y(t) &= \text{biomass of dead trees},\\
z(t) &= \text{biomass of living trees},\\
t &= \text{time in decades } (\textit{decade} = 10 \text{ years}).
\end{aligned}
$$

A typical biological model is

$$\begin{aligned}
x'(t) &= -x(t) + 3y(t),\\
y'(t) &= -3y(t) + 5z(t),\\
z'(t) &= -5z(t).
\end{aligned}$$

Suppose there are no dead trees and no humus at $t = 0$, with initially z_0 units of living tree biomass. These assumptions imply initial conditions $x(0) = y(0) = 0$, $z(0) = z_0$. The solution is

$$\begin{aligned}
x(t) &= \frac{15}{8} z_0 \left(e^{-5t} - 2e^{-3t} + e^{-t}\right),\\
y(t) &= \frac{5}{2} z_0 \left(-e^{-5t} + e^{-3t}\right),\\
z(t) &= z_0 e^{-5t}.
\end{aligned}$$

The live tree biomass $z(t) = z_0 e^{-5t}$ decreases according to a Malthusian decay law from its initial size z_0. It decays to 60% of its original biomass in one year. Interesting calculations that can be made from the other formulas include the future dates when the dead tree biomass and the humus biomass are maximum. The predicted dates are approximately 2.5 and 8 years hence, respectively.

The predictions made by this model are trends extrapolated from rate observations in the forest. Like weather prediction, it is a calculated guess that disappoints on a given day and from the outset has no predictable answer.

Total biomass is considered an important parameter to assess atmospheric carbon that is harvested by trees. Biomass estimates for forests since 1980 have been made by satellite remote sensing data with instances of 90% accuracy (*Science* 87(5), September 2004).

Pesticides in Soil and Trees

A Washington cherry orchard was sprayed with pesticides.

Figure 12. June Cherries.

Assume that a negligible amount of pesticide was sprayed on the soil. Pesticide applied to the trees has a certain outflow rate to the soil, and conversely, pesticide in the soil has a certain uptake rate into the trees. Repeated applications of the pesticide are required to control the insects, which implies the pesticide levels in the trees varies with time. Quantize the pesticide spraying as follows.

$$\begin{aligned}
x(t) &= \text{amount of pesticide in the trees,}\\
y(t) &= \text{amount of pesticide in the soil,}\\
r(t) &= \text{amount of pesticide applied to the trees,}\\
t &= \text{time in years.}
\end{aligned}$$

A typical model is obtained from input-output analysis, similar to the brine tank models:

$$\begin{aligned}
x'(t) &= 2x(t) - y(t) + r(t),\\
y'(t) &= 2x(t) - 3y(t).
\end{aligned}$$

In a pristine orchard, the initial data is $x(0) = 0$, $y(0) = 0$, because the trees and the soil initially harbor no pesticide. The solution of the model obviously depends on $r(t)$. The nonhomogeneous dependence is treated by the method of variation of parameters *infra*. Approximate formulas are

$$\begin{aligned}
x(t) &\approx \int_0^t \left(1.10e^{1.6(t-u)} - 0.12e^{-2.6(t-u)}\right) r(u)du,\\
y(t) &\approx \int_0^t \left(0.49e^{1.6(t-u)} - 0.49e^{-2.6(t-u)}\right) r(u)du.
\end{aligned}$$

The exponential rates 1.6 and -2.6 represent respectively the accumulation of the pesticide into the soil and the decay of the pesticide from the trees. The application rate $r(t)$ is typically a step function equal to a positive constant over a small interval of time and zero elsewhere, or a sum of such functions, representing periodic applications of pesticide.

Forecasting Prices

A manufacturer has a marketing policy based upon the price $x(t)$ of its product.

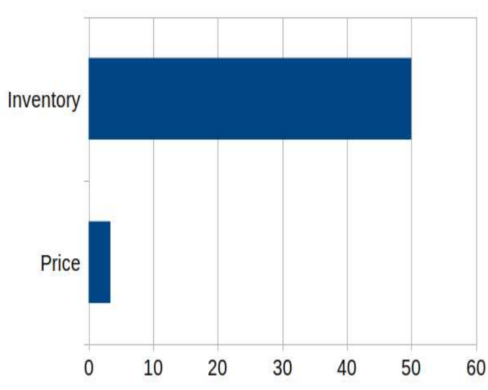

Figure 13. Pricing and Inventory.
Dynamic pricing reflects demand for the product, predicted by sales data.

The **Production** $P(t)$ and the **Sales** $S(t)$ are given in terms of the **Price** $x(t)$ and the **Change in Price** $x'(t)$ by the equations

$$\begin{aligned} P(t) &= 4 - \frac{3}{4}x(t) - 8x'(t) \quad \text{(Production)}, \\ S(t) &= 15 - 4x(t) - 2x'(t) \quad \text{(Sales)}. \end{aligned}$$

The differential equations for the price $x(t)$ and inventory level $I(t)$ are

$$\begin{aligned} x'(t) &= k(I(t) - I_0), \\ I'(t) &= P(t) - S(t). \end{aligned}$$

The inventory level $I_0 = 50$ represents the desired level. The equations can be written in terms of $x(t)$, $I(t)$ as follows.

$$\begin{array}{rcrcrcl} x'(t) &=& & & kI(t) &-& kI_0, \\ I'(t) &=& \frac{13}{4}x(t) &-& 6kI(t) &+& 6kI_0 - 11. \end{array}$$

If $k = 1$, $x(0) = 10$ and $I(0) = 7$, then the solution is given by

$$\begin{aligned} x(t) &= \frac{44}{13} + \frac{86}{13}e^{-13t/2}, \\ I(t) &= 50 - 43e^{-13t/2}. \end{aligned}$$

The **Forecast** of price $x(t) \approx 3.38$ dollars at inventory level $I(t) \approx 50$ is based upon the two limits

$$\lim_{t\to\infty} x(t) = \frac{44}{13}, \quad \lim_{t\to\infty} I(t) = 50.$$

Coupled Spring-Mass Systems

Three masses are attached to each other by four springs as in Figure 14.

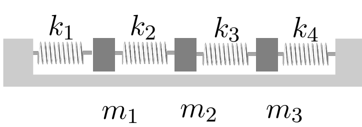

Figure 14. Three masses connected by springs. The masses slide along a frictionless horizontal surface.

The analysis uses the following constants, variables and assumptions.

Mass Constants	The masses m_1, m_2, m_3 are assumed to be point masses concentrated at their center of gravity.
Spring Constants	The mass of each spring is negligible. The springs operate according to Hooke's law: Force = k(elongation). Constants k_1, k_2, k_3, k_4 denote the Hooke's constants. The springs restore after compression and extension.
Position Variables	The symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the mass positions along the horizontal surface, measured from their equilibrium positions, plus right and minus left.

Fixed Ends The first and last spring are attached to fixed walls.

The **competition method** is used to derive the equations of motion. In this case, the law is

Newton's Second Law Force = Sum of the Hooke's Forces.

The model equations are

$$\begin{array}{rcl} m_1 x_1''(t) &=& -k_1 x_1(t) + k_2[x_2(t) - x_1(t)], \\ m_2 x_2''(t) &=& -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\ m_3 x_3''(t) &=& -k_3[x_3(t) - x_2(t)] - k_4 x_3(t). \end{array} \tag{6}$$

The equations are justified in the case of all positive variables by observing that the first three springs are elongated by x_1, $x_2 - x_1$, $x_3 - x_2$, respectively. The last spring is compressed by x_3, which accounts for the minus sign.

Another way to justify the equations is through mirror-image symmetry: interchange $k_1 \longleftrightarrow k_4$, $k_2 \longleftrightarrow k_3$, $x_1 \longleftrightarrow x_3$, then equation 2 should be unchanged and equation 3 should become equation 1.

Matrix Formulation. System (6) can be written as a second order vector-matrix system

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

More succinctly, the system is written as

$$M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t)$$

where the **displacement** $\vec{\mathbf{x}}$, **mass matrix** M and **stiffness matrix** K are defined by the formulas

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

Numerical example. Let $m_1 = 1$, $m_2 = 1$, $m_3 = 1$, $k_1 = 2$, $k_2 = 1$, $k_3 = 1$, $k_4 = 2$. Then the system is given by

$$\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The vector solution is given by the formula

$$\begin{aligned}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= (a_1\cos t + b_1 \sin t)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \\ &\quad + \left(a_2\cos\sqrt{3}t + b_2\sin\sqrt{3}t\right)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &\quad + (a_3\cos 2t + b_3\sin 2t)\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix},\end{aligned}$$

where a_1, a_2, a_3, b_1, b_2, b_3 are arbitrary constants.

Railway Cars

A special case of the coupled spring-mass system is three flatbed rail cars on a level frictionless track connected by springs, as in Figure 15.

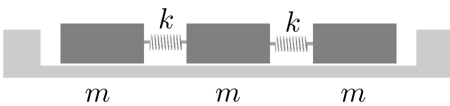

Figure 15. Three identical rail cars connected by identical springs.

Except for the springs on fixed ends, this problem is the same as the one in equation (6), therefore taking $k_1 = k_4 = 0$, $k_2 = k_3 = k$, $m_1 = m_2 = m_3 = m$ gives the system

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Take $k/m = 1$ to obtain the illustration

$$\vec{\mathbf{x}}'' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}\vec{\mathbf{x}},$$

which has vector solution

$$\begin{aligned}\vec{\mathbf{x}} &= (a_1 + b_1 t)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (a_2\cos t + b_2\sin t)\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &\quad + \left(a_3\cos\sqrt{3}t + b_3\sin\sqrt{3}t\right)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix},\end{aligned}$$

where a_1, a_2, a_3, b_1, b_2, b_3 are arbitrary constants.

The solution expression can be used to discover what happens to the rail cars when the springs act normally upon compression but disengage upon expansion. An interesting physical situation is when one car moves along the track, contacts two stationary cars, then transfers its momentum to the other cars, followed by disengagement.

Monatomic Crystals

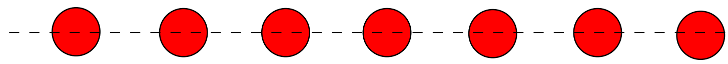

Figure 16. A Crystal Model.
The n crystals are identical masses m assumed connected by equal springs of Hooke's constant k. The last mass is connected to the first mass.

The scalar differential equations for Figure 16 are written for mass positions $x_1, \ldots, x_n$, with $x_0 = x_n$, $x_{n+1} = x_1$ to account for the ring of identical masses (periodic boundary condition). Then for $k = 1, \ldots, n$

$$m\frac{d^2x_k}{dt^2} = k(x_{k+1} - x_k) + k(x_{k-1} - x_k) = k(x_{k-1} - 2x_k + x_{k+1}).$$

These equations represent a system $x'' = Ax$, where the symmetric matrix of coefficients $A = M^{-1}K$ is given for $n = 5$ and $k/m = 1$ by

$$A = \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

If $n = 3$ and $k/m = 1$, then $A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ and the solutions x_1, x_2, x_3 are linear combinations of the functions 1, t, $\cos\sqrt{3}t$, $\sin\sqrt{3}t$.

Electrical LR–Network no EMF

Consider the LR-network of Figure 17.

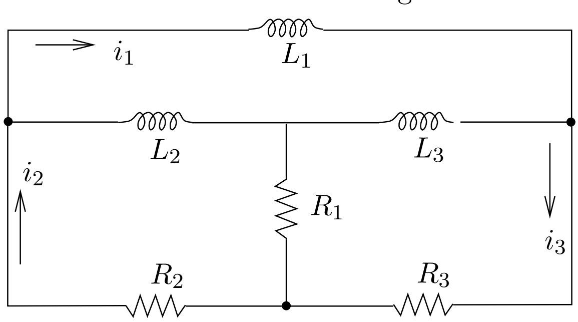

Figure 17. An electrical network.
There are three resistors R_1, R_2, R_3 and three inductors L_1, L_2, L_3. The currents i_1, i_2, i_3 are defined between nodes (black dots).

The derivation of the differential equations for the loop currents i_1, i_2, i_3 uses Kirchhoff's laws and the voltage drop formulas for resistors and inductors. The black dots in the diagram are the **nodes** that determine the beginning and end of each of the currents i_1, i_2, i_3. Currents are defined only on the outer boundary of the network. Kirchhoff's node law determines the currents across L_2, L_3

(arrowhead right) as $i_2 - i_1$ and $i_3 - i_1$, respectively. Similarly, $i_2 - i_3$ is the current across R_1 (arrowhead down). Using Ohm's law $V_R = RI$ and Faraday's law $V_L = LI'$ plus Kirchhoff's loop law *algebraic sum of the voltage drops is zero* around a closed loop (see the `maple` code below), we arrive at the model

$$
\begin{aligned}
i_1' &= - \left(\frac{R_2}{L_1}\right) i_2 - \left(\frac{R_3}{L_1}\right) i_3, \\
i_2' &= - \left(\frac{R_2}{L_2} + \frac{R_2}{L_1}\right) i_2 + \left(\frac{R_1}{L_2} - \frac{R_3}{L_1}\right) i_3, \\
i_3' &= \left(\frac{R_1}{L_3} - \frac{R_2}{L_1}\right) i_2 - \left(\frac{R_1}{L_3} + \frac{R_3}{L_1} + \frac{R_3}{L_3}\right) i_3
\end{aligned}
$$

A computer algebra system is helpful to obtain the differential equations from the closed loop formulas. Part of the theory is that the number of equations equals the number of *holes* in the network, called the **connectivity**. Here's some `maple` code for determining the equations in scalar and also in vector-matrix form.

```
loop1:=L1*D(i1)+R3*i3+R2*i2=0;
loop2:=L2*D(i2)-L2*D(i1)+R1*(i2-i3)+R2*i2=0;
loop3:=L3*D(i3)-L3*D(i1)+R3*i3+R1*(i3-i2)=0;
f1:=solve(loop1,D(i1));
f2:=solve(subs(D(i1)=f1,loop2),D(i2));
f3:=solve(subs(D(i1)=f1,loop3),D(i3));
with(linalg):
jacobian([f1,f2,f3],[i1,i2,i3]);
```

Electrical LR–Network with EMF

Consider the LR-network of Figure 18. This network produces only two differential equations, even though there are three *holes* (connectivity 3). The derivation of the differential equations parallels the previous network, so nothing will be repeated here.

A computer algebra system is used to obtain the differential equations from the closed loop formulas. Below is `maple` code to generate the equations $i_1' = f_1$, $i_2' = f_2$, $i_3 = f_3$.

```
loop1:=L1*D(i1)+R2*(i1-i2)+R1*(i1-i3)=0;
loop2:=L2*D(i2)+R3*(i2-i3)+R2*(i2-i1)=0;
loop3:=R3*(i3-i2)+R1*(i3-i1)=E;
f3:=solve(loop3,i3);
f1:=solve(subs(i3=f3,loop1),D(i1));
f2:=solve(subs(i3=f3,loop2),D(i2));
```

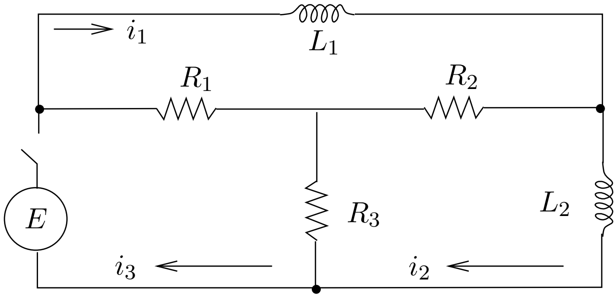

Figure 18. An electrical network.
There are three resistors R_1, R_2, R_3, two inductors L_1, L_2 and a battery E. The currents i_1, i_2, i_3 are defined between nodes (black dots).

The model, in the special case $L_1 = L_2 = 1$ and $R_1 = R_2 = R_3 = R$:

$$\begin{aligned}
i_1' &= -\frac{3R}{2}i_1 + \frac{3R}{2}i_2 + \frac{E}{2},\\
i_2' &= \frac{3R}{2}i_1 - \frac{3R}{2}i_2 + \frac{E}{2},\\
i_3 &= \frac{1}{2}i_1 + \frac{1}{2}i_2 + \frac{E}{2R}.
\end{aligned}$$

It is easily justified that the solution of the differential equations for initial conditions $i_1(0) = i_2(0) = 0$ is given by

$$i_1(t) = \frac{E}{2}t, \quad i_2(t) = \frac{E}{2}t.$$

Logging Timber by Helicopter

Certain sections of National Forest in the USA do not have logging access roads. In order to log the timber in these areas, helicopters are employed to move the felled trees to a nearby loading area, where they are transported by truck to the mill. The felled trees are slung beneath the helicopter on cables.

Figure 19. Helicopter logging.
Left: An Erickson helicopter lifts felled trees.
Right: Two trees are attached to the cable to lower transportation costs.

The payload for two trees approximates a double pendulum, which oscillates during flight. The angles of oscillation θ_1, θ_2 of the two connecting cables, measured from the gravity vector direction, satisfy the following differential equations, in which g is the gravitation constant, m_1, m_2 denote the masses of the two trees and L_1, L_2 are the cable lengths.

$$\begin{aligned}
(m_1 + m_2)L_1^2\theta_1'' + m_2L_1L_2\theta_2'' + (m1 + m_2)L_1g\theta_1 &= 0,\\
m_2L_1L_2\theta_1'' + m_2L_2^2\theta_2'' + m_2L_2g\theta_2 &= 0.
\end{aligned}$$

This model is derived assuming small displacements θ_1, θ_2, that is, $\sin\theta \approx \theta$ for both angles, using the following diagram.

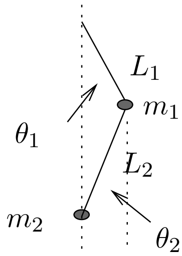

Figure 20. Logging Timber by Helicopter.
The cables have lengths L_1, L_2. The angles θ_1, θ_2 are measured from vertical.

The lengths L_1, L_2 are adjusted on each trip for the length of the trees, so that the trees do not collide in flight with each other nor with the helicopter. Sometimes, three or more smaller trees are bundled together in a package, which is treated here as identical to a single, very thick tree hanging on the cable.

Vector-matrix model. The angles θ_1, θ_2 satisfy the second-order vector-matrix equation

$$\begin{pmatrix} (m_1+m_2)L_1 & m_2L_2 \\ L_1 & L_2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}'' = - \begin{pmatrix} m_1g+m_2g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

This system is equivalent to the second-order system

$$\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}'' = \begin{pmatrix} -\dfrac{m_1g+m_2g}{L_1m_1} & \dfrac{m_2g}{L_1m_1} \\ \dfrac{m_1g+m_2\,g}{L_2m_1} & -\dfrac{(m_1+m_2)\,g}{L_2m_1} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}.$$

Earthquake Effects on Buildings

A horizontal earthquake oscillation $F(t) = F_0 \cos \omega t$ affects each floor of a 5-floor building; see Figure 21. The effect of the earthquake depends upon the natural frequencies of oscillation of the floors.

In the case of a single-floor building, the center-of-mass position $x(t)$ of the building satisfies $mx''+kx = E$ and the natural frequency of oscillation is $\sqrt{k/m}$. The earthquake force E is given by Newton's second law: $E(t) = -mF''(t)$. If $\omega \approx \sqrt{k/m}$, then the amplitude of $x(t)$ is large compared to the amplitude of the force E. The amplitude increase in $x(t)$ means that a small-amplitude earthquake wave can resonant with the building and possibly demolish the structure.

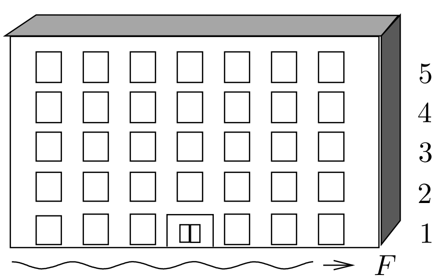

Figure 21. A 5-Floor Building.
A horizontal earthquake wave F affects every floor. The actual wave has wavelength many times larger than the illustration.

The following assumptions and symbols are used to quantize the oscillation of the 5-floor building.

- Each floor is considered a point mass located at its center-of-mass. The floors have masses $m_1, \ldots, m_5$.
- Each floor is restored to its equilibrium position by a linear restoring force or Hooke's force $-k$(elongation). The Hooke's constants are $k_1, \ldots, k_5$.
- The locations of masses representing the 5 floors are $x_1, \ldots, x_5$. The equilibrium position is $x_1 = \cdots = x_5 = 0$.
- Damping effects of the floors are ignored. This is a *frictionless* system.

The differential equations for the model are obtained by **competition**: the Newton's second law force is set equal to the sum of the Hooke's forces and the external force due to the earthquake wave. This results in the following system, where $k_6 = 0$, $E_j = m_j F''$ for $j = 1, 2, 3, 4, 5$ and $F = F_0 \cos \omega t$.

$$\begin{array}{rcl} m_1 x_1'' &=& -(k_1+k_2)x_1 + k_2 x_2 + E_1, \\ m_2 x_2'' &=& k_2 x_1 - (k_2+k_3)x_2 + k_3 x_3 + E_2, \\ m_3 x_3'' &=& k_3 x_2 - (k_3+k_4)x_3 + k_4 x_4 + E_3, \\ m_4 x_4'' &=& k_4 x_3 - (k_4+k_5)x_4 + k_5 x_5 + E_4, \\ m_5 x_5'' &=& k_5 x_4 - (k_5+k_6)x_5 + E_5. \end{array}$$

In particular, the equations for a floor depend only upon the neighboring floors. The bottom floor and the top floor are exceptions: they have just one neighboring floor.

Vector-matrix second order system. Define

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \vec{\mathbf{H}} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{pmatrix},$$

$$K = \begin{pmatrix} -k_1-k_2 & k_2 & 0 & 0 & 0 \\ k_2 & -k_2-k_3 & k_3 & 0 & 0 \\ 0 & k_3 & -k_3-k_4 & k_4 & 0 \\ 0 & 0 & k_4 & -k_4-k_5 & k_5 \\ 0 & 0 & 0 & k_5 & -k_5-k_6 \end{pmatrix}.$$

In the last row, $k_6 = 0$, to reflect the absence of a floor above the fifth. The second order system is

$$M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t) + \vec{\mathbf{H}}(t).$$

The matrix M is called the **mass matrix** and the matrix K is called the **Hooke's matrix**. The **external force** $\vec{\mathbf{H}}(t)$ can be written as a scalar function $E(t) =$

$-F''(t)$ times a constant vector:

$$\vec{\mathbf{H}}(t) = -\omega^2 F_0 \cos\omega t \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}.$$

Identical floors. Let us assume that all floors have the same mass m and the same Hooke's constant k. Then $M = mI$ and the equation becomes

$$\vec{\mathbf{x}}'' = m^{-1} \begin{pmatrix} -2k & k & 0 & 0 & 0 \\ k & -2k & k & 0 & 0 \\ 0 & k & -2k & k & 0 \\ 0 & 0 & k & -2k & k \\ 0 & 0 & 0 & k & -k \end{pmatrix} \vec{\mathbf{x}} - F_0\omega^2 \cos(\omega t) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

The Hooke's matrix K is symmetric ($K^T = K$) with negative entries only on the diagonal. The last diagonal entry is $-k$ (a common error is to write $-2k$).

Particular solution. The method of undetermined coefficients predicts a trial solution $\vec{\mathbf{x}}_p(t) = \vec{\mathbf{c}} \cos\omega t$, because each differential equation has nonhomogeneous term $-F_0\omega^2 \cos\omega t$. The constant vector $\vec{\mathbf{c}}$ is found by trial solution substitution. Cancel the common factor $\cos\omega t$ in the substituted equation to obtain the equation $\left(m^{-1}K + \omega^2 I\right) \vec{\mathbf{c}} = F_0\omega^2\vec{\mathbf{b}}$, where $\vec{\mathbf{b}}$ is column vector of ones in the preceding display. Let $B(\omega) = m^{-1}K + \omega^2 I$. Then the formula $B^{-1} = \dfrac{\mathbf{adj}(B)}{\det(B)}$ gives

$$\vec{\mathbf{c}} = F_0\omega^2 \frac{\mathbf{adj}(B(\omega))}{\det(B(\omega))} \vec{\mathbf{b}}.$$

The constant vector $\vec{\mathbf{c}}$ can have a large magnitude when $\det(B(\omega)) \approx 0$. This occurs when $-\omega^2$ is nearly an eigenvalue of $m^{-1}K$.

Homogeneous solution. The theory of this chapter gives the homogeneous solution $\vec{\mathbf{x}}_h(t)$ as the sum

$$\vec{\mathbf{x}}_h(t) = \sum_{j=1}^{5} (a_j \cos\omega_j t + b_j \sin\omega_j t) \vec{\mathbf{v}}_j$$

where $r = \omega_j$ and $\vec{\mathbf{v}} = \vec{\mathbf{v}}_j \neq \vec{\mathbf{0}}$ satisfy

$$\left(\frac{1}{m}K + r^2 I\right) \vec{\mathbf{v}} = \vec{\mathbf{0}}.$$

Special case $k/m = 10$. Then

$$\frac{1}{m}K = \begin{pmatrix} -20 & 10 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 10 & -10 \end{pmatrix}$$

and the values $\omega_1, \ldots, \omega_5$ are found by solving equation $\det((1/m)K + \omega^2 I) = 0$, to obtain the values in Table 1.

Table 1. Natural Frequencies for the Special Case $k/m = 10$.

Frequency	Value
ω_1	0.900078068
ω_2	2.627315231
ω_3	4.141702938
ω_4	5.320554507
ω_5	6.068366391

General solution. Superposition implies $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_h(t) + \vec{\mathbf{x}}_p(t)$. Both terms of the general solution represent bounded oscillations.

Resonance effects. The special solution $\vec{\mathbf{x}}_p(t)$ can be used to obtain some insight into practical resonance effects between the incoming earthquake wave and the building floors. When ω is close to one of the frequencies $\omega_1, \ldots, \omega_5$, then the amplitude of a component of $\vec{\mathbf{x}}_p$ can be very large, causing the floor to take an excursion that is too large to maintain the structural integrity of the floor.

The **physical interpretation** is that an earthquake wave of the proper frequency, having time duration sufficiently long, can demolish a floor and hence demolish the entire building. The amplitude of the earthquake wave does not have to be large: a fraction of a centimeter might be enough to start the oscillation of the floors.

Earthquakes and Tsunamis

Seismic wave shape was studied for first order equations in Chapter 2 Section 8. Recorded here are some historical notes about seismic waves and earthquake events.

The original **Richter scale**, with deprecated use in seismology, was invented by seismologist C. Richter to rank earthquake power.

The moment magnitude scale (M_W) has largely replaced the original Richter scale and its modified versions. The highest reported magnitude is 9.5 M_W by the United States Geological Survey for the Concepción, Chile earthquake of May 22, 1960. News reports and the general public still refer to earthquake magnitude using the term *Richter Scale*.

The Sumatra earthquake of December 26, 2004 occurred close to a deep-sea trench, a subduction zone where one tectonic plate slips beneath another. Most of the earthquake energy is released in these areas as the two plates grind towards each other. Estimates of magnitude 8.8 M_W to 9.3 M_W followed the event. The US Geological Survey estimated 9.2 M_W.

The largest earthquake ever recorded was the 1960 Chile earthquake. There were three earthquakes May 21-22, 1960, estimated magnitudes 9.4 to 9.6. The tsunami caused by the Chile earthquake has been well-documented by Dr. Gerard Fryer of the Hawaii Institute of Geophysics and Planetology in Honolulu.

> What happened in the earthquake was that a piece of the Pacific seafloor (or strictly speaking, the Nazca Plate) about the size of California slid fifty feet beneath the continent of South America. Like a spring, the lower slopes of the South American continent offshore snapped upwards as much as twenty feet while land along the Chile coast dropped about ten feet. This change in the shape of the ocean bottom changed the shape of the sea surface. Since the sea surface likes to be flat, the pile of excess water at the surface collapsed to create a series of waves — the tsunami.
>
> The tsunami, together with the coastal subsidence and flooding, caused tremendous damage along the Chile coast, where about 2,000 people died. The waves spread outwards across the Pacific. About 15 hours later the waves flooded Hilo, on the island of Hawaii, where they built up to 30 feet and caused 61 deaths along the waterfront. Seven hours after that, 22 hours after the earthquake, the waves flooded the coastline of Japan where 10-foot waves caused 200 deaths. The waves also caused damage in the Marquesas, in Samoa, and in New Zealand. Tidal gauges throughout the Pacific measured anomalous oscillations for about three days as the waves bounced from one side of the ocean to the other.

Image Source: Wikipedia 1960 Valdivia Chile Earthquakes

Exercises 11.1

There are no exercises for this section of examples. Later sections use this section for definitions, equations and key examples.

11.2 Fundamental System Methods

Solving 2×2 Systems

It is shown here that *any* constant linear system

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

can be solved by one of the following elementary methods.

(a) The integrating factor method for $y' = p(x)y + q(x)$.

(b) The second order constant coefficient formulas in Chapter 6, Theorem 6.1.

Triangular 2×2 Matrix A

Let's assume $b = 0$ in matrix $A = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix}$ making A lower triangular. The upper triangular case is handled similarly. Then $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has the scalar form

$$\begin{aligned} x_1' &= ax_1, \\ x_2' &= cx_1 + dx_2. \end{aligned}$$

The first differential equation is solved by the growth/decay formula:

$$x_1(t) = x_0 e^{at}.$$

Then substitute the answer just found into the second differential equation to give

$$x_2' = dx_2 + cx_0 e^{at}.$$

This is a linear first order equation of the form $y' = p(x)y + q(x)$, to be solved by the integrating factor method. Therefore, a triangular system can always be solved by the first order integrating factor method.

An illustration. Let us solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ for the triangular matrix

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad \text{representing} \quad \begin{cases} x_1' = x_1, \\ x_2' = 2x_1 + x_2. \end{cases}$$

The first equation $x_1' = x_1$ has solution $x_1 = c_1 e^t$. The second equation $x_2' = 2x_1 + x_2$ becomes upon substitution of $x_1 = c_1 e^t$ the new equation

$$x_2' = 2c_1 e^t + x_2,$$

which is a first order linear differential equation with linear integrating factor method solution $x_2 = (2c_1 t + c_2)e^t$. The general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ in scalar form is

$$x_1 = c_1 e^t, \quad x_2 = 2c_1 t e^t + c_2 e^t.$$

The **General Solution vector form** for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is

$$\vec{\mathbf{x}}(t) = c_1 \begin{pmatrix} e^t \\ 2te^t \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ e^t \end{pmatrix}.$$

A **vector basis** $\mathcal{B}$ for the solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is

$$\mathcal{B} = \left\{ \begin{pmatrix} e^t \\ 2te^t \end{pmatrix}, \begin{pmatrix} 0 \\ e^t \end{pmatrix} \right\}.$$

Non-Triangular 2×2 Matrix A

In order that A be non-triangular, both $b \neq 0$ and $c \neq 0$ must be satisfied. The scalar form of the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is

$$\left\{ \begin{array}{lcl} x_1' & = & ax_1 + bx_2, \\ x_2' & = & cx_1 + dx_2, \end{array} \right. \quad \vec{\mathbf{x}}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Theorem 11.1 (Solving 2×2 Non-Triangular $\vec{\mathbf{x}}\,' = A\,\vec{\mathbf{x}}$)
Solutions x_1, x_2 of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ are linear combinations of the list of Euler solution atoms obtained from roots r of $\det(A - rI) = 0$, which is the characteristic equation of A.

This result is called **Cayley-Hamilton-Ziebur** (abbreviated **CHZ**).

Proof: The method: differentiate the first equation, then use the equations to eliminate x_2, x_2'. The result is a second order differential equation for x_1. The same differential equation is satisfied also for x_2. The details:

$x_1'' = ax_1' + bx_2'$	Differentiate the first equation.
$= ax_1' + bcx_1 + bdx_2$	Use equation $x_2' = cx_1 + dx_2$.
$= ax_1' + bcx_1 + d(x_1' - ax_1)$	Use equation $x_1' = ax_1 + bx_2$.
$= (a + d)x_1' + (bc - ad)x_1$	Second order equation for x_1 found

The characteristic equation of $x_1'' - (a + d)x_1' + (ad - bc)x_1 = 0$ is

$$r^2 - (a + d)r + (bc - ad) = 0.$$

Finally, we show the expansion of $\det(A - rI)$ is the same characteristic polynomial:

$$\begin{array}{rcl} \det(A - rI) & = & \begin{vmatrix} a - r & b \\ c & d - r \end{vmatrix} \\ & = & (a - r)(d - r) - bc \\ & = & r^2 - (a + d)r + ad - bc. \end{array}$$

■

Proposition 11.1 (Differential Equation for x_1 and x_2)
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$:

$$\det(A - rI) = r^2 - \mathbf{trace}(A)r + \det(A)$$
$$u'' - \mathbf{trace}(A)u' + \det(A)u = 0 \text{ for } u = x_1, x_2$$

Proof: The trace of A is $a+d$ and $\det(A) = ad-bc$. Apply proof details from Theorem 11.1. ■

Assume below that A is non-triangular, meaning $b \neq 0$ and $c \neq 0$.

How to Find x_1. Apply Chapter 6 Theorem 6.1 for equation $Ay''+By'+Cy=0$ to solve for x_1. This involves writing a list of Euler solution atoms corresponding to the two roots of the characteristic equation $r^2-(a+d)r+ad-bc=0$, followed by expressing x_1 as a linear combination of the two Euler atoms.

How to Find x_2. Isolate x_2 in the first differential equation by division:

$$x_2 = \frac{1}{b}(x_1' - ax_1).$$

The two formulas for x_1, x_2 represent the general solution of the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, when A is 2×2.

An illustration. Let's solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ when

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \text{representing} \quad \begin{cases} x_1' &= x_1 + 2x_2, \\ x_2' &= 2x_1 + x_2. \end{cases}$$

The equation $\det(A-rI)=0$ is $(1-r)^2-4=0$ with roots $r=-1$ and $r=3$. The Euler solution atoms are e^{-t}, e^{3t}. Then $x_1 = c_1e^{-t} + c_2e^{3t}$, a linear combination of Euler solution atoms. The first equation $x_1' = x_1+2x_2$ implies $x_2 = \frac{1}{2}(x_1'-x_1)$ (we solve the first equation for symbol x_2). Insert $x_1 = c_1e^{-t}+c_2e^{3t}$ and simplify to find x_2 explicitly. The scalar general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is then

$$x_1 = c_1e^{-t} + c_2e^{3t}, \quad x_2 = -c_1e^{-t} + c_2e^{3t}.$$

In vector form, the general solution is

$$\vec{\mathbf{x}} = c_1\begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix} + c_2\begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}.$$

History. The fundamental idea in the illustration was developed by Ziebur using the classical **Cayley-Hamilton theorem**, which says that a square matrix satisfies its own characteristic equation. History suggests the name **Cayley-Hamilton-Ziebur** (abbreviated **CHZ**).

The Cayley-Hamilton theorem is the foundation for spectral methods developed in this chapter. Computer algebra systems provide algorithms for solving any system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$, possible because of the foundation provided by Cayley-Hamilton.

Method for $n \times n$ Diagonal A

If an $n \times n$ matrix A is diagonal, $A = \textbf{diag}(a_1, \ldots, a_n)$, then the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is a set of uncoupled scalar growth/decay equations:

$$\begin{array}{rcl} x_1'(t) &=& a_1 x_1(t), \\ x_2'(t) &=& a_2 x_2(t), \\ &\vdots& \\ x_n'(t) &=& a_n x_n(t). \end{array}$$

The solution to the system is given by the formulas

$$\begin{array}{rcl} x_1(t) &=& c_1 e^{a_1 t}, \\ x_2(t) &=& c_2 e^{a_2 t}, \\ &\vdots& \\ x_n(t) &=& c_n e^{a_n t}. \end{array}$$

The numbers $c_1, \ldots, c_n$ are arbitrary constants.

Method for $n \times n$ Lower Triangular A

Assume a linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has a square lower triangular matrix A. The system can be solved by first order scalar methods. To illustrate the ideas, consider the 3×3 lower triangular linear system

$$\vec{\mathbf{x}}' = \begin{pmatrix} 2 & 0 & 0 \\ 3 & 3 & 0 \\ 4 & 4 & 4 \end{pmatrix} \vec{\mathbf{x}}.$$

In scalar form, the system is given by the equations

$$\begin{array}{rcl} x_1'(t) &=& 2x_1(t), \\ x_2'(t) &=& 3x_1(t) + 3x_2(t), \\ x_3'(t) &=& 4x_1(t) + 4x_2(t) + 4x_3(t). \end{array}$$

A recursive method. The system is solved recursively by first order scalar methods only, starting with the first equation $x_1'(t) = 2x_1(t)$. This growth equation has general solution $x_1(t) = c_1 e^{2t}$. The second equation then becomes the first order linear equation

$$\begin{array}{rcl} x_2'(t) &=& 3x_1(t) + 3x_2(t) \\ &=& 3x_2(t) + 3c_1 e^{2t}. \end{array}$$

The integrating factor method applies: $x_2(t) = -3c_1 e^{2t} + c_2 e^{3t}$ is the general solution. The third and last equation becomes the first order linear equation

$$\begin{array}{rcl} x_3'(t) &=& 4x_1(t) + 4x_2(t) + 4x_3(t) \\ &=& 4x_3(t) + 4c_1 e^{2t} + 4(-3c_1 e^{2t} + c_2 e^{3t}). \end{array}$$

The integrating factor method is repeated to find the general solution $x_3(t) = 4c_1e^{2t} - 4c_2e^{3t} + c_3e^{4t}$.

In summary, the scalar general solution to the system is given by the formulas

$$\begin{aligned} x_1(t) &= c_1e^{2t}, \\ x_2(t) &= -3c_1e^{2t} + c_2e^{3t}, \\ x_3(t) &= 4c_1e^{2t} - 4c_2e^{3t} + c_3e^{4t}. \end{aligned}$$

Structure of solutions. A system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ for $n \times n$ triangular A has component solutions $x_1(t), \ldots, x_n(t)$ given as polynomials times exponentials. The exponential factors $e^{a_{11}t}, \ldots, e^{a_{nn}t}$ are expressed in terms of the diagonal elements $a_{11}, \ldots, a_{nn}$ of the matrix A. Fewer than n distinct exponential factors may appear, due to duplicate diagonal elements. These duplications cause the polynomial factors to appear. The reader is invited to work out the solution to the system below, which has duplicate diagonal entries $a_{11} = a_{22} = a_{33} = 2$.

$$\begin{aligned} x_1'(t) &= 2x_1(t), \\ x_2'(t) &= 3x_1(t) + 2x_2(t), \\ x_3'(t) &= 4x_1(t) + 4x_2(t) + 2x_3(t). \end{aligned}$$

The solution, given below, has polynomial factors t and t^2, appearing because of the duplicate diagonal entries $2, 2, 2$, and only one exponential factor e^{2t}.

$$\begin{aligned} x_1(t) &= c_1e^{2t}, \\ x_2(t) &= 3c_1te^{2t} + c_2e^{2t}, \\ x_3(t) &= 4c_1te^{2t} + 6c_1t^2e^{2t} + 4c_2te^{2t} + c_3e^{2t}. \end{aligned}$$

Method for $n \times n$ Upper Triangular A

A matrix differential system $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$ with T upper triangular splits into scalar equations which can be solved by elementary methods for first order scalar differential equations. To illustrate, consider the system

$$\begin{aligned} y_1' &= 3y_1 + y_2 + y_3, \\ y_2' &= 3y_2 + y_3, \\ y_3' &= 2y_3. \end{aligned}$$

The techniques that apply are the growth-decay formula for $u' = ku$ and the integrating factor method for $u' = ku + p(t)$. Working backwards from the last equation with back-substitution gives

$$\begin{aligned} y_3 &= c_3e^{2t}, \\ y_2 &= c_2e^{3t} - c_3e^{2t}, \\ y_1 &= (c_1 + c_2t)e^{3t}. \end{aligned}$$

Jordan's $n \times n$ Variable Change for $\vec{x}' = A\vec{x}$

What has been said above applies to any triangular system $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$, in order to write an exact formula for the solution $\vec{\mathbf{y}}(t)$.

If A is an $n \times n$ matrix, then Jordan's theorem gives $A = PTP^{-1}$ with T upper triangular and P invertible. The change of variable $\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t)$ changes $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ into the triangular system $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$.

There is no special condition on A to effect the change of variable $\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t)$. The solution $\vec{\mathbf{x}}(t)$ of $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ is a product of the invertible matrix P and a column vector $\vec{\mathbf{y}}(t)$; the latter is the solution of the triangular system $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$, obtained by growth-decay and integrating factor methods.

The *importance of this idea* is to provide a reliable method for solving any system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$. Later in this chapter, we outline how to find the matrix P and the matrix T in Jordan's theorem $A = PTP^{-1}$. The additional theory provides both desktop paper-and-pencil and computer matrix methods for solving any system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$.

Differential Equation Conversion to $\vec{x}' = A\vec{x}$

Considered here are source equations in scalar form or in vector form. The object is to define a new vector variable $\vec{\mathbf{x}}(t)$ and a matrix A which converts the source equations into the system form $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. The ideas apply as well to systems of nonlinear and/or non-homogeneous equations with higher derivatives, the converted system having the nonlinear form $\vec{\mathbf{x}}' = \vec{\mathbf{f}}(t, \vec{\mathbf{x}})$, a form precursor to applying computer numerical methods. The list of source equations to be considered:

Scalar linear 2nd order $au'' + bu' + cu = f$

Scalar linear 2nd order system $\left\{\begin{array}{rcl} a_1x_1'' + b_1x_1' + c_1x_1 & = & f_1, \\ a_2x_2'' + b_2x_2' + c_2x_2 & = & f_2. \end{array}\right.$

Coupled spring-mass $\begin{cases} m_1x_1''(t) = -k_1x_1(t) + k_2(x_2(t) - x_1(t)), \\ m_2x_2''(t) = -k_2(x_2(t) - x_1(t)) + k_3(u_3(t) - x_2(t)), \\ m_3u_3''(t) = -k_3(u_3(t) - x_2(t)) - k_4u_3(t). \end{cases}$

Scalar linear nth order $y^{(n)} = p_0y + \cdots + p_{n-1}y^{(n-1)}$

Scalar continuous coefficients $y^{iv} = a(x)y + b(x)y' + c(x)y'' + d(x)y'''$

Forced higher order $y^{iv} = 2y + \sin(x)y' + \cos(x)y'' + x^2y''' + f(x)$.

Second order system $M\vec{\mathbf{x}}'' = K\vec{\mathbf{x}}$

Forced second order system $M\vec{\mathbf{x}}'' = K\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

Damped Forced system $M\vec{\mathbf{x}}'' = B\vec{\mathbf{x}}' + K\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

Convert Scalar Linear 2nd Order to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Consider an equation $au'' + bu' + cu = f$ where $a \neq 0$, b, c, f are allowed to depend on t, $' = d/dt$. Define the **Position-Velocity substitution**

$$x_1(t) = u(t), \quad x_2(t) = u'(t).$$

Then $x_1' = u' = x_2$ and $x_2' = u'' = (-bu' - cu + f)/a = -(b/a)x_2 - (c/a)x_1 + f/a$. The resulting system is equivalent to the second order equation, in the sense that the position-velocity substitution transforms solutions of one system to the other:

$$\begin{cases} x_1'(t) &= \quad (0)\, x_1(t) \quad + \quad (1)\, x_2(t), \\ x_2'(t) &= -\left(\frac{c(t)}{a(t)}\right) x_1(t) - \left(\frac{b(t)}{a(t)}\right) x_2(t) + \frac{f(t)}{a(t)}. \end{cases}$$

The case of constant coefficients and f a function of t arises often enough to isolate the result for further reference.

Theorem 11.2 (Constant-Coefficient 2nd Order Conversion)
Let $a \neq 0$, b, c be constants and $f(t)$ continuous. Then $au'' + bu' + cu = f(t)$ is equivalent to the first order system

$$a\vec{\mathbf{x}}'(t) = \begin{pmatrix} 0 & a \\ -c & -b \end{pmatrix} \vec{\mathbf{w}}(t) + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad \vec{\mathbf{x}}(t) = \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}.$$

Convert Second Order Scalar Systems to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

A position-velocity substitution can be carried out on a system of two second order linear differential equations. Assume

$$\begin{cases} a_1 x_1'' + b_1 x_1' + c_1 x_1 &= f_1, \\ a_2 x_2'' + b_2 x_2' + c_2 x_2 &= f_2. \end{cases}$$

Then the preceding methods for the scalar case give the equivalence

$$\begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix}' = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -c_1 & -b_1 & 0 & 0 \\ 0 & 0 & 0 & a_2 \\ 0 & 0 & -c_2 & -b_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \\ x_2 \\ x_2' \end{pmatrix} + \begin{pmatrix} 0 \\ f_1 \\ 0 \\ f_2 \end{pmatrix}.$$

Convert Coupled Spring-Mass Systems to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Springs connecting undamped coupled masses were considered at the beginning of this chapter, page 827. Typical equations are

$$(1) \qquad \begin{cases} m_1 x_1''(t) &= -k_1 x_1(t) + k_2(x_2(t) - x_1(t)), \\ m_2 x_2''(t) &= -k_2(x_2(t) - x_1(t)) + k_3(u_3(t) - x_2(t)), \\ m_3 u_3''(t) &= -k_3(u_3(t) - x_2(t)) - k_4 u_3(t). \end{cases}$$

The equations can be represented by a second order linear system of dimension 3 of the form $M\vec{\mathbf{x}}'' = K\vec{\mathbf{x}}$, where the **Vector Position** $\vec{\mathbf{x}}$, the **mass matrix** M and the **Hooke's matrix** K are given by the equalities

$$\vec{\mathbf{x}}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ u_3(t) \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$

$$K = \begin{pmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & -k_3 & -(k_3+k_4) \end{pmatrix}.$$

Conversion to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ uses a position-velocity substitution to obtain the block matrix multiply equation (I = identity matrix, 0 = zero matrix)

$$\vec{\mathbf{x}}(t) = \begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix}, \quad \vec{\mathbf{x}}'(t) = \left(\begin{array}{c|c} 0 & I \\ \hline M^{-1}K & 0 \end{array} \right) \vec{\mathbf{x}}(t).$$

Convert Higher Order Linear Equations to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Every homogeneous nth order linear differential equation

$$y^{(n)} = p_0 y + \cdots + p_{n-1} y^{(n-1)}$$

with constant coefficients can be converted to a linear homogeneous vector-matrix system

$$\frac{d}{dx}\begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \\ p_0 & p_1 & p_2 & \cdots & p_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y' \\ y'' \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

This is a linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ where $\vec{\mathbf{x}}(t)$ is the $n \times 1$ column vector consisting of $y(t)$ and its successive derivatives, while the $n \times n$ matrix A is the classical **Companion Matrix**[5] of the characteristic polynomial

$$r^n = p_0 + p_1 r + p_2 r^2 + \cdots + p_{n-1} r^{n-1}.$$

To illustrate, the companion matrix (page 846) for $r^4 = a + br + cr^2 + dr^3$ is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & b & c & d \end{pmatrix}.$$

[5]The transpose of the companion matrix defined in Wikipedia. The companion matrix or its transpose appears in advanced topics in linear algebra, e.g. the Frobenius Rational Form.

The preceding companion matrix has the following block matrix form, which is representative of all companion matrices.

$$A = \left(\begin{array}{c|ccc} \vec{\mathbf{0}} & & I & \\ \hline a & b & c & d \end{array}\right).$$

Convert Scalar Continuous-Coefficient Equations to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

. Methods above apply equally to higher order linear differential equations with continuous coefficients. To illustrate, the fourth order linear equation $y^{iv} = a(x)y + b(x)y' + c(x)y'' + d(x)y'''$ has first order system form $\vec{\mathbf{x}}' = A(x)\vec{\mathbf{x}}$ where $A(x)$ is the companion matrix (page 846) for the polynomial $r^4 = a(x) + b(x)r + c(x)r^2 + d(x)r^3$, x held fixed:

$$A(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a(x) & b(x) & c(x) & d(x) \end{pmatrix}.$$

Convert Forced Higher Order Equations to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

All that has been said above applies equally to a forced linear equation like

$$y^{iv} = 2y + \sin(x)y' + \cos(x)y'' + x^2y''' + f(x).$$

It has a conversion to a first order nonhomogeneous linear system

$$\vec{\mathbf{x}}' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & \sin x & \cos x & x^2 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(x) \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} y \\ y' \\ y'' \\ y''' \end{pmatrix}.$$

Convert 2nd Order System to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

A second order system $M\vec{\mathbf{x}}'' = K\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ is called a **forced system** and $\vec{\mathbf{F}}$ is called the **external vector force**. Such a system can always be converted to a second order system where the mass matrix is the identity, by multiplying by M^{-1}:

$$\vec{\mathbf{x}}'' = M^{-1}K\vec{\mathbf{x}} + M^{-1}\vec{\mathbf{F}}(t).$$

The benign form $\vec{\mathbf{x}}'' = B\vec{\mathbf{x}} + \vec{\mathbf{G}}(t)$, where $B = M^{-1}K$ and $\vec{\mathbf{G}} = M^{-1}\vec{\mathbf{F}}$, admits a block matrix conversion into a forced first order system of the form $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$:

$$\vec{\mathbf{x}}(t) = \begin{pmatrix} \vec{\mathbf{x}}(t) \\ \vec{\mathbf{x}}'(t) \end{pmatrix}, \quad \frac{d}{dt}\vec{\mathbf{x}}(t) = \left(\begin{array}{c|c} 0 & I \\ \hline M^{-1}K & 0 \end{array}\right) \vec{\mathbf{x}}(t) + \begin{pmatrix} \vec{\mathbf{0}} \\ M^{-1}\vec{\mathbf{F}}(t) \end{pmatrix}.$$

Convert Damped 2nd Order System to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

The addition of a dashpot to each of the masses gives a **damped** second order system with forcing term

$$M\vec{\mathbf{x}}'' = B\vec{\mathbf{x}}' + K\vec{\mathbf{x}} + \vec{\mathbf{F}}(t).$$

In the case of one scalar equation, the matrices M, B, K are constants m, $-c$, $-k$ and the external force is a scalar function $f(t)$, hence the system becomes the classical damped spring-mass equation

$$mu'' + cu' + ku = f(t).$$

A standard way to write the first order system $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{G}}(t)$ is to introduce variable $\vec{\mathbf{u}} = M\begin{pmatrix}\vec{\mathbf{x}}\\ \vec{\mathbf{x}}'\end{pmatrix}$, in order to obtain

$$\vec{\mathbf{u}}' = M\frac{d}{dt}\begin{pmatrix}M\vec{\mathbf{x}}\\ \vec{\mathbf{x}}'\end{pmatrix} = M\begin{pmatrix}\vec{\mathbf{x}}'\\ \vec{\mathbf{x}}''\end{pmatrix} = M\begin{pmatrix}\vec{\mathbf{x}}'\\ B\vec{\mathbf{x}}' + K\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)\end{pmatrix}$$

Then a first order system in block matrix form is given by

$$\left(\begin{array}{c|c} M & 0 \\ \hline 0 & M\end{array}\right)\frac{d}{dt}\begin{pmatrix}\vec{\mathbf{x}}(t)\\ \vec{\mathbf{x}}'(t)\end{pmatrix} = \left(\begin{array}{c|c} 0 & M \\ \hline K & B\end{array}\right)\begin{pmatrix}\vec{\mathbf{x}}(t)\\ \vec{\mathbf{x}}'(t)\end{pmatrix} + \begin{pmatrix}\vec{\mathbf{0}}\\ \vec{\mathbf{F}}(t)\end{pmatrix}.$$

The benign form $\vec{\mathbf{x}}'' = M^{-1}B\vec{\mathbf{x}}' + M^{-1}K\vec{\mathbf{x}} + M^{-1}\vec{\mathbf{F}}(t)$, which is obtained from left-multiplication by M^{-1}, can be similarly written as a first order system in block matrix form.

$$\frac{d}{dt}\begin{pmatrix}\vec{\mathbf{x}}(t)\\ \vec{\mathbf{x}}'(t)\end{pmatrix} = \left(\begin{array}{c|c} 0 & I \\ \hline M^{-1}K & M^{-1}B\end{array}\right)\begin{pmatrix}\vec{\mathbf{x}}(t)\\ \vec{\mathbf{x}}'(t)\end{pmatrix} + \begin{pmatrix}\vec{\mathbf{0}}\\ M^{-1}\vec{\mathbf{F}}(t)\end{pmatrix}$$

Exercises 11.2

Solving 2×2 Systems

1. Solve $x_1' = 2x_1 + x_2$, $x_2' = x_2$. Ans: $x_1 = c_1 e^{2t} - c_2 e^t$, $x_2 = c_2 e^t$

2. Discuss how to solve $\vec{\mathbf{x}}' = \begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\vec{\mathbf{x}}$.

Triangular 2×2 Matrix A

3. Solve $\vec{\mathbf{x}}' = \begin{pmatrix}2 & 1\\ 0 & 3\end{pmatrix}\vec{\mathbf{x}}$.

4. Solve $\vec{\mathbf{x}}' = \begin{pmatrix}2 & 0\\ 2 & 3\end{pmatrix}\vec{\mathbf{x}}$.

Non-Triangular 2×2 Matrix A

5. Solve $\vec{\mathbf{x}}' = \begin{pmatrix}1 & 3\\ 3 & 1\end{pmatrix}\vec{\mathbf{x}}$.

6. Solve $\vec{\mathbf{x}}' = \begin{pmatrix}1 & 3\\ -3 & 1\end{pmatrix}\vec{\mathbf{x}}$.

Method for $n\times n$ Diagonal A

7. Solve $\vec{\mathbf{x}}' = \begin{pmatrix}1 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\vec{\mathbf{x}}$.

8. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1&0&0&0\\ 0&3&0&0\\ 0&0&4&0\\ 0&0&0&2 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Lower Triangular

9. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1&0&0\\ 1&3&0\\ 1&0&2 \end{pmatrix} \vec{\mathbf{x}}$.

10. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1&0&0\\ 0&3&0\\ 1&0&2 \end{pmatrix} \vec{\mathbf{x}}$.

Method for $n \times n$ Upper Triangular

11. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1&0&1\\ 0&3&1\\ 0&0&2 \end{pmatrix} \vec{\mathbf{x}}$.

12. Solve $\vec{\mathbf{x}}' = \begin{pmatrix} 1&1&0\\ 0&3&1\\ 0&0&2 \end{pmatrix} \vec{\mathbf{x}}$.

Jordan's $n \times n$ Variable Change

Let $A = PTP^{-1}$ with T upper triangular and P invertible. Define change of variable $\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t)$. Prove these results:

13. If $\vec{\mathbf{x}}(t)$ solves $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$, then $\vec{\mathbf{y}}(t) = P^{-1}\vec{\mathbf{x}}(t)$ solves $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$.

14. If $\vec{\mathbf{y}}'(t) = T\vec{\mathbf{y}}(t)$, then $\vec{\mathbf{x}}(t) = P\vec{\mathbf{y}}(t)$ solves $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$.

Convert Scalar Linear 2nd Order to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$

15. $x'' + 2x' + x = \sin t$

16. $2x'' + 3x' + 8x = 4\cos t$

Convert Second Order Scalar System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

17. $x'' = x + y$, $y'' = x - y$

18. $x'' = x + y + \sin t$, $y'' + y = x + \cos t$

Convert Coupled Spring-Mass System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}$

19. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2&1\\ 1&-1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0\\ \sin t \end{pmatrix}$

20. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2&1&0\\ 1&-2&1\\ 0&-1&-2 \end{pmatrix} \vec{\mathbf{x}}$

Convert Higher Order Linear Equations to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

21. $x''' = x$

22. $\dfrac{d^4y}{dx^4} + 16y = 0$

Convert Scalar Continuous-Coefficient Equation to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

23. $x^2y'' + 3xy' + 2y = 0$

24. $y''' + xy'' + x^2y + y = 0$

Convert Forced Higher Order Equation to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$

25. $\dfrac{d^4y}{dx^4} = y''' + y + \sin x$

26. $\dfrac{d^6y}{dx^6} = \dfrac{d^4y}{dx^4} + y + \cos t$

Convert 2nd Order System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{G}}(t)$

27. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2&1\\ 1&-1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1\\ -1 \end{pmatrix}$

28. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2&1&0\\ 1&-2&1\\ 0&-1&-2 \end{pmatrix} \vec{\mathbf{x}} + e^t \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix}$

Convert Damped 2nd Order System to $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{G}}(t)$

29. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2&1\\ 1&-1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0&1\\ 1&0 \end{pmatrix} \vec{\mathbf{x}}' + \begin{pmatrix} 1\\ -1 \end{pmatrix}$

30. $\vec{\mathbf{x}}'' = \begin{pmatrix} -2&1&0\\ 1&-2&1\\ 0&-1&-2 \end{pmatrix} \vec{\mathbf{x}} + \vec{\mathbf{x}}' + e^t \begin{pmatrix} 1\\ 1\\ 1 \end{pmatrix}$

11.3 Structure of Linear Systems

Notation for Linear Systems

A **linear system** is a system of differential equations of the form

$$
\begin{array}{rcccccccc}
x_1' & = & a_{11}x_1 & + & \cdots & + & a_{1n}x_n & + & f_1, \\
x_2' & = & a_{21}x_1 & + & \cdots & + & a_{2n}x_n & + & f_2, \\
\vdots & & & \vdots & \cdots & \vdots & & \vdots & \\
x_m' & = & a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & + & f_m,
\end{array}
\tag{1}
$$

where $' = d/dt$. Given are the functions $a_{ij}(t)$ and $f_j(t)$ on some interval $a < t < b$. The unknowns are the functions $x_1(t), \ldots, x_n(t)$.

The system is called **homogeneous** if all $f_j = 0$, otherwise it is called **non-homogeneous**.

Matrix Notation. A non-homogeneous system of linear equations (1) is written as the equivalent vector-matrix system

$$\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$$

where

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{\mathbf{F}} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Existence-Uniqueness

Special results are isolated to illustrate how Picard-Lindelöf theory is applied to linear systems. Proofs start on page 859.

Theorem 11.3 (Gronwall's Lemma)
Let $u(t), v(t)$ be continuous functions with $v(t) \geq 0$ on interval $t_0 \leq t \leq t_0 + H$. Assume $u(t) \leq c + \int_{t_0}^{t} u(r)v(r)dr$ for t for $t_0 \leq t \leq t_0 + H$. Then:

$$u(t) \leq c\, e^{-\int_{t_0}^{t} v(r)dr}, \quad t_0 \leq t \leq t_0 + H.$$

Theorem 11.4 (Unique Zero Solution)
Let $A(t)$ be an $m \times n$ matrix with entries continuous on $t_0 \leq t \leq t_0 + H$. Then the initial value problem

$$\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{0}}$$

has unique solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{0}}$ on $t_0 \leq t \leq t_0 + H$.

Theorem 11.5 (Picard-Lindelöf)
Let n-vector $\vec{\mathbf{F}}(t)$ and $n \times n$ matrix $A(t)$ be continuous on interval J: $a < t < b$. Let t_0 be in J. Let $\vec{\mathbf{x}}_0$ be in $\mathcal{R}^n$. Then the initial value problem

$$\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

has a unique solution $\vec{\mathbf{x}}(t)$ defined on all of J.

Theorem 11.6 (Existence-Uniqueness for Constant Linear Systems)
Let $A(t) = A$ be an $m \times n$ matrix with constant entries and let t_0 be any real number and let $\vec{\mathbf{x}}_0$ be any n-vector. Then the initial value problem

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}}, \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

has a unique solution $\vec{\mathbf{x}}(t)$ defined for all values of t.

Theorem 11.7 (Uniqueness and Solution Crossings)
Let $A(t)$ be an $m \times n$ matrix with entries continuous on $a < t < b$ and assume $\vec{\mathbf{F}}(t)$ is also continuous on $a < t < b$. If $\vec{\mathbf{x}}_1(t)$ and $\vec{\mathbf{x}}_2(t)$ are solutions of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ on $a < t < b$ and $\vec{\mathbf{x}}_1(t_0) = \vec{\mathbf{x}}_2(t_0)$ for some t_0, $a < t_0 < b$, then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{x}}_2(t)$ for $a < t < b$.

Linearity and Superposition

Linear homogeneous systems have **linear structure** and nonhomogeneous systems obey a **Principle of Superposition**.

Theorem 11.8 (Linear Structure)
Let $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ have two solutions $\vec{\mathbf{x}}_1(t)$, $\vec{\mathbf{x}}_2(t)$. If k_1, k_2 are constants, then $\vec{\mathbf{x}}(t) = k_1\,\vec{\mathbf{x}}_1(t) + k_2\,\vec{\mathbf{x}}_2(t)$ is also a solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$.

Theorem 11.9 (Basis)
The solution set of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ is an n-dimensional subspace of the vector space of all vector-valued functions $\vec{\mathbf{x}}(t)$ on $a < t < b$.

Let $a < t_0 < b$. A **standard basis** $\vec{\mathbf{w}}_1(t), \ldots, \vec{\mathbf{w}}_n(t)$ is defined by $\vec{\mathbf{w}}'_j(t) = A(t)\vec{\mathbf{w}}_j(t)$, $\vec{\mathbf{w}}_j(t_0) = \vec{\mathbf{e}}_j =$ column j of the identity matrix I, $1 \le j \le n$.

Every solution $\vec{\mathbf{x}}(t)$ of $\vec{\mathbf{x}}'(t) = A(t)\vec{\mathbf{x}}(t)$ has a unique basis expansion:

$$\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{w}}_1(t) + c_2\vec{\mathbf{w}}_2(t) + \cdots + c_n\vec{\mathbf{w}}_n(t)$$

Theorem 11.10 (Superposition Principle)
Let $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have a particular solution $\vec{\mathbf{x}}_p(t)$. If $\vec{\mathbf{x}}(t)$ is any solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$, then $\vec{\mathbf{x}}(t)$ can be decomposed as **homogeneous plus particular**:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_h(t) + \vec{\mathbf{x}}_p(t).$$

Term $\vec{\mathbf{x}}_h(t)$ is a certain solution of the homogeneous differential equation $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$, which means arbitrary constants c_1, c_2, ... have been assigned specific values. The shortest particular solution $\vec{\mathbf{x}}_p(t)$ excludes any term $\vec{\mathbf{y}}(t)$ satisfying $\vec{\mathbf{y}}'t() = A(t)\vec{\mathbf{y}}(t)$, such terms being absorbed into $\vec{\mathbf{x}}_h(t)$.

Theorem 11.11 (Difference of Solutions)
Let $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have two solutions $\vec{\mathbf{x}} = \vec{\mathbf{u}}(t)$ and $\vec{\mathbf{x}} = \vec{\mathbf{v}}(t)$. Define $\vec{\mathbf{y}}(t) = \vec{\mathbf{u}}(t) - \vec{\mathbf{v}}(t)$. Then $\vec{\mathbf{y}}(t)$ satisfies the homogeneous equation

$$\vec{\mathbf{y}}' = A(t)\vec{\mathbf{y}}.$$

General Solution

The general solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ is an expression involving arbitrary constants c_1, c_2, ... and certain functions. The expression may be given in vector notation, although scalar expressions are commonplace and perfectly acceptable. Required is that the expression represents all solutions of the differential equation, in the following sense:

Definition 11.1 (General Solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$)
An expression is called a **general solution** of system $\vec{\mathbf{x}}'(t) = A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t)$ provided:

> **(a)** Every **assignment of constants** in the expression produces a solution of the differential equation.
>
> **(b)** Every possible solution is **uniquely** obtained from the expression by **specializing the constants**.

Superposition Theorem 11.10 implies that the constants in the general solution are identified as multipliers against solutions of the homogeneous differential equation. The general solution has recognizable structure:

Theorem 11.12 (General Solution)
Let $A(t)$ be an $n \times n$ matrix. Let $\vec{\mathbf{F}}(t)$ be an $n \times 1$ vector. Assume $A(t)$ and $\vec{\mathbf{F}}(t)$ are continuous on an interval $a < t < b$. Then linear nonhomogeneous system $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ has general solution $\vec{\mathbf{x}}$ given by the expression

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_h(t) + \vec{\mathbf{x}}_p(t).$$

1. Term $\vec{\mathbf{y}} = \vec{\mathbf{x}}_h(t)$ is a general solution of the homogeneous equation $\vec{\mathbf{y}}' = A(t)\vec{\mathbf{y}}$ which contains n arbitrary constants $c_1, \ldots, c_n$.
2. Term $\vec{\mathbf{x}} = \vec{\mathbf{x}}_p(t)$ is a particular solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

Recognition of Homogeneous Solution Terms

Assume given an expression $\vec{\mathbf{x}}$ for the general solution of vector-matrix equation $\vec{\mathbf{x}}'(t) = A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t)$, either in scalar component form or in vector form. Expression $\vec{\mathbf{x}}$ contains arbitrary constants $c_1, \ldots, c_n$. It is possible to isolate both terms $\vec{\mathbf{x}}_h$ and $\vec{\mathbf{x}}_p$ by a simple procedure.

Finding $\vec{\mathbf{x}}_p$. The first step: **set to zero** all arbitrary constants $c_1, c_2, \ldots, c_n$. The resulting expression is free of unresolved constants. The answer sought for $\vec{\mathbf{x}}_p(t)$ has no term $\vec{\mathbf{y}}(t)$ with $A(t)\vec{\mathbf{y}}(t) = \vec{\mathbf{0}}$. If the expression contains such a term $\vec{\mathbf{y}}$, then remove it. Repeat inspection and removal until no such term $\vec{\mathbf{y}}$ appears. If the expression $\vec{\mathbf{x}}$ consists of equations in scalar component form, then assemble the modified equations into vector $\vec{\mathbf{x}}_p$. Otherwise, the modified $\vec{\mathbf{x}}$ is vector $\vec{\mathbf{x}}_p$.

Finding $\vec{\mathbf{x}}_h$. The first step: take partial derivatives on the general solution expression $\vec{\mathbf{x}}$ with respect to the symbols $c_1, \ldots, c_n$. The formula:

$$\vec{\mathbf{u}}_k(t) = \frac{\partial}{\partial c_k}\vec{\mathbf{x}}, \quad 1 \le k \le n.$$

A vector solution basis for $\vec{\mathbf{y}}' = A(t)\vec{\mathbf{y}}$ is $\{\vec{\mathbf{u}}_k\}_{k=1}^n$. The technique isolates the vector components of the homogeneous solution from any form of the general solution, including scalar formulas for the components of $\vec{\mathbf{x}}$. Then:

$$\vec{\mathbf{x}}_h(t) = c_1\vec{\mathbf{u}}_1(t) + c_2\vec{\mathbf{u}}_2(t) + \cdots + c_n\vec{\mathbf{u}}_n(t).$$

Vector General Solution. A general solution $\vec{\mathbf{x}}$ of the nonhomogeneous linear system $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ is given by the expression

$$\vec{\mathbf{x}} = c_1\vec{\mathbf{u}}_1(t) + c_2\vec{\mathbf{u}}_2(t) + \cdots + c_n\vec{\mathbf{u}}_n(t) + \vec{\mathbf{x}}_p(t).$$

In this expression, each *assignment* of the constants $c_1, \ldots, c_n$ produces a solution of the nonhomogeneous system, and conversely, each possible solution of the nonhomogeneous system is obtained by a unique *specialization* of the constants $c_1, \ldots, c_n$.

Independence

Constants $c_1, \ldots, c_n$ in the general solution $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$ appear exactly in the expression $\vec{\mathbf{x}}_h$, which has the form

$$\vec{\mathbf{x}}_h = c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + \cdots + c_n\vec{\mathbf{u}}_n.$$

A solution $\vec{\mathbf{x}}$ of $\vec{\mathbf{x}}'(t) = A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t)$ uniquely determines the constants. In particular, the zero solution of the homogeneous equation is uniquely represented, which can be stated this way:

$$c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + \cdots + c_n\vec{\mathbf{u}}_n = \vec{\mathbf{0}} \quad \text{implies} \quad c_1 = c_2 = \cdots = c_n = 0.$$

This statement equivalently says that the list of n vector-valued functions $\vec{\mathbf{u}}_1(t)$, ..., $\vec{\mathbf{u}}_n(t)$ is **Linearly Independent**, as defined in linear algebra.

Hand calculations might write down a candidate general solution to some 3×3 linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, the resulting equations looking like

$$\begin{aligned} x_1 &= c_1 e^t + c_2 e^t + c_3 e^{2t}, \\ x_2 &= c_1 e^t + c_2 e^t + 2c_3 e^{2t}, \\ x_3 &= c_1 e^t + c_2 e^t + 4c_3 e^{2t}. \end{aligned}$$

The example illustrates a classic mistake made in calculations: it is not a general solution, even though it satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$!

How can we detect the mistake, given only that this expression is supposed to represent the general solution? A required step is to test that $\vec{\mathbf{u}}_1 = \partial\vec{\mathbf{x}}/\partial c_1$, $\vec{\mathbf{u}}_2 = \partial\vec{\mathbf{x}}/\partial c_2$, $\vec{\mathbf{u}}_3 = \partial\vec{\mathbf{x}}/\partial c_3$ are indeed solutions. To insure the **unique representation requirement** of a general solution (**(b)** page 852), the vector functions $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$ must be linearly independent. Compute partial derivatives on symbols c_1, c_2, c_3:

$$\vec{\mathbf{u}}_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \vec{\mathbf{u}}_3 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}.$$

Then $\vec{\mathbf{u}}_1 = \vec{\mathbf{u}}_2$, which implies that the functions $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$ fail to be independent. While it is possible to test independence by a rudimentary test based upon the definition of independence, the preferred method uses following tests due to Norwegian mathematician N. H. Abel (1802-1829).

Definition 11.2 (Wronskian Determinant of Vector Functions)
Let $\vec{\mathbf{u}}_j(t) : a < t < b \to \mathcal{R}^n$ be given, $1 \le j \le n$. The **Wronskian determinant** is $W(t) = \det(U)$, where U is the augmented matrix of $\vec{\mathbf{u}}_1(t), \ldots, \vec{\mathbf{u}}_n(t)$. In terms of components u_{ij} of vector $\vec{\mathbf{u}}_j$, $1 \le i, j \le n$:

$$W(t) = \begin{vmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \cdots & u_{nn} \end{vmatrix}$$

Theorem 11.13 (Abel-Liouville Formula)
Let vector functions $\vec{\mathbf{u}}_1(t)$, ..., $\vec{\mathbf{u}}_n(t)$ be solutions of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$, $a < t < b$. Let $W(t)$ be the Wronskian determinant of these solutions. Assume $a < t_0 < b$. Then the **Abel-Liouville formula** holds:

$$W(t) = \mathrm{e}^{\int_{t_0}^t \mathbf{trace}(A(s))ds} W(t_0).$$ [6]

In particular, the Wronskian determinant $W(t)$ is either everywhere nonzero or everywhere zero, accordingly as $W(t_0) \neq 0$ or $W(t_0) = 0$.

[6] The **trace** of a square matrix is the sum of its diagonal elements.

Theorem 11.14 (Abel's Wronskian Independence Test)
Vector solutions $\vec{\mathbf{x}} = \vec{\mathbf{u}}_1, \ldots, \vec{\mathbf{u}}_n$ of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ are linearly independent on $a < t < b$ if and only if the Wronskian determinant $W(t_0)$ is nonzero for some $a < t_0 < b$.

Initial Value Problems and Reduced Echelon Form

An **initial value problem** is the problem of solving for $\vec{\mathbf{x}}$, given

$$\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0.$$

Assume general solution

$$\vec{\mathbf{x}} = c_1\vec{\mathbf{u}}_1(t) + \cdots + c_n\vec{\mathbf{u}}_n(t) + \vec{\mathbf{x}}_p(t),$$

then the problem of finding $\vec{\mathbf{x}}$ reduces to finding $c_1, \ldots, c_n$ in the relation

$$c_1\vec{\mathbf{u}}_1(t_0) + \cdots + c_n\vec{\mathbf{u}}_n(t_0) + \vec{\mathbf{x}}_p(t_0) = \vec{\mathbf{x}}_0.$$

This is a matrix equation for the unknown constants $c_1, \ldots, c_n$ of the form $B\vec{\mathbf{c}} = \vec{\mathbf{d}}$, where B is the augmented matrix of $\vec{\mathbf{u}}_1(t_0), \ldots, \vec{\mathbf{u}}_n(t_0)$:

$$B = \left\langle \vec{\mathbf{u}}_1(t_0) | \cdots | \vec{\mathbf{u}}_n(t_0) \right\rangle, \quad \vec{\mathbf{c}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad \vec{\mathbf{d}} = \vec{\mathbf{x}}_0 - \vec{\mathbf{x}}_p(t_0).$$

The reduced row echelon form or **rref** provides a method to find $\vec{\mathbf{c}}$. The method: perform swap, combination and multiply operations to the augmented matrix $C = \left\langle B | \vec{\mathbf{d}} \right\rangle$ until $\mathbf{rref}(C) = \left\langle I | \vec{\mathbf{c}} \right\rangle$.

Equilibria of $\vec{x}' = A(t)\vec{x}$

An equilibrium point $\vec{\mathbf{x}}_0$ of a linear system $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ is a constant solution, $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_0$ for all t. Equilibria make sense when $A(t)$ is constant, although the definition applies to continuous systems. For a solution $\vec{\mathbf{x}}$ to be constant means $\vec{\mathbf{x}}' = \vec{\mathbf{0}}$, hence all equilibria are determined from the equation

$$A(t)\vec{\mathbf{x}}_0 = \vec{\mathbf{0}} \quad \text{for all } t.$$

This homogeneous system of linear algebraic equations is to be solved for $\vec{\mathbf{x}}_0$. It is not allowed for the answer $\vec{\mathbf{x}}_0$ to depend on t: if it does, then it is **not** an equilibrium.

The theory for a constant matrix $A(t) \equiv A$ says that either $\vec{\mathbf{x}}_0 = \vec{\mathbf{0}}$ is the unique solution or else there are infinitely many nonzero answers for $\vec{\mathbf{x}}_0$. Expectations for any matrix $A(t)$ are similar but an algorithm is lacking for finding nonzero $\vec{\mathbf{x}}_0$.

Examples and Methods

Example 11.1 (Vector Form of the General Solution)

Consider a 3×3 linear system $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ with general solution $\vec{\mathbf{x}}$ (components x_1, x_2, x_3) given in scalar form by the expressions

$$\begin{array}{lrcl} (2) \qquad & x_1 &=& c_1e^t + c_2e^{-t} + t, \\ & x_2 &=& (c_1+c_2)e^t + c_3e^{2t}, \\ & x_3 &=& (2c_2 - c_1)e^{-t} + (4c_1 - 2c_3)e^{2t} + 2t. \end{array}$$

Find the vector form of the general solution.

Solution to Example 11.1

Find $\vec{\mathbf{x}}_p(t)$. Set $c_1 = c_2 = c_3 = 0$ in scalar equations (2):

$$\vec{\mathbf{x}}_p(t) = \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix}.$$

Find $\vec{\mathbf{x}}_h$. Take partial derivatives in scalar equations (2) with respect to the variable names c_1, c_2, c_3 to determine $\vec{\mathbf{u}}_k = \dfrac{\partial \vec{\mathbf{x}}}{\partial c_k}$:

$$\vec{\mathbf{u}}_1 = \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix}, \quad \vec{\mathbf{u}}_3 = \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix}.$$

The homogeneous system **vector** solution:

$$\vec{\mathbf{x}}_h(t) = c_1\vec{\mathbf{u}}_1(t) + c_2\vec{\mathbf{u}}_2(t) + c_3\vec{\mathbf{u}}_3(t)$$

The nonhomogeneous system **vector** general solution:

$$\begin{array}{rcl} \vec{\mathbf{x}}(t) &=& c_1\vec{\mathbf{u}}_1(t) + c_2\vec{\mathbf{u}}_2(t) + c_3\vec{\mathbf{u}}_3(t) + \vec{\mathbf{x}}_p(t) \\ &=& c_1 \begin{pmatrix} e^t \\ e^t \\ -e^{-t} + 4e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ e^t \\ 2e^{-t} \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ e^{2t} \\ -2e^{2t} \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix}. \end{array}$$

To be a general solution, expression $\vec{\mathbf{x}} = c_1\vec{\mathbf{u}}_1(t) + c_2\vec{\mathbf{u}}_2(t) + c_3\vec{\mathbf{u}}_3(t) + \vec{\mathbf{x}}_p(t)$ must satisfy required elements **(a)** and **(b)** in the definition of general solution (page 852). Already **(a)** is satisfied. Issue **(b)** is not settled: vectors $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$ must be independent, to be settled by Abel's formula and the Wronskian test *infra*, details delayed to a further example.

Example 11.2 (Dependence by Abel's Wronskian Test)

Assume a 3×3 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ was solved by hand for general solution $\vec{\mathbf{x}} = c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + c_3\vec{\mathbf{u}}_3$ where

$$\vec{\mathbf{u}}_1 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{pmatrix} e^t \\ e^t \\ e^t \end{pmatrix}, \quad \vec{\mathbf{u}}_3 = \begin{pmatrix} e^{2t} \\ 2e^{2t} \\ 4e^{2t} \end{pmatrix}.$$

Choose t_0 in Abel's Wronskian Test to establish **dependence**. The reported expression $\vec{\mathbf{x}}$ is **not** a general solution.

Details for Example 11.2

Wronskian determinant $W(t)$ is quite complicated, but $W(0)$ is zero because it has two duplicate columns. Choice $t_0 = 0$ in Abel's Wronskian test detects **dependence** of solutions $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$.

Example 11.3 (Abel's Wronskian Test Detects Independence)

Assume a 3×3 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ was solved by hand for general solution $\vec{\mathbf{x}} = c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + c_3\vec{\mathbf{u}}_3$ where

$$\vec{\mathbf{u}}_1 = \begin{pmatrix} 2e^{-t} \\ -e^{2t} + 2e^t \\ 4e^{-t} + 2e^{2t} \end{pmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{pmatrix} e^{-t} \\ e^{-t} - e^{2t} \\ 2e^{2t} + 2e^{-t} \end{pmatrix}, \quad \vec{\mathbf{u}}_3 = \begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix}.$$

Choose t_0 in Abel's Wronskian Test to establish independence. The expression $\vec{\mathbf{x}}$ is the general solution.

Details for Example 11.3

At $t = 0$ the solutions become the column vectors

$$\vec{\mathbf{u}}_1 = \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \quad \vec{\mathbf{u}}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Then $W(0) = \det\left(\left\langle \vec{\mathbf{u}}_1(0)|\vec{\mathbf{u}}_2(0)|\vec{\mathbf{u}}_3(0)\right\rangle\right) = -1$ is nonzero. Vectors $\vec{\mathbf{u}}_1$, $\vec{\mathbf{u}}_2$, $\vec{\mathbf{u}}_3$ are **independent** and $\vec{\mathbf{x}}$ is the general solution.

Example 11.4 (Find A and $\vec{\mathbf{F}}$ from a General Solution)

Assume a 3×3 system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t)$ has general solution $\vec{\mathbf{x}} = c_1\vec{\mathbf{u}}_1 + c_2\vec{\mathbf{u}}_2 + c_3\vec{\mathbf{u}}_3 + \vec{\mathbf{x}}_p$ where

$$\vec{\mathbf{u}}_1 = \begin{pmatrix} 2e^{-t} \\ -e^{2t} + 2e^t \\ 4e^{-t} + 2e^{2t} \end{pmatrix}, \vec{\mathbf{u}}_2 = \begin{pmatrix} e^{-t} \\ e^{-t} - e^{2t} \\ 2e^{2t} + 2e^{-t} \end{pmatrix}, \vec{\mathbf{u}}_3 = \begin{pmatrix} e^t \\ e^t \\ 3e^t \end{pmatrix}, \vec{\mathbf{x}}_p = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}.$$

Find matrix A and vector function $\vec{\mathbf{F}}(t)$.

Solution to Example 11.4

Superposition implies $\vec{\mathbf{u}}_k'(t) = A\vec{\mathbf{u}}_k(t)$, $1 \le k \le 3$. Let $t = 0$ in these equations and then re-assemble the equations into a single matrix equation:

$$\left\langle \vec{\mathbf{u}}_1'(0)|\vec{\mathbf{u}}_2'(0)|\vec{\mathbf{u}}_3'(0)\right\rangle = A\left\langle \vec{\mathbf{u}}_1(0)|\vec{\mathbf{u}}_2(0)|\vec{\mathbf{u}}_3(0)\right\rangle$$

$$\begin{pmatrix} -2 & -1 & 1 \\ -4 & -3 & 1 \\ 0 & 2 & 3 \end{pmatrix} = A \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 6 & 4 & 3 \end{pmatrix}$$

Solve the matrix equation by inversion:

$$A = \begin{pmatrix} -9 & 4 & 2 \\ -11 & 6 & 2 \\ -18 & 6 & 5 \end{pmatrix}$$

Vector $\vec{\mathbf{F}}(t)$ can be found from $\vec{\mathbf{x}}_p'(t) = A\vec{\mathbf{x}}_p(t) + \vec{\mathbf{F}}(t)$ by solving for $\vec{\mathbf{F}}$:

$$\vec{\mathbf{F}}(t) = \begin{pmatrix} -2t^2 - 4t + 9 \\ -2t^2 - 6t + 12 \\ -5t^2 - 4t + 18 \end{pmatrix}$$

Example 11.5 (Solve $\vec{\mathbf{x}}\,'(t) = A\,\vec{\mathbf{x}}\,(t) + \vec{\mathbf{F}}\,(t)$ with Initial Conditions)

Assume:

$$\vec{\mathbf{x}}'(t) = \begin{pmatrix} -3 & 4 & 2 \\ -2 & 6 & 2 \\ -12 & 6 & 7 \end{pmatrix} + \begin{pmatrix} t \\ 0 \\ 2t \end{pmatrix}$$

$$x_1(0) = 1,\ x_2(0) = 0,\ x_3(0) = -1$$

$$\begin{aligned} x_1 &= c_1 e^t + c_2 e^{-t} + t \\ x_2 &= (c_1 + c_2)e^t + c_3 e^{2t} \\ x_3 &= (2c_2 - c_1)e^{-t} + (4c_1 - 2c_3)e^{2t} + 2t \end{aligned}$$

Solve for c_1, c_2, c_3.

Solution to Example 11.5

The equations for x_1, x_2, x_3 evaluated at $t = 0$ give the system of linear algebraic equations

$$\begin{aligned} 1 &= c_1 e^0 + c_2 e^0 + 0, \\ 0 &= (c_1 + c_2)e^0 + c_3 e^0, \\ -1 &= (2c_2 - c_1)e^0 + (4c_1 - 2c_3)e^0 + 0. \end{aligned}$$

In standard form it is the 3×3 linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & & & = & 1, \\ c_1 & + & c_2 & + & c_3 & = & 0, \\ 3c_1 & + & 2c_2 & - & 2c_3 & = & -1. \end{array}$$

The augmented matrix C:

$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & -1 \end{pmatrix}, \quad \mathbf{rref}(C) = \begin{pmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Then $c_1 = -5$, $c_2 = 6$, $c_3 = -1$.

The final answer:

$$\begin{aligned} x_1 &= -5e^t + 6e^{-t} + t, \\ x_2 &= e^t - e^{2t}, \\ x_3 &= 17e^{-t} - 18e^{2t} + 2t. \end{aligned}$$

Example 11.6 (Equilibria for $\vec{\mathbf{x}}\,'(t) = A(t)\,\vec{\mathbf{x}}\,(t)$)

Find all equilibria for system

$$\begin{array}{rcrcr} x_1' &=& x_1 &+& x_2 \\ x_2' &=& \sin(t)\,x_1 &+& \sin(t)\,x_2 \end{array}$$

Solution to Example 11.6

Let $A(t) = \begin{pmatrix} 1 & 1 \\ \sin(t) & \sin(t) \end{pmatrix}$. Let vector $\vec{\mathbf{x}}_0$ have components x_1, x_2. Then $A(t)\vec{\mathbf{x}}_0 = \vec{\mathbf{0}}$ has scalar form:

$$\left\{ \begin{array}{rcrcl} x_1 &+& x_2 &=& 0 \\ \sin(t)\,x_1 &+& \sin(t)\,x_2 &=& 0 \end{array} \right.$$

The equations must hold for all values of t. Because $\sin(t) \neq 0$ except for $t = n\pi$, an equivalent system for x_1, x_2 is

$$\left\{ \begin{array}{rcl} x_1 + x_2 &=& 0 \\ x_1 + x_2 &=& 0 \end{array} \right.$$

Solve the linear system. Then all constant solutions of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ are:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad -\infty < t_1 < \infty$$

It is an error to report $t = \pi$, $x_1 = 1$, $x_2 = -1$ as an equilibrium solution. Reports of equilibria are constants for x_1, x_2 which produce a solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ **for all values of** t.

Proofs for Theorems 11.3 to 11.14

Proof of Theorem 11.3: Gronwall's Lemma

Let $w(t) = c + \displaystyle\int_{t_0}^{t} u(r)v(r)dr$ and $F(t) = e^{\int_{t_0}^{t} v(r)dr}$. Then:

$w'(t) = u(t)v(t)$	Fundamental Theorem of calculus.
$w'(t) \le v(t)w(t)$	Hypothesis $u(t) \le w(t)$.
$\dfrac{(F(t)w(t))'}{F(t)} \le 0$	Integrating factor identity, for $t > t_0$.
$(F(t)w(t))' \le 0$	For t in J.
$F(t)w(t) \le F(t_0)w(t_0)$	Integrate across the inequality on J.
$F(t)w(t) \le c$	Because $F(t_0) = 1$ and $w(t_0) = c$.
$w(t) \le c\,e^{-\int_{t_0}^{t} v(r)dr}$	Divide by $F(t)$.
$u(t) \le c\,e^{-\int_{t_0}^{t} v(r)dr}$	Hypothesis $u(t) \le w(t)$.

Proof of Theorem 11.4: Unique Zero Solution

Zero is a solution because it satisfies both the differential equation and the initial condition. It remains to prove that zero is the unique *global* solution.

Assume $\vec{\mathbf{x}}(t)$ is another solution to the initial value problem. Let $\|B\|$ denote the Euclidean matrix norm. Then $|\vec{\mathbf{x}}(t)| \le \int_0^t \|A(r)\||\vec{\mathbf{x}}(r)|dr$ for $t \ge t_0$. Define $u(t) = |\vec{\mathbf{x}}(t)|$ and $v(t) = \|A(t)\|$. Then $u(t) \le c + \int_{t_0}^t u(r)v(r)dr$ for $c = 0$. Apply Gronwall's Lemma 11.3. Then $u(t) \le 0$, which implies $\vec{\mathbf{x}}(t) = 0$ for $t_0 \le t \le t_0 + H$.

Proof of Theorem 11.5: Picard-Lindelöf

Uniqueness is proved by subtracting two possible solutions: $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_1(t) - \vec{\mathbf{x}}_2(t)$. Then $\vec{\mathbf{x}}$ satisfies the hypotheses of Theorem 11.4, implying $\vec{\mathbf{x}}(t) = 0$ and then $\vec{\mathbf{x}}_1(t) = \vec{\mathbf{x}}_2(t)$ for all t in J.

Existence is proved by modification of the classical Picard-Lindelöf proof. The Picard iterates are constructed for the associated integral equation:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}(t_0) + \int_{t_0}^t A(r)\vec{\mathbf{F}}(r)\,dr$$

The essential step proves that the iterates converge uniformly to a solution $\vec{\mathbf{x}}(t)$ on the **entire interval** J. Details are in the exercises (Advanced Calculus required).

Proof of Theorem 11.6: Existence-Uniqueness for Constant Linear Systems

Picard-Lindelöf Theorem 11.5 applies to any interval $a < t < b$. Therefore, the unique solution is defined for all values of t.

Proof of Theorem 11.7: Uniqueness and Solution Crossings

The crossing theorem restates uniqueness in Picard-Lindelöf Theorem 11.5.

Proof of Theorem 11.8: Linear Structure

Let $\vec{\mathbf{x}}(t) = k_1\,\vec{\mathbf{x}}_1(t) + k_2\,\vec{\mathbf{x}}_2(t)$. Then:

$$\begin{aligned} A(t)\vec{\mathbf{x}}(t) &= k_1\,A(t)\vec{\mathbf{x}}_1(t) + k_2\,A(t)\vec{\mathbf{x}}_2(t) && \text{Matrix multiply} \\ &= k_1\,\vec{\mathbf{x}}_1'(t) + k_2\,\vec{\mathbf{x}}_2'(t) && \text{Because } \vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2 \text{ are solutions.} \\ &= \vec{\mathbf{x}}'(t) && \text{Differential equation verified.} \end{aligned}$$

Proof of Theorem 11.9: Basis

Let V be the vector space of all real-valued vector functions $\vec{\mathbf{x}}(t)$ defined on $a < t < b$.

Let S be the set of all solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, a subset of V.

Construct a **standard basis** $\{\vec{\mathbf{w}}_k\}_{k=1}^n$ for S by applying the Picard-Lindelöf theorem to initial value problem $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$, with $\vec{\mathbf{x}}_0$ successively set equal to the columns of the $n \times n$ identity matrix. This produces n solutions $\vec{\mathbf{w}}_1, \ldots, \vec{\mathbf{w}}_n$ to the equation $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$, all of which exist on the same interval $a < t < b$.

It will be shown that the span in V of $W = \{\vec{\mathbf{w}} + 1, \ldots, \vec{\mathbf{w}}_n\}$ equals S. Then S has a basis of n elements, which proves the theorem.

span$(W) \subset S$: Let linear combination

$$\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{w}}_1(t) + c_2\vec{\mathbf{w}}_2(t) + \cdots + c_n\vec{\mathbf{w}}_n(t) \tag{3}$$

belong to **span**(W). Theorem 11.8 implies that the linear combination $\vec{\mathbf{x}}(t)$ is a solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$. Then $\vec{\mathbf{x}}(t)$ is in S.

$S \subset$ **span**(W): if $\vec{\mathbf{x}}(t)$ is in S, then $\vec{\mathbf{x}}(t_0)$ has components $c_1, \ldots, c_n$. Function $\vec{\mathbf{y}}(t) = c_1\vec{\mathbf{w}}_1(t) + c_2\vec{\mathbf{w}}_2(t) + \cdots + c_n\vec{\mathbf{w}}_n(t)$ is in **span**(W), hence in S, and it has the same initial condition: $\vec{\mathbf{y}}(t_0)$ equals $\vec{\mathbf{x}}(t_0)$. The Picard theorem says $\vec{\mathbf{x}}(t) = \vec{\mathbf{y}}(t)$, therefore $\vec{\mathbf{x}}(t)$ is in $\mathbf{span}(W)$.

Proof of Theorem 11.10: Superposition

Assume $\vec{\mathbf{x}}_h'(t) = A(t)\vec{\mathbf{x}}_h(t)$ and $\vec{\mathbf{x}}_p'(t) = A(t)\vec{\mathbf{x}}p(t) + \vec{\mathbf{F}}(t)$.

Let $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_h(t) + \vec{\mathbf{x}}_p(t)$. Let's prove $\vec{\mathbf{x}}(t)$ is a solution of the nonhomogeneous equation.

$$
\begin{aligned}
\vec{\mathbf{x}}'(t) &= \vec{\mathbf{x}}_h'(t) + \vec{\mathbf{x}}_p'(t) && \text{Differential calculus.} \\
&= A(t)\vec{\mathbf{x}}_h(t) + A(t)\vec{\mathbf{x}}_p(t) + \vec{\mathbf{F}}(t) && \text{Use the two differential equations.} \\
&= A(t)\left(\vec{\mathbf{x}}_h(t) + \vec{\mathbf{x}}_p(t)\right) + \vec{\mathbf{F}}(t) && \text{Matrix algebra.} \\
&= A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t) && \text{Definition of } \vec{\mathbf{x}}(t).
\end{aligned}
$$

Let $\vec{\mathbf{x}}(t)$ denote any solution of $\vec{\mathbf{x}}'(t) = A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t)$. To prove: $\vec{\mathbf{y}}(t) = \vec{\mathbf{x}}(t) - \vec{\mathbf{x}}_p(t)$ is a solution of the homogeneous equation $\vec{\mathbf{y}}'(t) = A(t)\vec{\mathbf{y}}(t)$. Then for some assignment of constants $\vec{\mathbf{y}}(t)$ equals $\vec{\mathbf{x}}_h(t)$ and $\vec{\mathbf{x}} = \vec{\mathbf{y}} + \vec{\mathbf{x}}_p = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$.

$$
\begin{aligned}
\vec{\mathbf{y}}'(t) &= \vec{\mathbf{x}}'(t) - \vec{\mathbf{x}}_p'(t) && \text{Differential calculus.} \\
&= A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t) - \vec{\mathbf{x}}_p'(t) && \text{Differential equation for } \vec{\mathbf{x}}(t). \\
&= A(t)\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t) - A(t)\vec{\mathbf{x}}_p(t) - \vec{\mathbf{F}}(t) && \text{Differential equation for } \vec{\mathbf{x}}_p(t). \\
&= A(t)\left(\vec{\mathbf{x}}(t) - \vec{\mathbf{x}}_p(t)\right) && \text{Matrix algebra.} \\
&= A(t)\vec{\mathbf{y}}(t) && \text{Definition of } \vec{\mathbf{y}}(t).
\end{aligned}
$$

Proof of Theorem 11.11: Difference of Solutions

$$
\begin{aligned}
\vec{\mathbf{y}}'(t) &= \vec{\mathbf{u}}'(t) - \vec{\mathbf{v}}'(t) && \text{Differential calculus.} \\
&= A(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{F}}(t) - \vec{\mathbf{v}}'(t) && \text{Differential equation for } \vec{\mathbf{u}}(t). \\
&= A(t)\vec{\mathbf{u}}(t) + \vec{\mathbf{F}}(t) - A(t)\vec{\mathbf{v}}(t) - \vec{\mathbf{F}}(t) && \text{Differential equation for } \vec{\mathbf{v}}(t). \\
&= A(t)\left(\vec{\mathbf{u}}(t) - \vec{\mathbf{v}}(t)\right) && \text{Matrix algebra.} \\
&= A(t)\vec{\mathbf{y}}(t) && \text{Definition of } \vec{\mathbf{y}}(t).
\end{aligned}
$$

Proof of Theorem 11.12: General Solution

Claim **1**. Term $\vec{\mathbf{y}} = \vec{\mathbf{x}}_h(t)$ is a general solution of the homogeneous equation $\vec{\mathbf{y}}' = A(t)\vec{\mathbf{y}}$ which contains n arbitrary constants $c_1, \ldots, c_n$.

Each solution $\vec{\mathbf{y}} = \vec{\mathbf{x}}_h(t)$ of $\vec{\mathbf{y}}' = A(t)\vec{\mathbf{y}}$ can be expanded uniquely as a linear combination of basis elements $\vec{\mathbf{w}}_1(t), \vec{\mathbf{w}}_2(t), \ldots, \vec{\mathbf{w}}_n(t)$ because of the Picard-Lindelöf Theorem 11.5 and Theorem 11.9. Then $\vec{\mathbf{y}}(t) = c_1\vec{\mathbf{w}}_1(t) + c_2\vec{\mathbf{w}}_2(t) + \cdots + c_n\vec{\mathbf{w}}_n(t)$ for weights $c_1, \ldots, c_n$ is a general solution of $\vec{\mathbf{y}}' = A(t)\vec{\mathbf{y}}$. The weights $c_1, \ldots, c_n$ are the n arbitrary constants required in the general solution.

Claim **2**. Term $\vec{\mathbf{x}} = \vec{\mathbf{x}}_p(t)$ is a particular solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

Let $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_h(t) + \vec{\mathbf{x}}_p(t)$ be a general solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$. Then $\vec{\mathbf{x}}_h' = A(t)\vec{\mathbf{x}}_h$ implies $\vec{\mathbf{x}}_p' = \vec{\mathbf{x}}' - \vec{\mathbf{x}}_h' = A(t)(\vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p) + \vec{\mathbf{F}}(t) - A(t)\vec{\mathbf{x}}_h = A(t)\vec{\mathbf{x}}_p + \vec{\mathbf{F}}(t)$. Then $\vec{\mathbf{x}}_p$ is a particular solution.

Proof of Theorem 11.13: Abel's Formula

Let determinant $D_j(t)$ equal $W(t)$ with row j replaced by its derivative, $1 \le j \le n$. The derivative of determinant $W(t)$ is the sum of these determinants:

$$W'(t) = D_1(t) + \cdots + D_n(t)$$

Determinant $D_1(t) = a_{11}W(t)$, discovered as follows. Element d_{1j} of $D_1(t)$ is expressed as a summation $\sum_{i=1}^n a_{1i}(t)u_{ij}(t)$. The details: $\vec{\mathbf{u}}_j'(t) = A(t)\vec{\mathbf{u}}_j(t)$ and $\vec{\mathbf{u}}_j(t)$ has components $u_{1j}, \ldots, u_{nj}$. Determinant $D_1(t)$ has value unchanged by adding to row 1 a linear combination of rows 2 to n. The selected combination adds $-\sum_{i=2}^n a_{1i}(t)u_{ij}(t)$ to d_{1j}, effectively replacing d_{1j} by $a_{11}u_{1j}$. Then $a_{11}(t)$ is a common factor in row 1 of the modified determinant $D_1(t)$. Factor out $a_{11}(t)$ from row 1, leaving determinant $W(t)$. Then $D_1(t) = a_{11}(t)W(t)$.

Proceeding similarly: $D_j(t) = a_{jj}(t)W(t)$ for $2 \le j \le n$. Then:

$$\begin{aligned} W'(t) &= D_1(t) + \cdots + D_n(t) \\ &= (a_{11}(t) + \cdots + a_{nn}(t))\, W(t) \\ &= \mathbf{trace}(A(t))\, W(t) \end{aligned}$$

The claimed expression for $W(t)$ is the solution of the first order linear differential equation $W' = \mathbf{trace}(A(t))W$, by the linear integrating factor method.

If $W(t_0) = 0$, then the formula implies $W(t) = 0$ for all t. Conversely, if $W(t_0) \ne 0$ for some t_0, then the formula implies $W(t)$ is never zero, because exponentials are never zero.

Proof of Theorem 11.14: Abel's Wronskian Test Linear combination $\sum_{i=1}^n c_i\vec{\mathbf{u}}_i(t)$ is the zero function if and only if the matrix equation $U(t)\vec{\mathbf{c}} = \vec{\mathbf{0}}$ has only the zero solution $\vec{\mathbf{c}} = \vec{\mathbf{0}}$, where $U(t)$ is the augmented matrix of $\vec{\mathbf{u}}_1(t), \ldots, \vec{\mathbf{u}}_n((t)$ and vector $\vec{\mathbf{c}}$ has components $c_1, \ldots, c_n$. The matrix equation has only the zero solution $\vec{\mathbf{c}} = \vec{\mathbf{0}}$ if and only if $\det(U(t)) \ne 0$. The Abel-Liouville formula completes the proof, because $\det(U(t)) = W(t)$, the Wronskian of the n solutions.

Exercises 11.3

Linear Systems

Convert to matrix notation $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$.

1. $x_1' = 2x_1 + x_2 + e^t$,
$x_2' + x_1 - 2x_2 = \sinh(t)$

2. $x_1' = x_1 + x_2 + x_3$,
$x_2' + x_1 - 2x_2 + x_3 = \ln|1 + t^2|$,
$x_3' = x_2 + x_3 + \cosh(t)$

Existence-Uniqueness

3. Apply Gronwall's inequality to $|y(t)| \le 4 + \int_0^t (1 + r^2)|y(r|\,dr$, $t \ge 0$.

4. Solve with $x_1(0) = x_2(0) = 0$:
$x_1' = e^t x + e^{-t}x_2$,
$x_2' = \ln|1 + \sinh^2(t)|\, x_1 + x_2$

5. Find the interval on which the solution is defined:
$x_1' = tx_1 + x_2$, $x_2' = x_1 + \tan(t)\, x_2$

6. Let matrix A be 2×2 constant. Find A, given $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has general solution $x_1 = c_1e^t + c_2e^{2t}$, $x_2 = 5c_12e^t + 4c_2e^{2t}$.

7. Let $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ have two solutions : $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} e^t \\ e^t \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$.

8. Let $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

9. Let constant matrix A be 10×10. Two solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ have equal value at $t = 100$. Are they the same solution?

10. Solutions y_1, y_2 of $y' + p(x)y = q(x)$ are zero at $x = -2$. What assumptions on p, q imply $y_1 \equiv y_2$?

Superposition

11. Explain: e^t is a solution of $y'' - y = 0$ because $\cosh(t)$, $\sinh(t)$ are a solution basis.

12. Explain: $e^t + 10$ is a solution of $y'' - y = -10$, therefore 10 is a particular solution.

13. The shortest solution of $y' + y = 100$ is $y = 100$. Explain why.

14. Let $x_1' = 2x_1$, $x_2' = -x_2$. Report the matrix form $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ and the vector general solution.

15. Let 2-dimensional $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have general solution $x_1 = c_1 e^t + c_2 e^{3t}$, $x_2 = (c_1 + c_2)e^t + 2c_2 e^{3t} + \cos(t)$. Find formulas for vectors $\vec{\mathbf{x}}_h$ and $\vec{\mathbf{x}}_p$.

16. Let $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have two solutions $x_1 = e^t + e^{3t}$, $x_2 = 2e^t + \sin(t)$ and $x_1 = e^{3t}$, $x_2 = e^{3t} + \sin(t)$. Find a solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

Superposition $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$

17. Let $\vec{\mathbf{u}}_1(t), \ldots, \vec{\mathbf{u}}_k(t)$ be solutions of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$. Let $c_1, \ldots, c_k$ be constants. Prove: $\vec{\mathbf{u}}(t) = \sum_{i=1}^k c_i \vec{\mathbf{u}}_i(t)$ is a solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$.

18. Find the **standard basis** $\vec{\mathbf{w}}_1(t), \vec{\mathbf{w}}_2(t)$:
$$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}}$$

19. Let matrix A be 2×2. For $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$, find $\vec{\mathbf{x}}_h(t)$, $\vec{\mathbf{x}}_p(t)$:
$x_1 = c_1 + c_2 t + e^t$, $x_2 = (c_1 - c_2)t + e^{2t}$

20. Let matrix $A(t)$ be 2×2. Let $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ have two solutions $\begin{pmatrix} 1 + e^t \\ 1 \end{pmatrix}$, $\begin{pmatrix} 1 + e^{-t} \\ -1 \end{pmatrix}$. Find a solution of $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$.

General Solution

21. Assume A is 2×2 and $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has solutions $e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Find the general solution and explain.

22. Assume $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Prove that zero is not a solution.

23. Assume $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_0 =$ constant. Find an equation for $\vec{\mathbf{x}}_0$.

24. Find the vector general solution:
$$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

25. Given 3 $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ with scalar general solution $x_1 = c_1 + c_2 t + c_3 t^2$, $x_2 = c_2 + c_3 t$, $x_3 = c_3$, find the vector general solution.

26. Given 3 $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}}$ with scalar general solution $x_1 = c_1 + c_2 t + c_3 t^2$, $x_2 = c_2 + c_3 t$, $x_3 = c_3$, find $A(t)$.

27. Find the vector general solution:
$$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

28. Find the vector general solution:
$$\vec{\mathbf{x}}' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Independence

29. Assume A is 2×2 and $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has solutions $e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Prove they are independent directly from the definition.

30. Compute the Wronskian:
$$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Abel-Liouville Formula

31. Apply Abel's Independence Test:
$$e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

32. Let $\Phi(t)$ an invertible matrix satisfying $\Phi'(t) = A\Phi(t)$. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

33. Let $\Phi(t)$ an invertible matrix satisfying $\Phi'(t) = A\Phi(t)$. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

34. Let $\Phi(t)$ any matrix satisfying $\Phi'(t) = A\Phi(t)$. Assume the determinant of $\Phi(t_0)$ is nonzero. Prove that the columns of $\Phi(t)$ are independent solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

35. Let $\Phi(t)$ any matrix satisfying $\Phi'(t) = A\Phi(t)$. Let C be a constant matrix. Prove that the columns of $\Phi(t)C$ are solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

36. Assume continuous coefficients:
$y^{(n)}+p_{n-1}y^{(n-1)}+\cdots+p_0y=0$
Prove from the Abel-Liouville formula for the companion system
that the Wronskian $W(t)$ of
solutions $y_1, \ldots, y_n$ satisfies
$W' + p_{n-1}(t)W = 0$.

Initial Value Problem

37. Let matrix A be 3×3. Assume $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ has scalar general solution $x_1 = c_1e^t + c_2e^{-t} + t$, $x_2 = (c_1 + c_2)e^t + c_3e^{2t}$, $x_3 = (c_1 + c_2)e^t - 2c_2e^{-t} + c_3e^{2t} + t$. Given initial conditions $x_1(0) = x_2(0) = 0$, $x_3(0) = 1$, solve for c_1, c_2, c_3.

38. Let matrix A be 3×3. Assume $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ has scalar general solution $x_1 = c_1 + c_2t + c_3t^2 + e^t$, $x_2 = c_2 + c_3t + e^{2t}$, $x_3 = c_3$. Find the **vector** particular solution $\vec{\mathbf{x}}$ for initial conditions $x_1(0) = x_2(0) = 0$, $x_3(0) = 1$.

Equilibria

39. Find all equilibria:
$$\vec{\mathbf{x}}' = \begin{pmatrix} \cos(t) & \cos(t) \\ 2 & 2 \end{pmatrix} \vec{\mathbf{x}}$$

40. Find all equilibria:
$$\vec{\mathbf{x}}' = \begin{pmatrix} \sin(t) & \sin^2(t) \\ 2 & 2 \end{pmatrix} \vec{\mathbf{x}}$$

11.4 Matrix Exponential

The problem

$$\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t), \quad \vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$$

has a unique solution, according to the Picard-Lindelöf theorem. Solve the problem n times, when $\vec{\mathbf{x}}_0$ equals a column of the identity matrix, and write $\vec{\mathbf{w}}_1(t)$, $\ldots$, $\vec{\mathbf{w}}_n(t)$ for the n solutions so obtained. The solutions form the **standard basis**. Define the **matrix exponential** e^{At} by packaging these n solutions into the columns of a matrix:

$$e^{At} \equiv \left\langle \vec{\mathbf{w}}_1(t) | \ldots | \vec{\mathbf{w}}_n(t) \right\rangle.$$

By construction, any possible solution of $\frac{d}{dt}\vec{\mathbf{x}} = A\vec{\mathbf{x}}$ can be uniquely expressed in terms of the matrix exponential e^{At} by the formula

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0).$$

Matrix Exponential Identities

Announced here are formulas and identities for e^{At}, the matrix exponential. Most details are delayed to page 869.

$e^{At} = e^{\lambda_1 t} I + \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 \neq \lambda_2$ real, Theorem page 866.
$e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 = \lambda_2$ real.
$e^{At} = e^{at}\cos bt\, I + \dfrac{e^{at}\sin bt}{b}(A - aI)$	A is 2×2, $\lambda_1 = \overline{\lambda_2} = a + ib$, $b > 0$.
$e^{At} = r_1(t)P_1 + \cdots + r_n(t)P_n$	Putzer's $n \times n$ spectral formula, Theorem page 868.
$\dfrac{d}{dt}\left(e^{At}\right) = Ae^{At}$	Columns of e^{At} satisfy $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Page 869.
$e^{\mathbf{0}} = I$	Where $\mathbf{0}$ is the zero matrix.
$Be^{At} = e^{At}B$	If $AB = BA$.
$e^{At}e^{Bt} = e^{(A+B)t}$	If $AB = BA$.
$e^{At}e^{As} = e^{A(t+s)}$	Since At and As commute.
$\left(e^{At}\right)^{-1} = e^{-At}$	Equivalently, $e^{At}e^{-At} = I$.
$e^{At} = P^{-1}e^{Jt}P$	Jordan form $J = PAP^{-1}$
$e^{At} = \sum_{n=0}^{\infty} A^n \dfrac{t^n}{n!}$	Picard series identity, proof on page 870

Putzer's Spectral Formula

The spectral formula of Putzer applies to a system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ to find its general solution. The method uses matrices $P_1, \ldots, P_n$ constructed from A, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, matrix multiplication and the solution $\vec{\mathbf{r}}(t)$ of the first order $n \times n$ initial value problem

$$\vec{\mathbf{r}}'(t) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & \lambda_n \end{pmatrix} \vec{\mathbf{r}}(t), \quad \vec{\mathbf{r}}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The system is solved by first order scalar methods and back-substitution. The formula will be derived separately for the 2×2 case (the one used most often) and the $n \times n$ case.

Theorem 11.15 (Putzer's 2×2 Spectral Formula)
Let A be a 2×2 matrix. Let $r = \lambda_1, \lambda_2$ be the two real or complex roots of the characteristic equation $\det(A - rI) = 0$. Let $P_1 = I$, $P_2 = A - \lambda_1 I$. Let functions $r_1(t)$, $r_2(t)$ be defined by the scalar system

$$\begin{cases} r_1' &= \lambda_1 r_1, & r_1(0) = 1, \\ r_2' &= \lambda_2 r_2 + r_1, & r_2(0) = 0. \end{cases}$$

Then the 2×2 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$ has solution

$$\vec{\mathbf{x}}(t) = (r_1(t)P_1 + r_2(t)P_2)\,\vec{\mathbf{x}}_0$$

Proof: The Cayley-Hamilton formula $(A - \lambda_1 I)(A - \lambda_2 I) = \vec{\mathbf{0}}$ is valid for any 2×2 matrix A, if $r = \lambda_1, \lambda_2$ are the two roots of the determinant equation $\det(A - rI) = 0$. See page **??**. The Cayley-Hamilton formula is the same as $(A - \lambda_2 I)P_2 = \vec{\mathbf{0}}$, which implies the identity $AP_2 = \lambda_2 P_2$. Compute as follows.

$$\begin{aligned} \vec{\mathbf{x}}'(t) &= (r_1'(t)P_1 + r_2'(t)P_2)\,\vec{\mathbf{x}}_0 \\ &= (\lambda_1 r_1(t)P_1 + r_1(t)P_2 + \lambda_2 r_2(t)P_2)\,\vec{\mathbf{x}}_0 \\ &= (r_1(t)A + \lambda_2 r_2(t)P_2)\,\vec{\mathbf{x}}_0 \\ &= (r_1(t)A + r_2(t)AP_2)\,\vec{\mathbf{x}}_0 \\ &= A\,(r_1(t)I + r_2(t)P_2)\,\vec{\mathbf{x}}_0 \\ &= A\vec{\mathbf{x}}(t). \end{aligned}$$

This proves that $\vec{\mathbf{x}}(t)$ is a solution. Because $\Phi(t) \equiv r_1(t)P_1 + r_2(t)P_2$ satisfies $\Phi(0) = I$, then $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$ is satisfied. ■

Real Distinct Eigenvalues

Suppose A is 2×2 having real distinct eigenvalues λ_1, λ_2 and $\vec{\mathbf{x}}(0)$ is real. Then

$$r_1 = e^{\lambda_1 t}, \quad r_2 = \frac{e^{\lambda_1 t} - e^{\lambda_2 T}}{\lambda_1 - \lambda_2}$$

and

$$\vec{\mathbf{x}}(t) = \left(e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)\right)\vec{\mathbf{x}}(0).$$

The matrix exponential formula for real distinct eigenvalues:

$$e^{At} = e^{\lambda_1 t}I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

Real Equal Eigenvalues

Suppose A is 2×2 having real equal eigenvalues $\lambda_1 = \lambda_2$ and $\vec{\mathbf{x}}(0)$ is real. Then $r_1 = e^{\lambda_1 t}$, $r_2 = te^{\lambda_1 t}$ and

$$\vec{\mathbf{x}}(t) = \left(e^{\lambda_1 t}I + te^{\lambda_1 t}(A - \lambda_1 I)\right)\vec{\mathbf{x}}(0).$$

The matrix exponential formula for real equal eigenvalues:

$$e^{At} = e^{\lambda_1 t}I + te^{\lambda_1 t}(A - \lambda_1 I).$$

Complex Eigenvalues

Suppose A is 2×2 having complex eigenvalues $\lambda_1 = a + bi$ with $b > 0$ and $\lambda_2 = a - bi$. If $\vec{\mathbf{x}}(0)$ is real, then a real solution is obtained by taking the real part of the spectral formula. This formula is formally identical to the case of real distinct eigenvalues. Then

$$\begin{aligned}
\mathcal{R}\mathrm{e}(\vec{\mathbf{x}}(t)) &= (\mathcal{R}\mathrm{e}(r_1(t))I + \mathcal{R}\mathrm{e}(r_2(t)(A - \lambda_1 I)))\,\vec{\mathbf{x}}(0) \\
&= \left(\mathcal{R}\mathrm{e}(e^{(a+ib)t})I + \mathcal{R}\mathrm{e}(e^{at}\frac{\sin bt}{b}(A - (a+ib)I))\right)\vec{\mathbf{x}}(0) \\
&= \left(e^{at}\cos bt\, I + e^{at}\frac{\sin bt}{b}(A - aI)\right)\vec{\mathbf{x}}(0)
\end{aligned}$$

The matrix exponential formula for complex conjugate eigenvalues:

$$e^{At} = e^{at}\left(\cos bt\, I + \frac{\sin bt}{b}(A - aI)\right).$$

How to Remember Putzer's 2×2 Formula

The expressions

$$
\begin{aligned}
e^{At} &= r_1(t)I + r_2(t)(A-\lambda_1 I),\\
r_1(t) &= e^{\lambda_1 t},\quad r_2(t) = \frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}
\end{aligned}
\tag{1}
$$

are enough to generate all three formulas. Fraction r_2 is the $d/d\lambda$-Newton quotient for r_1. It has limit $te^{\lambda_1 t}$ as $\lambda_2 \to \lambda_1$, therefore the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$, then the fraction r_2 is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

$$r_2(t) = \frac{z-\overline{z}}{w-\overline{w}} = \frac{\sin bt}{b}.$$

Taking real parts of expression (1) gives the complex case formula.

Theorem 11.16 (Putzer's $n \times n$ Spectral Formula)
Let A be an $n\times n$ matrix. Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of A, the real or complex roots r of $\det(A - rI) = 0$. Let

$$P_1 = I,\quad P_k = P_{k-1}(A-\lambda_{k-1}I) = \Pi_{j=1}^{k-1}(A-\lambda_j I),\quad k = 2,\ldots,n.$$

Let functions $r_1(t)$, ..., $r_n(t)$ be defined by the differential system

$$
\begin{array}{rcll}
r_1' &=& \lambda_1 r_1, & r_1(0) = 1,\\
r_2' &=& \lambda_2 r_2 + r_1, & r_2(0) = 0,\\
&\vdots&&\\
r_n' &=& \lambda_n r_n + r_{n-1}, & r_n(0) = 0.
\end{array}
$$

Then system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$ has solution

$$\vec{\mathbf{x}}(t) = \left(r_1(t)P_1 + r_2(t)P_2 + \cdots + r_n(t)P_n\right)\vec{\mathbf{x}}_0,$$

Proof on page 871

Theorem 11.17 (Compute e^{Jt} for J Triangular)
If J is an upper triangular matrix, then a column $\vec{\mathbf{u}}(t)$ of e^{Jt} can be computed by solving the system $\vec{\mathbf{u}}'(t) = J\vec{\mathbf{u}}(t)$, $\vec{\mathbf{u}}(0) = \vec{\mathbf{v}}$, where $\vec{\mathbf{v}}$ is the corresponding column of the identity matrix. This problem can always be solved by first-order scalar methods of growth-decay theory and the integrating factor method. Proof on page 872.

Theorem 11.18 (Exponential of a Diagonal Matrix)
For real or complex constants λ_1, ..., λ_n,

$$e^{\mathbf{diag}(\lambda_1,\ldots,\lambda_n)t} = \mathbf{diag}\left(e^{\lambda_1 t},\ldots,e^{\lambda_n t}\right).$$

Proof on page 872.

Theorem 11.19 (Block Diagonal Matrix)
If $A = \mathbf{diag}(B_1, \ldots, B_k)$ and each of $B_1, \ldots, B_k$ is a square matrix, then

$$e^{At} = \mathbf{diag}\left(e^{B_1 t}, \ldots, e^{B_k t}\right).$$

Proof on page 872.

Theorem 11.20 (Complex Exponential)

Given real a, b, then $e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} t} = e^{at} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}$.

Proof on page 872

Proofs of Matrix Exponential Identities

The 2×2 Putzer identities have proofs in the text page 866. Proofs of theorems are on page 871. The remaining proofs are here.

Verify $\left(e^{At}\right)' = Ae^{At}$.

Let $\vec{\mathbf{x}}_0$ denote a column of the identity matrix. Define $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0$. Then

$$\begin{aligned} \left(e^{At}\right)' \vec{\mathbf{x}}_0 &= \vec{\mathbf{x}}'(t) \\ &= A\vec{\mathbf{x}}(t) \\ &= Ae^{At}\vec{\mathbf{x}}_0. \end{aligned}$$

Because this identity holds for all columns of the identity matrix, then $(e^{At})'$ and Ae^{At} have identical columns. Identity $\left(e^{At}\right)' = Ae^{At}$ is proved. ■

Verify $e^{\mathbf{0}} = I$.

$e^{\mathbf{0}} = \left\langle \vec{\mathbf{w}}_1(0) | \ldots | \vec{\mathbf{w}}_n(0) \right\rangle = I$. ■

Verify $Be^{At} = e^{At}B$ if $AB = BA$.

Define $\vec{\mathbf{w}}_1(t) = e^{At}B\vec{\mathbf{w}}_0$ and $\vec{\mathbf{w}}_2(t) = Be^{At}\vec{\mathbf{w}}_0$. Calculate $\vec{\mathbf{w}}_1'(t) = A\vec{\mathbf{w}}_1(t)$ and $\vec{\mathbf{w}}_2'(t) = BAe^{At}\vec{\mathbf{w}}_0 = ABe^{At}\vec{\mathbf{w}}_0 = A\vec{\mathbf{w}}_2(t)$, due to $BA = AB$. Because $\vec{\mathbf{w}}_1(0) = \vec{\mathbf{w}}_2(0) = \vec{\mathbf{w}}_0$, then the uniqueness assertion of the Picard-Lindelöf theorem implies that $\vec{\mathbf{w}}_1(t) = \vec{\mathbf{w}}_2(t)$. Because $\vec{\mathbf{w}}_0$ is any vector, then $e^{At}B = Be^{At}$. ■

Verify $e^{At}e^{Bt} = e^{(A+B)t}$.

Let $\vec{\mathbf{x}}_0$ be a column of the identity matrix. Define $\vec{\mathbf{x}}(t) = e^{At}e^{Bt}\vec{\mathbf{x}}_0$ and $\vec{\mathbf{y}}(t) = e^{(A+B)t}\vec{\mathbf{x}}_0$. We must show that $\vec{\mathbf{x}}(t) = \vec{\mathbf{y}}(t)$ for all t. Define $\vec{\mathbf{u}}(t) = e^{Bt}\vec{\mathbf{x}}_0$. We will apply the result $e^{At}B = Be^{At}$, valid for $BA = AB$. The details:

$$\begin{aligned} \vec{\mathbf{x}}'(t) &= \left(e^{At}\vec{\mathbf{u}}(t)\right)' \\ &= Ae^{At}\vec{\mathbf{u}}(t) + e^{At}\vec{\mathbf{u}}'(t) \\ &= A\vec{\mathbf{x}}(t) + e^{At}B\vec{\mathbf{u}}(t) \\ &= A\vec{\mathbf{x}}(t) + Be^{At}\vec{\mathbf{u}}(t) \\ &= (A+B)\vec{\mathbf{x}}(t). \end{aligned}$$

Also known is that $\vec{\mathbf{y}}'(t) = (A+B)\vec{\mathbf{y}}(t)$ and since $\vec{\mathbf{x}}(0) = \vec{\mathbf{y}}(0) = \vec{\mathbf{x}}_0$, then the Picard-Lindelöf theorem implies that $\vec{\mathbf{x}}(t) = \vec{\mathbf{y}}(t)$ for all t. ■

Verify $e^{At}e^{As} = e^{A(t+s)}$.

Let t be a variable and consider s fixed. Define $\vec{\mathbf{x}}(t) = e^{At}e^{As}\vec{\mathbf{x}}_0$ and $\vec{\mathbf{y}}(t) = e^{A(t+s)}\vec{\mathbf{x}}_0$. Then $\vec{\mathbf{x}}(0) = \vec{\mathbf{y}}(0)$ and both satisfy the differential equation $\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t)$. By the uniqueness in the Picard-Lindelöf theorem, $\vec{\mathbf{x}}(t) = \vec{\mathbf{y}}(t)$, which implies $e^{At}e^{As} = e^{A(t+s)}$. ■

Verify $\left(e^{At}\right)^{-1} = e^{-At}$.

Let $s = -t$ in the preceding identity $e^{At}e^{As} = e^{A(t+s)}$. The right side is $e^{\mathbf{0}} = I$. The inverse test Chapter 5 Section 2, Theorem 5.9, implies that the two matrices e^{At} and e^{-At} are inverses of one another. ■

Verify $e^{At} = P^{-1}e^{Jt}P$ if $J = PAP^{-1}$.

The proof uses the Picard series identity $e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$, which is proved below. The issue is the simplification of A^n using $A = P^{-1}JP$. Induction is used to derive the following identities, in which $Q = P^{-1}$ (then $QP = PQ = I$):

$$\begin{array}{lcllcl} A & = & P^{-1}JP & & = & QJP \\ A^2 & = & QJPQJP & & = & QJ^2P \\ & \vdots & & & & \\ A^n & = & (QJP)\cdots(QJP) & & = & QJ^nP \end{array}$$

Then the infinite series simplifies:

$$\begin{aligned} e^{At} &= \sum_{n=0}^{\infty} A^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} QJ^nP\frac{t^n}{n!} \\ &= Q\left(\sum_{n=0}^{\infty} J^n \frac{t^n}{n!}\right)P \\ &= Qe^{Jt}P \\ &= P^{-1}e^{Jt}P \end{aligned}$$

■

Verify $e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$.

The idea of the proof is to apply Picard iteration. By definition, the columns of e^{At} are vector solutions $\vec{\mathbf{w}}_1(t), \ldots, \vec{\mathbf{w}}_n(t)$ whose values at $t=0$ are the corresponding columns of the $n \times n$ identity matrix. According to the theory of Picard iterates, a particular iterate is defined by

$$\vec{\mathbf{y}}_{n+1}(t) = \vec{\mathbf{y}}_0 + \int_0^t A\vec{\mathbf{y}}_n(r)dr, \quad n \geq 0.$$

Vector $\vec{\mathbf{y}}_0$ equals some column k of the identity matrix. The Picard iterates can be found explicitly, as follows.

$$
\begin{aligned}
\vec{\mathbf{y}}_1(t) &= \vec{\mathbf{y}}_0 + \int_0^t A\vec{\mathbf{y}}_0 dr \\
&= (I + At)\,\vec{\mathbf{y}}_0, \\
\vec{\mathbf{y}}_2(t) &= \vec{\mathbf{y}}_0 + \int_0^t A\vec{\mathbf{y}}_1(r)dr \\
&= \vec{\mathbf{y}}_0 + \int_0^t A\,(I + At)\,\vec{\mathbf{y}}_0 dr \\
&= \left(I + At + A^2t^2/2\right)\vec{\mathbf{y}}_0, \\
&\ \ \vdots \\
\vec{\mathbf{y}}_n(t) &= \left(I + At + A^2\tfrac{t^2}{2} + \cdots + A^n\tfrac{t^n}{n!}\right)\vec{\mathbf{y}}_0.
\end{aligned}
$$

The Picard-Lindelöf theorem implies

$$\lim_{n\to\infty} \vec{\mathbf{y}}_n(t) = \vec{\mathbf{w}}_k(t).$$

This being valid for each index k, then the columns of the matrix converge as $N \to \infty$ to $\vec{\mathbf{w}}_1(t), \ldots, \vec{\mathbf{w}}_n(t)$. The matrix limit is formally the infinite series

$$\sum_{m=0}^{\infty} A^m \frac{t^m}{m!} = \lim_{N\to\infty} \sum_{m=0}^{N} A^m \frac{t^m}{m!} = \left\langle \vec{\mathbf{w}}_1(t)|\ldots|\vec{\mathbf{w}}_n(t)\right\rangle$$

but also $e^{At} \equiv \left\langle \vec{\mathbf{w}}_1(t)|\ldots|\vec{\mathbf{w}}_n(t)\right\rangle$. This proves the matrix identity

$$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}. \quad \blacksquare$$

Proofs of Theorems 11.16–11.20

Theorem 11.16, Proof of Putzer's $n \times n$ Formula:

The Cayley-Hamilton formula $(A - \lambda_1 I)\cdots(A - \lambda_n I) = \vec{\mathbf{0}}$ is valid for any $n \times n$ matrix A and the n roots $r = \lambda_1, \ldots, \lambda_n$ of the determinant equality $\det(A - rI) = 0$. Two facts will be used: (1) The Cayley-Hamilton formula implies $AP_n = \lambda_n P_n$; (2) The definition of P_k implies $\lambda_k P_k + P_{k+1} = AP_k$ for $1 \le k \le n-1$. Compute as follows.

$$
\begin{aligned}
\boxed{1}\quad \vec{\mathbf{x}}'(t) &= (r_1'(t)P_1 + \cdots + r_n'(t)P_n)\,\vec{\mathbf{x}}(0) \\
\boxed{2}\quad &= \left(\sum_{k=1}^{n} \lambda_k r_k(t)P_k + \sum_{k=2}^{n} r_{k-1}P_k\right)\vec{\mathbf{x}}_0 \\
\boxed{3}\quad &= \left(\sum_{k=1}^{n-1} \lambda_k r_k(t)P_k + r_n(t)\lambda_n P_n + \sum_{k=1}^{n-1} r_k P_{k+1}\right)\vec{\mathbf{x}}_0 \\
\boxed{4}\quad &= \left(\sum_{k=1}^{n-1} r_k(t)(\lambda_k P_k + P_{k+1}) + r_n(t)\lambda_n P_n\right)\vec{\mathbf{x}}_0 \\
\boxed{5}\quad &= \left(\sum_{k=1}^{n-1} r_k(t)AP_k + r_n(t)AP_n\right)\vec{\mathbf{x}}_0
\end{aligned}
$$

$$\boxed{6} \qquad = A\left(\sum_{k=1}^{n} r_k(t)P_k\right)\vec{\mathbf{x}}_0$$

$$\boxed{7} \qquad = A\vec{\mathbf{x}}(t).$$

Details: $\boxed{1}$ Differentiate the formula for $\vec{\mathbf{x}}(t)$. $\boxed{2}$ Use the differential equations for $r_1,\ldots,r_n$. $\boxed{3}$ Split off the last term from the first sum, then re-index the last sum. $\boxed{4}$ Combine the two sums. $\boxed{5}$ Use the recursion for P_k and the Cayley-Hamilton formula $(A - \lambda_n I)P_n = \vec{\mathbf{0}}$. $\boxed{6}$ Factor out A on the left. $\boxed{7}$ Apply the definition of $\vec{\mathbf{x}}(t)$.

Then $\vec{\mathbf{x}}(t)$ is a solution. Because $\Phi(t) \equiv \sum_{k=1}^{n} r_k(t)P_k$ satisfies $\Phi(0) = I$, then $\vec{\mathbf{x}}(t)$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$. ■

Proof of Theorem 11.17, Compute e^{Jt} for J Triangular:

The first statement computes the solution of the problem $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) =$ column j of I, $1 \le j \le n$. These are the columns of e^{At}, by definition.

Each such problem is known to be solvable by linear first order integrating factor methods, using the variable list in reverse order.

An example for such a scalar system:

$$\begin{aligned}
&x_1' = 2x_1 + x_3,\\
&x_2' = 3x_2 + x_3,\\
&x_3' = 4x_3,\\
&x_1(0) = 1, x_2(0) = x_3(0) = 0.
\end{aligned}$$

The variable list reversed is x_3, x_2, x_1. The solution starts with $x_3' = 4x_3$, $x_3(0) = 0$. The solution is $x_3 = 0$. Then the equation for x_2 becomes $x_2' = 3x_2 + 0, x_2(0) = 0$. Again the solution is $x_2 = 0$. The last equation is $x_1' = 2x_1 + 0$, $x_1(0) = 1$ with solution $x_1 = e^{2t}$. ■

Proof of Theorem 11.18, Exponential of a Diagonal Matrix:

It suffices to prove that $\Phi(t) = \textbf{diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right)$ satisfies $\Phi'(t) = A\Phi(t)$, $\Phi(0) = I$. Because $e^{0t} = 1$, then $\Phi(0) = I$. The differential equation is satisfied by the following steps:

$$\begin{aligned}
\Phi'(t) &= \begin{pmatrix} \lambda_1 e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n e^{\lambda_n t} \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} e^{\lambda_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{\lambda_n t} \end{pmatrix} \\
&= A\Phi(t)
\end{aligned}$$

■

Proof of Theorem 11.19, Block Diagonal Matrix Exponential:

Let $\Phi(t) = \textbf{diag}\left(e^{B_1 t}, \ldots, e^{B_k t}\right)$. To prove $\Phi(t)$ equals e^{At}, it suffices to prove identities $\Phi'(t) = A\Phi(t)$, $\Phi(0) = I$, Already $\Phi(0) = I$. Details for identity $\Phi'(t) = A\Phi(t)$ will use the formula $\frac{d}{dt}e^{Ct} = C\,e^{Ct}$. Apply block differentiation to show $\Phi'(t) = A\Phi(t)$:

$$\begin{aligned}
\Phi'(t) &= \begin{pmatrix} B_1\,e^{B_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_k\,e^{B_k t} \end{pmatrix} \\
&= \begin{pmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_k \end{pmatrix} \begin{pmatrix} e^{B_1 t} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{B_k t} \end{pmatrix} \\
&= A\Phi(t)
\end{aligned}$$

■

Proof of Theorem 11.20, Complex Exponential:

Assume $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $b > 0$. Then A has eigenvalues $a \pm bi$. Putzer's 2×2 formula will be used, page 867:

$$e^{At} = e^{at}\left(\cos bt\, I + \frac{\sin bt}{b}(A - aI)\right).$$

Simplify the matrix expression on the right:

$$\begin{aligned}
e^{At} &= e^{at}\left(\begin{pmatrix} \cos bt & 0 \\ 0 & \cos bt \end{pmatrix} + \tfrac{\sin bt}{b}\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right) \\
&= e^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}
\end{aligned}$$

■

Exercises 11.4

Matrix Exponential.

1. **(Picard)** Let A be real 2×2. Write out the two initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$ of e^{At}.

2. **(Picard)** Let A be real 3×3. Write out the three initial value problems which define the columns $\vec{\mathbf{w}}_1(t)$, $\vec{\mathbf{w}}_2(t)$, $\vec{\mathbf{w}}_3(t)$ of e^{At}.

3. Let A be real 2×2. Show that $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{u}}_0$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{u}}_0$.

4. Let A be real $n \times n$. Show that $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0$ satisfies $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$.

Matrix Exponential 2×2. Find e^{At} from representation $e^{At} = \langle \vec{\mathbf{w}}_1 | \vec{\mathbf{w}}_2 \rangle$. Use first-order scalar methods.

5. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

6. $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

7. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

8. $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.

Matrix Exponential Identities. Verify from exponential identities.

9. $e^A\,e^{-A} = I$

10. $e^{-A} = \left(e^A\right)^{-1}$

11. $A = \frac{d}{dt}e^{At}$ evaluated at $t = 0$

12. If $A^3 = \mathbf{0}$, then $e^A = I + A + \frac{1}{2}A^2$.

13. Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Verify $N^2 = \mathbf{0}$ and $e^{At+Nt} = e^{At}(I + Nt)$.

14. Let A be 3×3 diagonal and $N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Prove $N^3 = \mathbf{0}$ and $e^{At+Nt} = e^{At}(I + Nt + N^2\frac{t^2}{2})$.

15. $e^{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} t} = \begin{pmatrix} e^t & e^{2t} - e^t \\ 0 & e^{2t} \end{pmatrix}$

16. $e^{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} t} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$

Putzer's Spectral Formula.

17. Apply Picard-Lindelöf theory to conclude that $r_1, \ldots, r_n$ are everywhere defined,

18. Prove that $P_1, \ldots, P_k$ commute.

Putzer's Formula 2×2 .

19. Find a formula for $\frac{d}{dt} e^{At}$ for a 2×2 matrix A with eigenvalues $1, 2$.

20. Let $2{\times}2$ matrix A have duplicate eigenvalues $0, 0$. Compute r_1, r_2 and then report e^{At}.

Putzer: Real Distinct. Find the matrix exponential.

21. $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

22. $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$

Putzer: Real Equal. Find the matrix exponential.

23. $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

24. $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Putzer: Complex Eigenvalues. Find the matrix exponential.

25. $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

26. $A = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

How to Remember Putzer's 2×2 Formula.

27. Find $\lim_{\lambda \to \lambda_1} \dfrac{e^{\lambda t} - e^{\lambda_1 t}}{\lambda - \lambda_1}$.

28. Let matrix A be 2×2 real. Take the real part: $e^{At} = I + \dfrac{e^{it} - e^{-it}}{2i} A$.

Classical $n \times n$ Spectral Formula. Find e^{At}.

29. $A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

30. $A = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

Proofs of Matrix Exponential Properties.

31. Let $A\vec{\mathbf{u}} = B\vec{\mathbf{u}}$ for all vectors $\vec{\mathbf{u}}$. Prove $A = B$.

32. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$. Compute the first four Picard iterates for $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$.

Special Cases e^{At}.

33. Show the details to solve
$x_1' = 2x_1 + x_3$,
$x_2' = 3x_2 + x_3$,
$x_3' = 4x_3$,
$x_1(0) = 1, x_2(0) = x_3(0) = 0$.

34. Let $A = \mathbf{diag}(1, 2, 3, 4)$. Find e^{At}.

35. Let $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $A = \mathbf{diag}(B, B)$. Find e^{At}.

36. Let $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$ and $A = \mathbf{diag}(B, B)$. Find e^{At}.

11.5 Cayley-Hamilton-Ziebur, Spectral and Eigenanalysis Methods

Established earlier in this chapter:

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0 \quad \text{solves} \quad \vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t), \quad \vec{\mathbf{x}}(0) - \vec{\mathbf{x}}_0$$

Matrix e^{At} is the augmented matrix of solutions $\vec{\mathbf{w}}_i(t)$ to $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ with $\vec{\mathbf{w}}_i(0) =$ column i of the identity matrix, $1 \le i \le n$.

Presented in this section are three premier methods for finding e^{At}:

Eigenanalysis Method
Spectral Method
Cayley-Hamilton-Ziebur (CHZ) Method

Eigenanalysis Method Requirements. The $n \times n$ real matrix A is required to have n independent eigenvectors in its list of eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$, $\ldots$, $(\lambda_n, \vec{\mathbf{v}}_n)$. Briefly, matrix A is **diagonalizable**. It is not required that the eigenvalues $\lambda_1, \ldots, \lambda_n$ be distinct and eigenvalues can be real or complex. The method uses independence of the Euler substitution solutions $e^{\lambda_i t}\vec{\mathbf{v}}_i$, $1 \le i \le n$, which are assembled into augmented matrix $\Phi(t)$. The general solution is $\vec{\mathbf{x}}(t) = e^{At}\,\vec{\mathbf{x}}(0)$, using identity $e^{At} = \Phi(t)\,\Phi(0)^{-1}$. A negative of the method occurs with complex eigenvalues: real solutions are found with extra effort via opaque identities. The method works best on diagonalizable matrices with only real eigenvalues, e.g., symmetric matrices.

Spectral Method Requirements. The method applies to any real $n \times n$ matrix A. Classical spectral theory of A provides a formula for e^{At} similar to Putzer's formula, thereby finding the solution $\vec{\mathbf{x}} = e^{At}\,\vec{\mathbf{x}}(0)$ of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Emphasis is on theory. Computational details are left to computer algebra systems, which efficiently implement the formulas. Hand computation is possible for low dimensions $n = 2, 3$ with time impact similar to Putzer's algorithm for e^{At}.

Cayley-Hamilton-Ziebur Method Requirements. The method applies to any real $n \times n$ matrix A. It provides a basis of n real vector solutions to the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, which are found from characteristic equation $|A - \lambda I| = 0$ and Euler solution atom theory developed for scalar differential equations. The connection to e^{At} is direct and simple: $e^{At} = \Phi(t)\,\Phi(0)^{-1}$ where $\Phi(t)$ is the $n \times n$ augmented matrix of the vector solutions. Hand computation is possible for low dimensional examples ($n = 2, 3$) with the lowest time impact of the three methods. A feature of the Cayley-Hamilton-Ziebur method is minimization of encounters with complex numbers. One important consequence of the method:

Solutions of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ are vector linear combinations of Euler solution atoms.

Eigenanalysis Method: 2×2 Matrix

Theorem 11.21 (Eigenanalysis Method 2×2)
Let matrix A be 2×2 real with eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$. Assume eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are independent.

Then the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ can be written as

$$\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2.$$

Proof:

Eigenvalues λ_1, λ_2 are either both real or a complex conjugate pair $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$. Derivatives and calculations below apply in both cases.

$$\begin{aligned}
\vec{\mathbf{x}}' &= c_1(e^{\lambda_1 t})'\vec{\mathbf{v}}_1 + c_2(e^{\lambda_2 t})'\vec{\mathbf{v}}_2 && \text{Differentiate the formula for } \vec{\mathbf{x}}. \\
&= c_1 e^{\lambda_1 t}\lambda_1\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\lambda_2\vec{\mathbf{v}}_2 \\
&= c_1 e^{\lambda_1 t}A\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}A\vec{\mathbf{v}}_2 && \text{Use } \lambda_1\vec{\mathbf{v}}_1 = A\vec{\mathbf{v}}_1,\ \lambda_2\vec{\mathbf{v}}_2 = A\vec{\mathbf{v}}_2. \\
&= A\left(c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2\right) && \text{Factor } A \text{ left.} \\
&= A\vec{\mathbf{x}} && \text{Definition of } \vec{\mathbf{x}}.
\end{aligned}$$

Re-write the solution $\vec{\mathbf{x}}$ in the vector-matrix form

$$\vec{\mathbf{x}}(t) = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \right\rangle \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{1}$$

Because eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are assumed independent, then $\left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \right\rangle$ is invertible and setting $t = 0$ in (1) gives

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \right\rangle^{-1} \vec{\mathbf{x}}(0). \tag{2}$$

Because c_1, c_2 can be chosen to produce any initial condition $\vec{\mathbf{x}}(0)$, then $\vec{\mathbf{x}}(t)$ is the *general solution* of the system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

Proposition 11.2 (Exponential Matrix: 2×2) Let matrix A be 2×2 real with eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$. Assume eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ are independent.

Then:

$$e^{At} = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \right\rangle \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 \right\rangle^{-1} \tag{3}$$

Proof: Combine (1) and (2). ■

Formula (3) is immediately useful when the eigenpairs are real. It is problematic when the eigenvalues are complex. The complex arithmetic inherited by complex eigenpairs can be minimized by applying results collected into a Proposition.

Proposition 11.3 (Exponential Matrix: Complex $\lambda_2 = \overline{\lambda_1}$)
Assume matrix A is 2×2 real with eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$. Let eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$ be independent.

Assume $\lambda_2 = \overline{\lambda}_1$ and λ_1 is not real. Define for eigenpair $(\lambda_1, \vec{\mathbf{v}}_1)$ symbols a, b, P:

$$\lambda_1 = a + ib, \quad b > 0, \quad P = \left\langle \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) | \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1) \right\rangle$$

Then

$$e^{At} = e^{at} P \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} P^{-1} \tag{4}$$

Proof on page 886.

Eigenanalysis Method: 3×3 Matrix

Theorem 11.22 (Eigenanalysis Method: 3×3)
Let matrix A be 3×3 real with eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1), (\lambda_2, \vec{\mathbf{v}}_2), (\lambda_3, \vec{\mathbf{v}}_3)$. Assume $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are independent.

Then the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is:

$$\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t} \vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t} \vec{\mathbf{v}}_2 + c_3 e^{\lambda_3 t} \vec{\mathbf{v}}_3.$$

Proof on page 887.

Proposition 11.4 (Exponential Matrix: 3×3 Complex Form)
Let matrix A be 3×3 real with eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1), (\lambda_2, \vec{\mathbf{v}}_2), (\lambda_3, \vec{\mathbf{v}}_3)$. Let $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ be independent. Then:

$$e^{At} = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \right\rangle \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \right\rangle^{-1}.$$

The formula applies when the eigenpairs are real and also when the eigenpairs are complex. Proof on page 887.

Proposition 11.5 (Exponential Matrix: 3×3 Real Form)
Let matrix A be 3×3 real with eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$, $(\lambda_3, \vec{\mathbf{v}}_3)$. Let $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ be independent. Assume one eigenvalue λ_3 is real and the other eigenvalues are a complex conjugate pair $\lambda_1 = \overline{\lambda}_2 = a + ib$, $b > 0$. Define matrix $P = \left\langle \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) | \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1) | \vec{\mathbf{v}}_3 \right\rangle$. Then P is invertible and the exponential matrix is:

$$e^{At} = P \begin{pmatrix} e^{at}\cos bt & e^{at}\sin bt & 0 \\ -e^{at}\sin bt & e^{at}\cos bt & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} P^{-1} \tag{5}$$

Proof on page 887.

Eigenanalysis Method: $n \times n$ Matrix

The general solution formula and the formula for e^{At} generalize routinely from the 2×2 and 3×3 cases to the general case of an $n\times n$ matrix. Proofs are left as an exercise, guided by the 3×3 case.

Theorem 11.23 (The Eigenanalysis Method)
Let the $n\times n$ real matrix A have eigenpairs

$$(\lambda_1, \vec{\mathbf{v}}_1), \quad (\lambda_2, \vec{\mathbf{v}}_2), \quad \ldots, \quad (\lambda_n, \vec{\mathbf{v}}_n)$$

with n independent eigenvectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$. Then the general solution of the linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is given by

$$\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2\vec{\mathbf{v}}_2 e^{\lambda_2 t} + \cdots + c_n\vec{\mathbf{v}}_n e^{\lambda_n t}. \tag{6}$$

Proposition 11.6 (General Solution: $n\times n$ Complex Matrix Form)
General solution (6) can be expressed as a matrix product:

$$\vec{\mathbf{x}}(t) = \left\langle \vec{\mathbf{v}}_1 | \cdots | \vec{\mathbf{v}}_n \right\rangle \mathbf{diag}(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

Definition 11.3 (Real Diagonal Form)
Assume $n\times n$ matrix A is diagonalizable. List all complex eigenvalues of A in pairs λ_1, $\overline{\lambda}_1$, …, λ_p, $\overline{\lambda}_p$. Then list the real eigenvalues $r_1, \ldots, r_q$, $2p+q=n$. List the eigenpairs as $(\lambda_i, \vec{\mathbf{v}}_i)$, $(\overline{\lambda}_i, \overline{\vec{\mathbf{v}}}_i)$, $1 \le i \le p$ and $(r_j, \vec{\mathbf{v}}_{2p+j})$, $1 \le j \le q$. Define

$$\begin{aligned} P &= \left\langle \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) | \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1) | \cdots | \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_{2p-1}) | \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_{2p-1}) | \vec{\mathbf{v}}_{2p+1} | \cdots | \vec{\mathbf{v}}_n \right\rangle \\ J_\lambda &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad \lambda = a+ib, \quad b>0 \end{aligned}$$

The **real diagonal form**:

$$A = P\,\mathbf{diag}\left(J_{\lambda_1}, \cdots, J_{\lambda_p}, r_1, \cdots, r_q\right) P^{-1}$$

Proposition 11.7 (Exponential Matrix: $n\times n$ Real Matrix Form)
Define

$$R_\lambda(t) = e^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}, \quad \lambda + a + ib, \quad b > 0.$$

Let $A = P\,\mathbf{diag}\left(J_{\lambda_1}, \cdots, J_{\lambda_p}, r_1, \cdots, r_q\right) P^{-1}$ be the **real diagonal form** of diagonalizable matrix A.

Then:

$$e^{At} = P\,\mathbf{diag}(R_{\lambda_1}(t), \ldots, R_{\lambda_p}(t), e^{r_1 t}, \ldots, e^{r_q t})P^{-1}.$$

Remark on Euler Solution Atoms. If the characteristic equation is $(\lambda-1)^3 = 0$ and there are three independent eigenvectors, then the general solution $\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + c_3 e^{\lambda_3 t}\vec{\mathbf{v}}_3$ contains no terms with te^t nor t^2e^t. Intuition from $(\lambda-1)^3 = 0$ suggests that solution components should be linear combinations of e^t, te^t, t^2e^t. How is that possible? The answer is contained in the linear combination $2e^t + 0te^t + 0t^2e^t$: it is indeed a linear combination of Euler atoms e^t, te^t, t^2e^t.

Classical Spectral Theory Method

The simplicity of Putzer's spectral method for computing e^{At} is appreciated, but we also recognize that the literature has an algorithm to compute e^{At}, devoid of differential equations, which is of fundamental importance in linear algebra. The parallel algorithm computes e^{At} directly from the eigenvalues λ_j of A and certain products of the nilpotent matrices $A - \lambda_j I$. Called **spectral formulas**, they can be implemented in a numerical laboratory or computer algebra system, in order to efficiently compute e^{At}, even in the case of multiple eigenvalues.

Theorem 11.24 (Spectral Formula for e^{At}: Simple Eigenvalues)
Let the $n \times n$ matrix A have n simple eigenvalues $\lambda_1, \ldots, \lambda_n$ (possibly complex) and define constant matrices $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_n$ by the formulas

$$\boldsymbol{Q}_j = \Pi_{i\neq j}\frac{A-\lambda_i I}{\lambda_j - \lambda_i}, \quad j = 1, \ldots, n.$$

Then

$$e^{At} = e^{\lambda_1 t}\boldsymbol{Q}_1 + \cdots + e^{\lambda_n t}\boldsymbol{Q}_n.$$

Theorem 11.25 (Spectral Formula for e^{At}: Multiple Eigenvalues)
Let the $n \times n$ matrix A have k distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of algebraic multiplicities $m_1, \ldots, m_k$. Let $p(\lambda) = \det(A - \lambda I)$ and define polynomials $a_1(\lambda), \ldots, a_k(\lambda)$ by the partial fraction identity

$$\frac{1}{p(\lambda)} = \frac{a_1(\lambda)}{(\lambda-\lambda_1)^{m_1}} + \cdots + \frac{a_k(\lambda)}{(\lambda-\lambda_k)^{m_k}}.$$

Define constant matrices $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_k$ by the formulas

$$\boldsymbol{Q}_j = a_j(A)\Pi_{i\neq j}(A - \lambda_i I)^{m_i}, \quad j = 1, \ldots, k.$$

Then

$$e^{At} = \sum_{i=1}^{k} e^{\lambda_i t}\,\boldsymbol{Q}_i \sum_{j=0}^{m_i-1}(A-\lambda_i I)^j\,\frac{t^j}{j!}. \tag{7}$$

Proof: Let $\boldsymbol{N}_i = \boldsymbol{Q}_i(A - \lambda_i I)$, $1 \le i \le k$. First:

Lemma 11.1 (Properties)
1. $\boldsymbol{Q}_1 + \cdots + \boldsymbol{Q}_k = I$,
2. $\boldsymbol{Q}_i \boldsymbol{Q}_j = \boldsymbol{0}$ for $i \neq j$,
3. $\boldsymbol{Q}_i \boldsymbol{Q}_i = \boldsymbol{Q}_i$,
4. $\boldsymbol{N}_i \boldsymbol{N}_j = \boldsymbol{0}$ for $i \neq j$,
5. $\boldsymbol{N}_i^{m_i} = \boldsymbol{0}$,
6. $A = \sum_{i=1}^k (\lambda_i \boldsymbol{Q}_i + \boldsymbol{N}_i)$.

To prove exponential formula (7), use Lemma 11.1 as follows:

$$\begin{array}{rll} e^{At} & = \sum_{i=1}^k \boldsymbol{Q}_i e^{At} & \text{Lemma 11.1, item } \mathbf{1} \\ & = \sum_{i=1}^k \boldsymbol{Q}_i e^{\lambda_i I t + (A-\lambda_i I)t} & \\ & = \sum_{i=1}^k \boldsymbol{Q}_i e^{\lambda_i t} e^{(A-\lambda_i I)t} & \\ & = \sum_{i=1}^k \boldsymbol{Q}_i e^{\lambda_i t} e^{\boldsymbol{Q}_i (A-\lambda_i I)t} & \text{Lemma 11.1, items } \mathbf{2}, \mathbf{3} \\ & = \sum_{i=1}^k \boldsymbol{Q}_i e^{\lambda_i t} e^{\boldsymbol{N}_i t} & \text{Definition of } \boldsymbol{N}_i \\ & = \sum_{i=1}^k \boldsymbol{Q}_i e^{\lambda_i t} \sum_{j=0}^{m_1 - 1} (A - \lambda_i I)^j \frac{t^j}{j!} & \text{Lemma 11.1, item } \mathbf{6} \end{array}$$

Proof of Lemma 11.1:
Identity 1: Clear fractions in the partial fraction expansion of $1/p(\lambda)$:

$$1 = \sum_{i=1}^k a_i(\lambda) \frac{p(\lambda)}{(\lambda - \lambda_i)^{m_i}}.$$

Identity 2: Observe that $\boldsymbol{Q}_i$ and $\boldsymbol{Q}_j$ together contain all the factors of $p(A)$, therefore $\boldsymbol{Q}_i \boldsymbol{Q}_j = q(A)p(A)$ for some polynomial q. The Cayley-Hamilton theorem $p(A) = \boldsymbol{0}$ finishes the details.

Identity 3: Multiply identity **1** by $\boldsymbol{Q}_i$ and then use **2**.

Identity 4: Write $\boldsymbol{N}_i \boldsymbol{N}_j = (A - \lambda_i I)(A - \lambda_j I)\boldsymbol{Q}_i \boldsymbol{Q}_j$ and apply **3**.

Identity 5: Identity **2** implies $\boldsymbol{Q}_i^{m_i} = \boldsymbol{Q}_i$, then $\boldsymbol{N}_i^{m_i} = (A - \lambda_i I)^{m_i} \boldsymbol{Q}_i = p(A) = \boldsymbol{0}$.

Identity 6: Multiply identity **1** by A and rearrange:

$$\begin{array}{rcl} A & = & \sum_{i=1}^k A\boldsymbol{Q}_i \\ & = & \sum_{i=1}^k \lambda_i \boldsymbol{Q}_i + (A - \lambda_i I)\boldsymbol{Q}_i \\ & = & \sum_{i=1}^k \lambda_i \boldsymbol{Q}_i + \boldsymbol{N}_i \end{array}$$

■

Cayley-Hamilton-Ziebur for $\vec{x}'(t) = A\vec{x}(t)$

Given $n \times n$ matrix A, determinant $|A - rI|$ is formed by subtracting r from the diagonal of A. The **characteristic polynomial** is $p(r) = |A - rI|$ and $|A - rI| = 0$ is the **characteristic equation**.

The famous result of Cayley and Hamilton is restated in Theorem 11.26. An elementary proof appears in linear algebra Chapter 5, Theorem 5.20, page **??**.

Theorem 11.26 (Cayley-Hamilton)
Every square matrix A satisfies its own characteristic equation.

Let $|A - r\boldsymbol{I}| = (-r)^n + a_{n-1}(-r)^{n-1} + \cdots + a_0$ be the characteristic polynomial of $n \times n$ matrix A. Let $\boldsymbol{I}$ and $\mathbf{0}$ denote the $n \times n$ identity and zero matrix. Then:

$$(-A)^n + a_{n-1}(-A)^{n-1} + \cdots + a_1(-A) + a_0\boldsymbol{I} = \mathbf{0}$$

Theorem 11.27 (Cayley-Hamilton-Ziebur Theorem: Scalar Form)
Let A be an $n \times n$ real matrix. Each of the components $x_1(t), \ldots, x_n(t)$ of a real vector solution $\vec{\mathbf{x}}(t)$ of system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ is a solution of an nth order scalar linear homogeneous constant-coefficient differential equation with characteristic equation $|A - rI| = 0$. The result remains true for complex solutions $\vec{\mathbf{x}}(t)$ and complex A. Proof on page 888.

Theorem 11.28 (Cayley-Hamilton-Ziebur Theorem: Vector Form)
Let A be an $n \times n$ real matrix. Let $A_1(t), \ldots, A_n(t)$ be Euler solution atoms constructed from the roots of $|A - rI| = 0$. The solution of system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ is a **vector** linear combination of $A_1(t), \ldots, A_n(t)$:

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{d}}_1\, A_1(t) + \cdots + \vec{\mathbf{d}}_n\, A_n(t).$$

Constant vectors $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$ are determined by A and $\vec{\mathbf{x}}(0)$ (see identity (8) *infra*). The result holds for complex A provided $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$ are complex. Euler atoms may be replaced by complex exponentials times powers of t. Proof on page 888.

Theorem 11.29 (Cayley-Hamilton-Ziebur Identity: Real)
Let $W(t)$ be the Wronskian matrix of Euler solution atoms $\{A_j\}_{j=1}^n$ constructed from the roots of $|A - rI| = 0$. Let $V = W(0)^T$. Constant vectors $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$ in Cayley-Hamilton-Ziebur Theorem 11.28 are determined by:

$$\left\langle \vec{\mathbf{d}}_1 | \cdots | \vec{\mathbf{d}}_n \right\rangle = \left\langle \vec{\mathbf{x}}(0) | A\vec{\mathbf{x}}(0) | \cdots | A^{n-1}\vec{\mathbf{x}}(0) \right\rangle V^{-1}. \tag{8}$$

Proof on page 888.

Theorem 11.30 (Cayley-Hamilton-Ziebur Identity: Complex)
Identity (8) remains valid if set $\{A_j\}_{j=1}^n$ is replaced by complex independent linear combinations $\{B_j\}_{j=1}^n$ of $\{A_j\}_{j=1}^n$ with $\{\vec{\mathbf{d}}_j\}_{j=1}^n$ possibly complex and $W(t)$ is replaced by the Wronskian matrix of $\{B_j\}_{j=1}^n$. Proof on page 888.

Theorem 11.31 (Vandermonde Matrix and Identity (8))
Assume the results of Theorems 11.29 and 11.30. If roots $\lambda = \lambda_1, \ldots, \lambda_n$ of $|A - \lambda I| = 0$ are distinct, then matrix $V = W(0)^T$ is the Vandermonde matrix of the roots:

$$V = \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{pmatrix}. \tag{9}$$

Proof on page 889.

Theorem 11.32 (Eigenvectors and Identity (8))
Assume the results of Theorems 11.29, 11.30. If $|A - \lambda I| = 0$ has distinct roots $\lambda_1, \ldots, \lambda_n$, then vector $\vec{\mathbf{d}}_j$ is a scalar multiple of eigenvector $\vec{\mathbf{v}}_j$ for eigenvalue λ_j, $1 \le j \le n$ (Warning: $\vec{\mathbf{d}}_j = \vec{\mathbf{0}}$ is possible). Proof on page 889.

Theorem 11.33 (Eigenvectors by Matrix Multiply)
Let A have distinct eigenvalues $\{\lambda_j\}_{j=1}^n$ and define for any n-vector $\vec{\mathbf{U}}$

$$V = \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{pmatrix}, \quad P = \left\langle \vec{\mathbf{U}} | A\vec{\mathbf{U}} | \cdots | A^{n-1}\vec{\mathbf{U}} \right\rangle V^{-1}.$$

Then column j of P is either zero or else an eigenvector of A for λ_j.

> **Notation**: $\left\langle \vec{\mathbf{y}}_1 | \cdots | \vec{\mathbf{y}}_n \right\rangle$ = augmented matrix of $\vec{\mathbf{y}}_1, \ldots, \vec{\mathbf{y}}_n$. To determine all eigenvectors experimentally, start with all $\vec{\mathbf{U}}$-components one, then change some ones in $\vec{\mathbf{U}}$ to zero or minus one and repeat.

Proof on page 889.

Example 11.7 (Eigenvectors by Matrix Multiply)
Compute by Theorem 11.33 all eigenvectors of matrix $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$.

Details for Example 11.7:
The matrix of eigenvectors is $P = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$. Solve $|A - \lambda I| = 0$ for complex eigenvalues $\lambda_1, \lambda_2 = 1 \pm 2i$, then define $V = \begin{pmatrix} 1 & \lambda_1 \\ 1 & \lambda_2 \end{pmatrix}$, $U = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Compute $B = \left\langle U | AU \right\rangle = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$ and $V^{-1} = \frac{1}{|V|}\,\mathbf{adj}(V) = \frac{1}{-4i}\begin{pmatrix} \lambda_2 & -\lambda_1 \\ -1 & 1 \end{pmatrix}$. Multiply to find $P = BV^{-1} = \frac{1}{2}\begin{pmatrix} 1-i & 1+i \\ 1+i & 1-i \end{pmatrix}$. Maple code to check the computation:

```
with(LinearAlgebra):A:=<1,2|-2,1>^+;EV:=Eigenvalues(A);
U:=<1,1>;V:=VandermondeMatrix(EV);P:=<U|A.U>.(1/V);
J:=DiagonalMatrix(EV);A.P-P.J; # Check eigenvectors
```

Inverse of a Vandermonde Matrix

Notation: Symmetric function $e_k(r_1, \ldots, r_N) = \displaystyle\sum_{1 \le i_1 < \cdots < i_k \le N} r_{i_1} \cdots r_{i_k}$

Vieta's formulas[7] supply coefficients $a_k = (-1)^{N-k} e_k(r_1, \ldots, r_N)$ of degree N polynomial $\sum_{k=0}^{N-1} a_k y^k + y^N = \prod_{p=1}^{N}(y - r_p)$ with roots $r_1, \ldots, r_N$.

[7] See https://en.wikipedia.org/wiki/Vieta%27s_formulas.

Theorem 11.34 (Vandermode Inverse)

Let $A = \begin{pmatrix} 1 & \cdots & x_1^{n-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & x_n^{n-1} \end{pmatrix}$ with $x_1, \ldots, x_n$ distinct. Then $A^{-1} = B = (b_{i\,j})$:

$$b_{i\,j} = \frac{(-1)^{n-j} e_{n-j}(\{x_1, \ldots, x_n\} \setminus \{x_i\})}{\displaystyle\prod_{p=1, p\neq i}^{n} (x_j - x_p)}, \quad 1 \le i, j \le n. \tag{10}$$

Proof on page 890.

Vieta's Formulas: `maple`

```
# Vieta's formulas, monic polynomial: maple library
n:=3;q:=expand(product((y-x[i]),i=1..n));
ListTools[Reverse]([coeffs(q,y)]);

# Vieta's formulas: basic algorithm, no library
# Monic polynomial, roots x[1] to x[n]
F:=proc(r) local A,n,i,j;
n:=nops(r);A:=[seq(0,i=1..n+2)];A[n+1]:=1;
for i from 1 to n do  for j from n+1-i to n+1 do
   A[j]:=A[j]+(-1)*r[i]*A[j+1]; od; od;
return simplify([seq(A[i],i=1..n+1)]); end proc:
F([seq(x[i],i=1..3)]); # Test n=3
```

Solving Planar Systems $\vec{x}'(t) = A\vec{x}(t)$

A 2×2 real system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ can be solved in terms of matrix A and the two roots of the characteristic equation $\det(A - \lambda I) = 0$.

Two distinct methods are explored below, both with minimal use of complex numbers.

The most-used method on paper is the Cayley-Hamilton-Ziebur **Scalar Shortcut**. Implementations for embedded systems might use the formulas obtained from the **Matrix Shortcut**. The only requirement on matrix A is that it **not** be a diagonal matrix.

Theorem 11.35 (Cayley-Hamilton-Ziebur Scalar 2×2 Shortcut)
Let $b \neq 0$ in the scalar system

$$\begin{aligned} x_1' &= a\,x_1 + b\,x_2 \\ x_2' &= c\,x_1 + d\,x_2 \end{aligned} \tag{11}$$

Define $x_1(t){=}c_1y_1(t) + c_2y_2(t)$. Solve for $x_2(t)$ in the first equation, then replace x_1 by $c_1y_1 + c_2y_2$ on the right of $bx_2 = x_1' - ax_1$ and simplify to find $x_2 = k_1y_1 + k_2y_2$.
Proof on page 890.

Theorem 11.36 (Cayley-Hamilton-Ziebur Matrix 2×2 Shortcut)

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b \neq 0$. Let $y_1(t), y_2(t)$ be the Euler solution atoms found from the roots of $|A - \lambda I| = 0$. Define constant matrix B by identity $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. Then the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ with arbitrary constants c_1, c_2 is

$$\begin{cases} x_1(t) = c_1 y_1(t) + c_2 y_2(t), \\ x_2(t) = k_1 y_1(t) + k_2 y_2(t), \end{cases} \quad \text{where} \quad \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{b}(B^T - aI)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Proof on page 890.

> **Remark**. Theorems 11.35, 11.36 solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ when A is not a diagonal matrix (meaning either $b \neq 0$ or $c \neq 0$). The case $b = 0$ and $c \neq 0$ is treated by swapping b, c and x_1, x_2 in both of Theorems 11.35, 11.36.

Example 11.8 ()
(Scalar and Matrix 2×2 Shortcuts for Real Roots)
Solve the system

$$\begin{cases} x_1'(t) = x_1(t) + 2x_2(t), \\ x_2'(t) = 2x_1(t) + x_2(t), \end{cases} \quad A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad \vec{\mathbf{x}}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

verifying the general solution $\begin{cases} x_1(t) = c_1 e^{-t} + c_2 e^{3t}, \\ x_2(t) = -c_1 e^{-t} + c_2 e^{3t}. \end{cases}$

Details Example 11.8:

The characteristic polynomial $\begin{vmatrix} 1-r & 2 \\ 2 & 1-r \end{vmatrix} = (1-r)^2 - 4 = (r+1)(r-3)$ has roots $r = -1, r = 3$ and Euler solution atoms e^{-t}, e^{3t}.

Scalar Shortcut Details. To apply Theorem 11.35, define $x_1 = c_1 e^{-t} + c_2 e^{3t}$. Solve the first differential equation $x_1' = x_1 + 2x_2$ for $2x_2 = x_1' - x_1 = (c_1 e^{-t} + c_2 e^{3t})' - x_1 = -2c_1 e^{-t} + 2e^{3t}$. Then $x_2 = -e^{-t} + e^{3t}$.

Matrix Shortcut Details. To apply Theorem 11.36, first compute matrix $B = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$ from $\frac{d}{dt}\begin{pmatrix} e^{-t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} -e^{-t} \\ e^{3t} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} e^{-t} \\ e^{3t} \end{pmatrix}$.
Theorem 11.36 implies

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{b}(B^T - aI)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2}(B^T - I)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then $x_2(t) = -c_1 y_1 + c_2 y_2 = -c_1 e^{-t} + c_2 e^{3t}$.

Example 11.9 ()
(Scalar and Matrix 2×2 Shortcuts for Complex Roots)
Solve the system

$$\begin{cases} x_1'(t) = x_1(t) + 2x_2(t), \\ x_2'(t) = -2x_1(t) + x_2(t), \end{cases} \quad A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad \vec{\mathbf{x}}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix},$$

verifying the general solution $\begin{cases} x_1(t) = c_1 e^t \cos(2t) + c_2 e^t \sin(2t), \\ x_2(t) = c_2 e^t \cos(2t) - c_1 e^t \sin(2t). \end{cases}$

Details Example 11.9: Characteristic polynomial $\begin{vmatrix} 1-r & 2 \\ -2 & 1-r \end{vmatrix} = (1-r)^2 + 4$ has roots $r = 1 \pm 2i$ and Euler solution atoms $e^t \cos(2t)$, $e^t \sin(2t)$.

Scalar Shortcut Details. To apply Theorem 11.35, let $x_1 = c_1 e^t \cos(2t) + c_2 e^t \sin(2t)$, then solve the first differential equation $x_1' = x_1 + 2x_2$ for $2x_2 = x_1' - x_1 = (c_1 e^t \cos(2t) + c_2 e^t \sin(2t))' - x_1 = 2c_2 e^t \cos(2t) - 2c_1 e^t \sin(2t)$. Then $x_2 = c_2 e^t \cos(2t) - c_1 e^t \sin(2t)$.

Matrix Shortcut Details. To apply Theorem 11.36, first compute matrix $B = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$:

$\frac{d}{dt}\begin{pmatrix} e^t \cos(2t) \\ e^t \sin(2t) \end{pmatrix} = \begin{pmatrix} e^t \cos(2t) - 2e^t \sin(2t) \\ e^t \sin(2t) + 2e^t \cos(2t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^t \cos(2t) \\ e^t \sin(2t) \end{pmatrix}$

Theorem 11.36 implies

$$\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{b}(B^T - aI)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2}(B^T - I)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then $x_2(t) = c_2 y_1 - c_1 y_2 = c_2 e^t \cos(2t) - c_1 e^t \sin(2t)$.

Theorem 11.37 (Putzer's Spectral Formula: 2×2)

Consider the real planar system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$. Let λ_1, λ_2 be the roots of the characteristic equation $\det(A - \lambda I) = 0$. The real general solution is $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0)$ where the 2×2 exponential matrix e^{At} is given by

Real $\lambda_1 \neq \lambda_2$	$e^{At} = e^{\lambda_1 t} I + \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I).$
Real $\lambda_1 = \lambda_2$	$e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t}(A - \lambda_1 I).$
Complex $\lambda_1 = \overline{\lambda}_2$, $\lambda_1 = a + bi$, $b > 0$	$e^{At} = e^{at} \cos bt\, I + \dfrac{e^{at} \sin(bt)}{b}(A - aI).$

Proof: The formulas are from Putzer's algorithm page 868 or equivalently from the spectral formulas with rearranged terms. The complex case is formally the real part of the distinct root case when $\lambda_2 = \overline{\lambda}_1$. The three formulas are analogous to the second order equation formulas Chapter 6 Section 1, Theorem 6.1. ■

Example 11.10 (Classical and Putzer Spectral Formulas)

Typical cases are represented by the following 2×2 matrices A, which correspond to roots λ_1, λ_2 of the characteristic equation $\det(A - \lambda I) = 0$ which are real distinct, real double or complex conjugate. The solution $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0)$ is given here in two forms, by writing e^{At} using $\boxed{1}$ a **classical spectral formula** from Theorems 11.24–11.25 and $\boxed{2}$ Putzer's **spectral formula** from Theorem 11.37.

$\lambda_1 = 5$, $\lambda_2 = 2$ Real distinct roots.

$A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}$

$\boxed{1}$ $e^{At} = \dfrac{e^{5t}}{3}\begin{pmatrix} -3 & 3 \\ -6 & 6 \end{pmatrix} + \dfrac{e^{2t}}{-3}\begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$

$\boxed{2}$ $e^{At} = e^{5t} I + \dfrac{e^{2t} - e^{5t}}{2 - 5}\begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$

$\lambda_1 = \lambda_2 = 3$ Real double root.

$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ $\boxed{1}$ $e^{At} = e^{3t}\left(I + t\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\right)$

$\boxed{2}$ $e^{At} = e^{3t}I + te^{3t}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$

$\lambda_1 = \overline{\lambda}_2 = 2 + 3i$ Complex conjugate roots.

$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$ $\boxed{1}$ $e^{At} = 2\,\mathcal{R}\mathrm{e}\left(\dfrac{e^{2t+3it}}{2(3i)}\begin{pmatrix} 3i & 3 \\ -3 & 3i \end{pmatrix}\right)$

$\boxed{2}$ $e^{At}{=}e^{2t}\cos(3t)I{+}\dfrac{e^{2t}\sin(3t)}{3}\begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$

The complex eigenvalue example is typical for real $n \times n$ matrices A with a complex conjugate pair of eigenvalues $\lambda_1 = \overline{\lambda}_2$. Then $\boldsymbol{Q}_2 = \overline{\boldsymbol{Q}}_1$ for $\boxed{1}$. The result is that λ_2 is not used and a simpler expression results by using the college algebra equality $z + \overline{z} = 2\,\mathcal{R}\mathrm{e}(z)$:

$$e^{\lambda_1 t}\boldsymbol{Q}_1 + e^{\lambda_2 t}\boldsymbol{Q}_2 = 2\,\mathcal{R}\mathrm{e}\left(e^{\lambda_1 t}\boldsymbol{Q}_1\right).$$

This observation explains why e^{At} is real when A is real, by pairing complex conjugate eigenvalues in Theorems 11.24–11.25,

Proofs and Methods

Proof of Proposition 11.3:

Eigenpair $(\lambda_2, \vec{\mathbf{v}}_2)$ is never computed or used, because $A\vec{\mathbf{v}}_1 = \lambda_1\vec{\mathbf{v}}_1$ implies $A\overline{\vec{\mathbf{v}}}_1 = \overline{\lambda}_1\overline{\vec{\mathbf{v}}}_1$, which implies λ_2 $(= \overline{\lambda}_1)$ has eigenvector $\vec{\mathbf{v}}_2 = \overline{\vec{\mathbf{v}}}_1$.

If A is real, then e^{At} is real. Take real parts across the formula for e^{At} to give a real formula. Due to the unpleasantness of the complex algebra, we will justify the answer with minimal use of complex numbers.

The formula is established by showing that the matrix $\Phi(t)$ on the right of equation (4) satisfies $\Phi(0) = I$ and $\Phi' = A\Phi$. Then by definition, $e^{At} = \Phi(t)$. For exposition, let

$$R(t) = e^{at}\begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}, \quad \Phi(t) = PR(t)P^{-1}.$$

Identity $\Phi(0) = I$ is verified as follows.

$$\begin{aligned} \Phi(0) &= PR(0)P^{-1} \\ &= Pe^0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}P^{-1} \\ &= I \end{aligned}$$

Express $\vec{\mathbf{v}}_1 = \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) + i\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)$. Expand eigenpair relation $A\vec{\mathbf{v}}_1 = \lambda_1\vec{\mathbf{v}}_1$ into real and imaginary parts:

$$A\left(\mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) + i\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)\right) = (a + ib)\left(\mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1) + i\,\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)\right)$$

Match real and imaginary parts left and right in this equation to obtain:

$$AP = P\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

Then:

$$\begin{aligned}
\Phi'(t)\Phi^{-1}(t) &= PR'(t)P^{-1}PR^{-1}(t)P^{-1} \\
&= PR'(t)R^{-1}(t)P^{-1} \\
&= P\left(aI + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}\right)P^{-1} \\
&= P\begin{pmatrix} a & b \\ -b & a \end{pmatrix}P^{-1} \\
&= A
\end{aligned}$$

Because $\Phi'(t) = A\Phi(t)$, $\Phi(0) = I$, then $\Phi(t) = e^{At}$. The general solution is $\vec{\mathbf{x}}(t) = \Phi(t)\vec{\mathbf{x}}(0)$. Then

$$\vec{\mathbf{x}}(t) = e^{at}\Big\langle \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1)|\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)\Big\rangle \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

where values c_1, c_2 are related to the initial condition $\vec{\mathbf{x}}(0)$ by identity

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \Big\langle \mathcal{R}\mathrm{e}(\vec{\mathbf{v}}_1)|\mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1)\Big\rangle^{-1}\vec{\mathbf{x}}(0)$$

Proof of Theorem 11.22:

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ can be all real or eigenvalue λ_3 is real and the other eigenvalues are complex: $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$.

The proposed solution $\vec{\mathbf{x}}$ can be written in vector-matrix form:

$$\vec{\mathbf{x}}(t) = \Big\langle \vec{\mathbf{v}}_1|\vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\Big\rangle \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

Because the three eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are assumed independent, then $\Big\langle \vec{\mathbf{v}}_1|\vec{\mathbf{v}}_2|\vec{\mathbf{v}}_3\Big\rangle$ is invertible. Setting $t = 0$ in the previous display gives

$$\begin{pmatrix} c_1 \\ c_2 \\ c_2 \end{pmatrix} = \Big\langle \vec{\mathbf{v}}_1|\vec{\mathbf{v}}_2|\vec{\mathbf{v}}_3\Big\rangle^{-1}\vec{\mathbf{x}}(0).$$

Constants c_1, c_2, c_3 can be chosen to produce any initial condition $\vec{\mathbf{x}}(0)$, therefore $\vec{\mathbf{x}}(t)$ is the *general solution* of the 3×3 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

Proofs of Propositions 11.4 and 11.5:

The proof of Theorem 11.22 supplies the proof details for Proposition 11.4.

Proposition 11.5 is proved in two steps: (1) Show P has independent columns, hence P is invertible; (2) The exponential matrix is given by equation (5).

(1) Let $\vec{\mathbf{v}}_2 = \overline{\vec{\mathbf{v}}}_1$. Replace the first two column vectors in P by

$$Re(\vec{\mathbf{v}}_1) = \frac{1}{2}(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2), \quad \mathcal{I}\mathrm{m}(\vec{\mathbf{v}}_1) = -\frac{i}{2}(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2).$$

Let d_1, d_2, d_3 be constants. Assume dependency relation $P\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \vec{\mathbf{0}}$. Then:

$$\frac{1}{2}\,d_1(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2) - \frac{i}{2}\,d_2(\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2) + d_3\vec{\mathbf{v}}_3 = \vec{\mathbf{0}}.$$

Independence of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ implies all linear combination weights are zero:

$$\frac{1}{2}d_1 - \frac{i}{2}d_2 = 0, \quad \frac{1}{2}d_1 + \frac{i}{2}d_2 = 0, \quad d_3 = 0.$$

Solve this system to prove $d_1 = d_2 = d_3 = 0$. Conclude that the columns of P are independent.

(2) Let $B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. Define block matrix $J = \begin{pmatrix} B & 0 \\ \mathbf{0} & \lambda_3 \end{pmatrix}$. Diagonalization theory for matrices implies $AP = PJ$. Then:

$e^{Jt} = \begin{pmatrix} e^{Bt} & \vec{\mathbf{0}} \\ \mathbf{0} & e^{\lambda_3 t} \end{pmatrix}$	Theorem 11.19, page 869
$e^{Bt} = \begin{pmatrix} e^{at}\cos bt & e^{at}\sin bt \\ -e^{at}\sin bt & e^{at}\cos bt \end{pmatrix}$	Theorem 11.20, page 869
$e^{At} = P\,e^{Jt}\,P^{-1}$	Identities page 865
$e^{At} = P\begin{pmatrix} e^{at}\cos bt & e^{at}\sin bt & 0 \\ -e^{at}\sin bt & e^{at}\cos bt & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} P^{-1}$	

Proof of Theorem 11.27. Consider first the case $n = 2$, which has a routine generalization to higher dimensions.

$r^2 + a_1 r + a_0 = 0$	Expanded characteristic equation
$A^2 + a_1 A + a_0 I = 0$	Cayley-Hamilton matrix equation, where I and 0 are the identity and zero matrix.
$A^2\vec{\mathbf{x}} + a_1 A\vec{\mathbf{x}} + a_0\vec{\mathbf{x}} = \vec{\mathbf{0}}$	Right-multiply by $\vec{\mathbf{x}} = \vec{\mathbf{x}}(t)$
$\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}' = A^2\vec{\mathbf{x}}$	Differentiate $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$
$\vec{\mathbf{x}}'' + a_1\vec{\mathbf{x}}' + a_0\vec{\mathbf{x}} = \vec{\mathbf{0}}$	Replace $A^2\vec{\mathbf{x}} \to \vec{\mathbf{x}}''$, $A\vec{\mathbf{x}} \to \vec{\mathbf{x}}'$

Multiply the vector relation by the rows of the identity matrix to show the components $x_1(t)$, $x_2(t)$ of $\vec{\mathbf{x}}(t)$ satisfy the two differential equations

$$\begin{aligned} x_1''(t) + a_1 x_1'(t) + a_0 x_1(t) &= 0, \\ x_2''(t) + a_1 x_2'(t) + a_0 x_2(t) &= 0. \end{aligned}$$

This system implies that the components of $\vec{\mathbf{x}}(t)$ are solutions of the second order differential equation with characteristic equation $|A - rI| = 0$.

The proof remains valid if real solution $\vec{\mathbf{x}}(t)$ is replaced by a complex solution, no changes required in the above text. Because the Cayley-Hamilton theorem is valid for complex A, the proof is complete for $n = 2$. Details for any n are left to the reader.

Proofs of Theorems 11.27, 11.28, 11.29 and 11.30. The scalar form Theorem 11.27 can be written

$$x_i(t) = c_{i1}A_1 + \cdots + c_{in}A_n, \quad i = 1, \ldots, n.$$

In matrix form:

$$\vec{\mathbf{x}}(t) = \begin{pmatrix} c_{11} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots \\ c_{11} & \cdots & c_{1,n} \end{pmatrix} \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}.$$

Then $\vec{\mathbf{d}}_1$ is the first column of matrix (c_{ij}) above, and so on, which proves

$$\vec{\mathbf{x}}(t) = \vec{\mathbf{d}}_1 A_1 + \cdots + \vec{\mathbf{d}}_n A_n \tag{12}$$

Left to prove is that column vectors $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$ depend only on A and initial data $\vec{\mathbf{x}}(0)$. We proceed as in the theory of Wronskian determinants by differentiation $n-1$ times of equation (12), then replace t by zero to obtain these formulas:

$$\left\{\begin{array}{lcccccc} \vec{\mathbf{x}}(0) & = & A_1(0)\,\vec{\mathbf{d}}_1 & + & \cdots & + & A_n(0)\,\vec{\mathbf{d}}_n \\ \vec{\mathbf{x}}'(0) & = & A_1'(0)\,\vec{\mathbf{d}}_1 & + & \cdots & + & A_n'(0)\,\vec{\mathbf{d}}_n \\ & \vdots & & & & & \\ \vec{\mathbf{x}}^{(n-1)}(0) & = & A_1^{(n-1)}(0)\,\vec{\mathbf{d}}_1 & + & \cdots & + & A_n^{(n-1)}(0)\,\vec{\mathbf{d}}_n \end{array}\right. \tag{13}$$

The derivatives on the left in equation (13) can be cleverly rewritten as $\vec{\mathbf{x}}(0)$, $A\vec{\mathbf{x}}(0)$, $\ldots$, $A^{n-1}\vec{\mathbf{x}}(0)$ by successive differentiation of $\vec{\mathbf{x}}'(t) = Xx(t)$. For instance, $\vec{\mathbf{x}}''(t) = (A\vec{\mathbf{x}}(t))' = A\vec{\mathbf{x}}'(t) = AA\vec{\mathbf{x}}(t) = A^2\vec{\mathbf{x}}(t)$, then $t = 0$ gives $\vec{\mathbf{x}}''(0) = A^2\vec{\mathbf{x}}(0)$. The result in matrix form:

$$\left\langle \vec{\mathbf{x}}(0)|\cdots|A^{n-1}\vec{\mathbf{x}}(0)\right\rangle = \left\langle \vec{\mathbf{d}}_1|\cdots|\vec{\mathbf{d}}_n\right\rangle \begin{pmatrix} A_1(0) \cdots A_1^{(n-1)}(0) \\ \vdots \quad \cdots \quad \vdots \\ A_n(0) \cdots A_n^{(n-1)}(0) \end{pmatrix} \tag{14}$$

The augmented matrix $\left\langle \vec{\mathbf{d}}_1|\cdots|\vec{\mathbf{d}}_n\right\rangle$ of vectors $\vec{\mathbf{d}}_1,\ldots,\vec{\mathbf{d}}_n$ is then obtained by matrix inversion: $\left\langle \vec{\mathbf{d}}_1|\cdots|\vec{\mathbf{d}}_n\right\rangle = \left\langle \vec{\mathbf{x}}(0)|\cdots|A^{n-1}\vec{\mathbf{x}}(0)\right\rangle (W(0)^T)^{-1}$, where $W(t)$ is the Wronskian matrix of the n Euler solution atoms.

Suppose A_j is replaced by B_j which are independent linear combinations of atoms A_j with complex coefficients. Assume given a solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, then $\vec{\mathbf{x}}(t) = \sum_{j=1}^n \vec{\mathbf{d}}_j\, B_j(t)$ for some column vectors $\vec{\mathbf{d}}_j$. Let's differentiate this relation $n-1$ times and substitute $t = 0$, as before. The same analysis with matrix multiply, Wronskians and inverses applies, therefore identity (8) remains valid. Related details appear in the proof of Theorem 11.31.

Proof of Theorem 11.31. If all roots are real distinct, then the Euler solution atoms are $e^{\lambda_1 t}$, $\ldots$, $e^{\lambda_n t}$. Find the Wronskian matrix of these functions, then let $t = 0$, which makes all exponentials equal to one. The first row is all ones, therefore the transpose matrix has first column all ones. If complex roots $a \pm bi$ appear, then in affected atoms $\cos bt = \frac{1}{2}(e^{ibt} + e^{-ibt})$, $\sin bt = \frac{1}{2i}(e^{ibt} - e^{-ibt})$. Collect terms into complex exponentials $e^{(a+ib)t}$ multiplied by vectors (complex entries allowed). Identity (8) is unchanged except for replacement of atoms by exponentials. Proceed to identify $W(t)$, $W(0)$ and $W(0)^T$ in the same manner as for real roots.

Proofs of Theorems 11.32, 11.33. Assume the CHZ results of previous theorems and that A has distinct eigenvalues, complex numbers allowed. Let $(\lambda_j, \vec{\mathbf{v}}_j)$, $1 \le j \le n$ be a list of eigenpairs of A. Let $A_j = e^{\lambda_j t}$, $1 \le j \le n$: they are independent functions with invertible Wronskian matrix $W(t)$ (see the Exercises). The Eigenanalysis method supplies solution $\vec{\mathbf{x}}(t) = \sum_{j=1}^{n} c_j \vec{\mathbf{v}}_j\, A_j(t)$ whereas CHZ supplies $\vec{\mathbf{x}}(t) = \sum_{j=1}^{n} \vec{\mathbf{d}}_j\, A_j(t)$. Independence of list $\{A_j(t)\}_{j=1}^{n}$ implies $\vec{\mathbf{d}}_j = c_j \vec{\mathbf{v}}_j$. However, $c_j = 0$ is possible, therefore $\vec{\mathbf{d}}_j = \vec{\mathbf{0}}$ or else $\vec{\mathbf{d}}_j$ is a nonzero multiple of eigenvector $\vec{\mathbf{v}}_j$, $1 \le j \le n$.

Theorem 11.33 directly applies Theorem 11.32, which implies the columns of augmented matrix $P = \left\langle \vec{\mathbf{d}}_1 | \cdots | \vec{\mathbf{d}}_n \right\rangle$ are either zero or a nonzero multiple of an eigenvector of A. Examples choose $\vec{\mathbf{U}}$ initially to be the column vector of ones, then ones are modified to zero or minus one: then re-apply the formula to find all eigenvectors.

Proof of Theorem 11.34, Vandermonde Inverse:

Case for $n = 3$. The inverse matrix $B = \begin{pmatrix} a_0 & \cdot & \cdot \\ a_1 & \cdot & \cdot \\ a_2 & \cdot & \cdot \end{pmatrix}$ of Vandermonde matrix $A = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{pmatrix}$ satisfies $AB = I$. Match column one on both sides of $AB = I$ using matrix multiply, then for polynomial $p_1(y) = a_0 + a_1 y + a_2 y^2$ there are three interpolation equations to be satisfied:

$$a_0 + a_1 x_1 + a_2 x_1^2 = 1,\ a_0 + a_1 x_2 + a_2 x_2^2 = 0,\ a_0 + a_1 x_3 + a_2 x_3^2 = 0.$$

Degree 2 polynomial $q_1(y) = \frac{1}{y - x_1} \prod_{i=1}^{3} (y - x_i)$ constructs the interpolation problem unique solution $p_1(y) = \frac{q_1(y)}{q_1(x_1)}$. Coefficients a_0, a_1, a_2 are found by matching y-coefficients after expanding equation $a_0 + a_1 y + a_2 y^2 = \frac{q_1(y)}{q_1(x_1)}$. Define q_2, q_3, p_2, p_3 analogously and repeat for columns 2, 3. Then inverse B equals:

$$\begin{pmatrix} x_2 x_3 & x_1 x_3 & x_1 x_2 \\ -x_2 - x_3 & -x_1 - x_3 & -x_1 - x_2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{(x_1 - x_2)(x_1 - x_3)} & 0 & 0 \\ 0 & \frac{1}{(x_2 - x_1)(x_2 - x_3)} & 0 \\ 0 & 0 & \frac{1}{(x_3 - x_1)(x_3 - x_2)} \end{pmatrix}$$

Case for General n. For i from 1 to n, define degree $n-1$ polynomials $q_i(y) = \frac{1}{y - x_i} \prod_{p=1}^{n} (y - x_p)$, then expand $q_i(y) = \sum_{j=1}^{n} q_{ij} y^{j-1}$ to obtain

$$B = (b_{ij}) = \begin{pmatrix} q_{11} & \cdots & q_{n1} \\ \vdots & \cdots & \vdots \\ q_{1n} & \cdots & q_{nn} \end{pmatrix} \begin{pmatrix} \frac{1}{q_1(x_1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{q_n(x_n)} \end{pmatrix}.$$

Formula for $q_i(x_i)$. Cancel factor $y - x_i$, then $q_i(x_i) = \prod_{p=1, p \ne i}^{n} (x_i - x_p)$.

Formula for q_{ij}. Let $N = n - 1$. Vieta's formulas applied to degree N polynomial $q_i(y) = \sum_{j=1}^{n} q_{i\,j} y^{j-1}$ give $q_{i\,j} = (-1)^{N-j+1} e_{N-j+1}(\{x_1, \ldots, x_n\} \setminus \{x_i\})$, for $j = 1, \ldots, n-1$. Equality $b_{ij} = \frac{q_{ij}}{q_i(x_i)}$ then establishes equation (10).

Proof of Theorem 11.35. The details are in the proof of Theorem 11.36, which discusses the application of Theorem 11.27 and solving the first differential equation for variable x_2. This is the preferred shortcut on paper.

Proof of Theorem (11.36). The formula for $x_1(t)$ follows directly from Cayley-Hamilton-Ziebur Theorem 11.27. Equation $x_2(t) = k_1 y_1(t) + k_2 y_2(t)$ follows from the same theorem, for some constants k_1, k_2. It remains to prove that the constants are $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \frac{1}{b}(B^T - aI) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$. Details:

$x_2 = \frac{1}{b} x_1' - \frac{a}{b} x_1$ Solve $x_1' = a x_1 + b x_2$ for x_2.

$x_2 = \frac{1}{b}(c_1 y_1' + c_2 y_2') - \frac{a}{b}(c_1 y_1 + c_2 y_2)$	Replace $x_1 = c_1 y_1 + c_2 y_2$.
$x_2 = \frac{1}{b}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} - \frac{a}{b}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	Rewrite as matrix multiply.
$x_2 = \frac{1}{b}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T B \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \frac{a}{b}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	Definition of B.
$x_2 = \frac{1}{b}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T (B - aI) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	Factor out $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ right, $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}^T$ left.
$x_2 = \frac{1}{b}\left((B^T - aI)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\right)^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	Matrix transpose properties $(CD)^T = D^T C^T$ and $(C+D)^T = C^T + D^T$.
$x_2 = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}^T \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$	Theorem's definition of k_1, k_2.
$x_2 = k_1 y_1 + k_2 y_2$	Verification complete.

Exercises 11.5

Determinant $|A - rI|$

Justify these statements.

1. Subtract r from the diagonal of A to form $|A - rI|$.

2. If A is 2×2, then $|A - rI|$ is a quadratic.

3. If A is 3×3, then $|A - rI|$ is a cubic.

4. Expansion of $|A - rI|$ by the cofactor rule often preserves factorizations.

5. If A is triangular, then $|A - rI|$ is the product of diagonal entries.

6. The *combo*, *mult* and *swap* rules for determinants are generally counter-productive for expansion of $|A - rI|$.

Characteristic Polynomial

Show expansion details for $|A - rI|$.

7. $A = \begin{pmatrix} 2 & 3 \\ 0 & 4 \end{pmatrix}$.

Ans: $(2 - r)(4 - r)$

8. $A = \begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$.

Ans: $(2 - r)(5 - r)(7 - r)$

Eigenanalysis Method: 2×2

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

9. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

10. $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

Eigenanalysis Method: 3×3

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

11. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

12. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Eigenanalysis Method: $n \times n$

Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$.

13. $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

14. $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 2 & 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

e^{At} for Simple Eigenvalues

Find a^{At} using classical spectral theory. Check by computer.

15. $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$

16. $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

e^{At} for Multiple Eigenvalues

Find a^{At} using classical spectral theory. Check by computer.

17. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

18. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Cayley-Hamilton Theorem

Prove the identity by applying the Cayley-Hamilton Theorem.

19. Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a_0=|A|=ad-bc$, $a_1=\textbf{trace}(A)=a+d$. Then $A^2 + a_1(-A) + a_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

20. Let $A=\begin{pmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{pmatrix}$. Then:

$(2I-A)(5I-A)(7I-A)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

CHZ Theorem: Scalar Form

21. Write Theorem 11.27 proof missing details for $n = 3$.

22. Write Theorem 11.27 proof missing details for any n.

CHZ Theorem: Vector Form

23. Write Theorem 11.28 proof details for $n = 2$.

24. Write Theorem 11.28 proof details for $n = 3$.

CHZ Identity: Vandermonde

Find matrix $D = \left\langle \vec{\mathbf{d}}_1 | \cdots | \vec{\mathbf{d}}_n \right\rangle$ using Theorems 11.29, 11.31, given $\vec{\mathbf{x}}(0)=\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$.

25. $A=\begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$. Ans: $W(0)^T, D=$ $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & c_1 \\ 2c_1 + c_2 & -2c_1 \end{pmatrix}$

26. $A=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Ans: $W(0)^T, D=$ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$, $\begin{pmatrix} c_1 & 0 & 0 \\ -2c_1 & 2c_1 + c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix}$

CHZ and Eigenvectors

Supply details for the following.

27. Find a scalar 3rd order linear differential equation that has e^t, e^{2it}, e^{-2it} as solutions. Apply theorems to conclude that the Wronskian of the exponentials is invertible for every t.

28. Assume $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ are independent exponentials . Apply theorems to conclude that the Wronskian of the exponentials is invertible for every t.

29. If $\vec{\mathbf{d}}_1 e^t + \vec{\mathbf{d}}_2 e^{-t} + \vec{\mathbf{d}}_3 e^{2t} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, then $\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_2 = \vec{\mathbf{d}}_3 = \vec{\mathbf{0}}$.

30. Independence of atoms applied to the n-vector equation $\vec{\mathbf{d}}_1 e^t + \vec{\mathbf{d}}_2 e^{-t} = c_1 \vec{\mathbf{v}}_1 e^t + c_2 \vec{\mathbf{v}}_2 e^{-t}$ implies $\vec{\mathbf{d}}_1 = c_1 \vec{\mathbf{v}}_1$ and $\vec{\mathbf{d}}_2 = c_2 \vec{\mathbf{v}}_2$.

31. There is a 2×2 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ for which CHZ vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ are not eigenvectors of A.

32. Let A be the 3×3 identity matrix. For $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$, two of the CHZ vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3$ are zero.

Eigenvectors by Matrix Multiply

Find the eigenvectors of A by Theorem 11.33. Report the choice of $\vec{\mathbf{U}}$.

33. $A=\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Ans: $\vec{\mathbf{U}}=\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

34. $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Ans: $\vec{\mathbf{U}}=\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$.

CHZ 2×2 Matrix Shortcut Find the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ using Theorem 11.36.

35. $A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$, $r = -2, 4$

36. $A = \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$, $r = 1 \pm 3i$

CHZ Scalar 2×2 Shortcut Find the general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ using Theorem 11.35.

37. $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$, $r = -3, 5$

38. $A = \begin{pmatrix} 1 & 4 \\ -4 & 1 \end{pmatrix}$, $r = 1 \pm 4i$

Putzer's 2×2 Spectral Formula Verify the identity.

39. $A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}$

$$e^{At} = e^{5t}I + \frac{e^{5t} - e^{2t}}{3} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix}$$

40. $A = \begin{pmatrix} 0 & 1 \\ 6 & 1 \end{pmatrix}$

$$e^{At} = e^{-2t}I + \frac{e^{3t} - e^{-2t}}{5} \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$$

41. $A = \begin{pmatrix} 0 & 1 \\ -16 & 8 \end{pmatrix}$

$$e^{At} = e^{4t}I + te^{4t} \begin{pmatrix} -4 & 1 \\ -16 & 4 \end{pmatrix}$$

42. $A = \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix}$, $e^{At} =$

$$e^{3t}\cos(2t)I + e^{3t}\sin(2t) \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

11.6 Jordan Form and Eigenanalysis

Generalized Eigenanalysis

The main result is **Jordan's decomposition**

$$A = PJP^{-1},$$

valid for any real or complex square matrix A. Described here is how to compute the invertible matrix P of generalized eigenvectors and the upper triangular matrix J, called a **Jordan form** of A.

Jordan Block

An $m \times m$ upper triangular matrix $B(\lambda, m)$ is called a **Jordan block** provided all m diagonal elements are the same eigenvalue λ and all super-diagonal elements are one:

$$B(\lambda, m) = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (m \times m \text{ matrix})$$

Jordan Form

Given an $n \times n$ matrix A, a **Jordan form** J for A is a block diagonal matrix

$$J = \textbf{diag}(B(\lambda_1, m_1), B(\lambda_2, m_2), \ldots, B(\lambda_k, m_k)),$$

where $\lambda_1, \ldots, \lambda_k$ are eigenvalues of A (duplicates possible) and $m_1 + \cdots + m_k = n$. The eigenvalues of J are on the diagonal of J and J has exactly k eigenpairs. If $k < n$, then J is non-diagonalizable. Relation $AP = PJ$ implies A has exactly k eigenpairs and A fails to be diagonalizable for $k < n$.

The relation $A = PJP^{-1}$ is called a **Jordan decomposition** of A. The $n \times n$ matrix P is an augmented matrix of column vectors, i.e., $P = \left\langle \vec{\mathbf{v}}_1 | \ldots | \vec{\mathbf{v}}_n \right\rangle$, which is called the **matrix of generalized eigenvectors** of A. It defines a coordinate system $\vec{\mathbf{x}} = P\vec{\mathbf{y}}$ in which the vector function $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is transformed to the simpler vector function $\vec{\mathbf{y}} \to J\vec{\mathbf{y}}$.

If equal eigenvalues are adjacent in J, then Jordan blocks with equal diagonal entries can be adjacent. Zeros can appear on the super-diagonal of J, because adjacent Jordan blocks join on the super-diagonal with a zero. A complete specification of how to build J from A appears below.

Organizing a Jordan Form

One scheme to organize J first lists distinct eigenvalues low to high $\lambda_1, \ldots, \lambda_k$. Then the Jordan blocks appear in J in that order, with block size high to low for those blocks with the same eigenvalue.

For instance, suppose $\lambda_1 = -4$, $\lambda_1 = 2$, $\lambda_1 = 7$ with respective multiplicities 5, 1 and 3. Then one possible Jordan form is:

$$
\begin{aligned}
J &= \mathbf{diag}(B(\lambda_1, 3), B(\lambda_1, 2), B(\lambda_2, 1), B(\lambda 3, 3)) \\
&= \begin{pmatrix}
-4 & 1 & 0 & & & & & & \\
0 & -4 & 1 & & & & & & \\
0 & 0 & -4 & & & & & & \\
 & & & -4 & 1 & & & & \\
 & & & 0 & -4 & & & & \\
 & & & & & 2 & & & \\
 & & & & & & 7 & 1 & 0 \\
 & & & & & & 0 & 7 & 1 \\
 & & & & & & 0 & 0 & 7
\end{pmatrix}
\end{aligned}
$$

Decoding a Jordan Decomposition $A = PJP^{-1}$

If J contains $m \times m$ Jordan block $B(\lambda, m)$, consuming rows 1 to m of J, then $P = \left\langle \vec{\mathbf{v}}_1 | \ldots | \vec{\mathbf{v}}_n \right\rangle$ and $AP = PJ$ implies m vector equations:

$$
\begin{aligned}
A\vec{\mathbf{v}}_1 &= \lambda \vec{\mathbf{v}}_1, \\
A\vec{\mathbf{v}}_2 &= \lambda \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_1, \\
\vdots \quad & \vdots \qquad \vdots \\
A\vec{\mathbf{v}}_m &= \lambda \vec{\mathbf{v}}_m + \vec{\mathbf{v}}_{m-1}.
\end{aligned}
$$

To justify this, start with $AP = PJ$. Expand $AP = \left\langle A\vec{\mathbf{v}}_1 | \ldots | A\vec{\mathbf{v}}_n \right\rangle$ and match its first m columns to those of PJ. This exploded view of the relation $AP = PJ$ according to the Jordan block $B(\lambda, m)$ is called a **Jordan chain**. The formulas can be compacted via matrix $N = A - \lambda I$ into the **Jordan chain relations**

(1) $$N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}, \quad N\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1, \quad \ldots, \quad N\vec{\mathbf{v}}_m = \vec{\mathbf{v}}_{m-1}.$$

The first vector $\vec{\mathbf{v}}_1$ is an eigenvector. The remaining vectors $\vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_m$ are **not eigenvectors**, they are called **generalized eigenvectors**. Similar formulas can be written for each Jordan block in matrix J. A given eigenvalue may appear multiple times in the chain relations, due to the appearance of two or more Jordan blocks with the same eigenvalue. It is known that the vectors $\{\vec{\mathbf{v}}_i\}|_{i=1}^m$ in a Jordan chain are independent from vectors appearing in a different chain.

Theorem 11.38 (Jordan Decomposition)
Every $n \times n$ matrix A has a Jordan decomposition $A = PJP^{-1}$.
Induction proof on page 907.

Proposition 11.8 (Jordan's Extension) Any $n \times n$ matrix A can be represented in the block triangular form

$$A = PTP^{-1}, \quad T = \mathbf{diag}(T_1, \ldots, T_k),$$

where P is invertible and each matrix T_i is upper triangular with diagonal entries equal to a single eigenvalue of A.

See also Theorem **??** page **??**. The theorem is proved from the Jordan decomposition theorem by defining $T_i = J_i$, a Jordan Block. A shorter, simpler induction proof exists for Jordan's extension, but the structure of the blocks T_i is unknown with no practical algorithm for their construction.

Geometric and Algebraic Multiplicity

Symbol $\mathbf{GeoMult}(\lambda) = \dim(\mathbf{kernel}(A - \lambda I))$ is called the **geometric multiplicity**. It is defined as the number of basis vectors in a solution to $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$, or, equivalently, the number of free variables for this homogeneous problem.

The integer $k = \mathbf{AlgMult}(\lambda)$ is called the **algebraic multiplicity**, defined by the condition that $(r - \lambda)^k$ divides the characteristic polynomial $\det(A - rI)$, but larger powers do not.

Eigenvalue λ is called a **defective eigenvalue** provided inequality $\mathbf{GeoMult}(\lambda) < \mathbf{AlgMult}(\lambda)$ holds. If matrix A has a defective eigenvalue, then is called a **defective matrix**. Defective matrices are not diagonalizable, but they do admit a Jordan decomposition $A = PJP^{-1}$.

Theorem 11.39 (Algebraic and Geometric Multiplicity)
Let A be a square real or complex matrix. Then

$$1 \le \mathbf{GeoMult}(\lambda) \le \mathbf{AlgMult}(\lambda). \tag{2}$$

In addition, there are the following relationships between the Jordan form J and algebraic and geometric multiplicities.

$\mathbf{GeoMult}(\lambda)$	Equals the number of Jordan blocks in J with eigenvalue λ,
$\mathbf{AlgMult}(\lambda)$	Equals the number of times λ is repeated along the diagonal of J.

Proof: Let $d = \mathbf{GeoMult}(\lambda_0)$. Construct a basis $v_1, \ldots, v_n$ of $\mathcal{R}^n$ such that $v_1, \ldots, v_d$ is a basis for $\mathbf{kernel}(A - \lambda_0 I)$. Define $S = \left\langle v_1 | \ldots | v_n \right\rangle$ and $B = S^{-1}AS$. The first d columns of AS are $\lambda_0 v_1, \ldots, \lambda_0 v_d$. Then $B = \left(\begin{array}{c|c} \lambda_0 I & C \\ \hline 0 & D \end{array}\right)$ for some matrices C and D. Cofactor expansion implies some polynomial g satisfies

$$\det(A - \lambda I) = \det(S(B - \lambda I)S^{-1}) = \det(B - \lambda I) = (\lambda - \lambda_0)^d g(\lambda)$$

and therefore $d \le \mathbf{AlgMult}(\lambda_0)$. Other details of proof are omitted. ■

Number of Jordan Blocks

Calculation of generalized eigenvectors of matrix A for eigenvalue λ is organized by computing only the Jordan chains of a certain size k. The sizes are found by computing ranks of the powers N^j of the nilpotent matrix $N = A - \lambda I$.

Theorem 11.40 (Number of Jordan Blocks)
Let matrix A have eigenvalue λ. Define $N = A - \lambda I$. Let p be the least integer such that $N^p = N^{p+1}$. Then the number $M(j)$ of Jordan blocks $B(\lambda, j)$ is given by

$$M(j) = \mathbf{rank}(N^{j+1}) + \mathbf{rank}(N^{j-1}) - 2\,\mathbf{rank}(N^j), \quad j = 2, \ldots, p.$$

The proof of the theorem[8] is in the exercises, where more detail appears for $p = 1$ and $p = 2$.

Chains of Generalized Eigenvectors

Given an eigenvalue λ of the matrix A, the topic of generalized eigenanalysis determines all Jordan blocks $B(\lambda, m)$ in J and the corresponding columns of P. The ordering of the blocks in J is not unique. The corresponding columns of P are not unique.

Let $N = A = \lambda I$. Suppose an m-chain is known to exist because of Theorem 11.40, $m \le \textbf{AlgMult}(\lambda)$. How exactly do we find $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_m$ in Jordan chain relations (1)?

A first step might be to write the chain relations (1) in reverse order using a new symbol $\vec{\mathbf{w}}$ that stands for $\vec{\mathbf{v}}_m$:

$$\text{(3)} \qquad \vec{\mathbf{v}}_1 = N^{m-1}\vec{\mathbf{w}}, \quad \ldots, \quad \vec{\mathbf{v}}_{m-1} = N\vec{\mathbf{w}}, \vec{\mathbf{v}}_m = \vec{\mathbf{w}}$$

For instance, if $m = 3$ then the equations are $\vec{\mathbf{v}}_1 = N^2\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_2 = N\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_3 = \vec{\mathbf{w}}$. The impact of (3) is to change the problem of finding an m-chain into finding a suitable vector $\vec{\mathbf{w}}$. Clearly $\vec{\mathbf{w}}$ is not unique. Generally, $\vec{\mathbf{w}}$ is not an eigenvector.

How to Choose Vector $\vec{\mathbf{w}}$

The requirements on $\vec{\mathbf{w}}$ are:

(1) $N^{m-1}\vec{\mathbf{x}} = \vec{\mathbf{w}}$ has no solution $\vec{\mathbf{x}}$.
(2) $N^m\vec{\mathbf{w}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{w}} \in \mathbf{nullspace}(N^m)$
(3) $N^{m-1}\vec{\mathbf{w}} \ne \vec{\mathbf{0}}$ or $\vec{\mathbf{w}} \notin \mathbf{nullspace}(N^{m-1})$

[8] Jordan matrix. Encyclopedia of Mathematics. URL:
https://encyclopediaofmath.org/index.php?title=Jordan_matrix&oldid=17628
An equivalent formula is $M(j) = 2\,\mathbf{nullity}(N^j) - \mathbf{nullity}(N^{j+1}) - \mathbf{nullity}(N^{j-1})$.

Proposition 11.9 (Choosing Vector $\vec{\mathbf{w}}$)
Let $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ belong to the nullspace of $(N^{m-1})^T$. Then $N^{m-1}\vec{\mathbf{x}} = \vec{\mathbf{w}}$ has no solution $\vec{\mathbf{x}}$.

Proof: Assume a solution $\vec{\mathbf{x}}$ exists. Let $B = N^{m-1}$ and $S = \mathbf{nullspace}(B^T)$. Given: $\vec{\mathbf{w}} \in S$. Equation $B\vec{\mathbf{x}} = \vec{\mathbf{w}}$ implies $\vec{\mathbf{w}} \in \mathbf{Image}(B)$. The Fundamental Theorem of Linear Algebra (FTLA) gives $\mathbf{Image}(B) = \mathbf{nullspace}((B^T)^\perp = S^\perp$. Then $\vec{\mathbf{w}} \in S \cap S^\perp$. The intersection of S and $S^\perp$ is the zero vector. Then $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{w}} = \vec{\mathbf{0}}$, a contradiction. ■

Because of the **chain relations** of equation (1) the very first vector $\vec{\mathbf{v}}_1$ of the chain is an eigenvector: $(A - \lambda I)\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$. The others $\vec{\mathbf{v}}_2, \ldots, \vec{\mathbf{v}}_k$ are not eigenvectors.

Table 2. Shortcut: How to Choose $\vec{\mathbf{w}}$

1. Let $B = (N^{m-1})^T$. Choose a nonzero vector $\vec{\mathbf{w}}$ in the nullspace of B which also satisfies $N^m\vec{\mathbf{w}} = \vec{\mathbf{0}}$. See Proposition 11.9.
2. Require vector $\vec{\mathbf{w}}$ to satisfy $B\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, it is not in the nullspace of N^{m-1}.

Jordan Decomposition using `maple`

The matrix

$$A = \begin{pmatrix} 4 & -2 & 5 \\ -2 & 4 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

has a Jordan decomposition

$$A = PJP^{-1} = \begin{pmatrix} 1 & 4 & -7 \\ -1 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 1 & -\frac{7}{4} \\ -\frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

```
# Maple, Find Jordan Form of matrix A
 A := Matrix([[4, -2, 5], [-2, 4, -3], [0, 0, 2]]);
 factor(LinearAlgebra[CharacteristicPolynomial](A,lambda));
 # Answer == (lambda-6)*(lambda-2)^2
 J,P:=LinearAlgebra[JordanForm](A,output=['J','Q']);
 zero:=A.P-P.J; # zero matrix expected
```

The `maple` algorithm for the Jordan Form employs the Frobenius Normal Form, which in 2022 differs from Wikipedia and Wolfram references in the ordering of the diagonal blocks. Expect `maple` and `mathematica` to deliver Jordan forms for a given matrix A with different ordering of Jordan blocks.

Examples: Jordan Form and m-Chain

Calculation of generalized eigenvectors of matrix A for eigenvalue λ is organized by computing only the Jordan chains of a certain size k. The sizes are found by rank computation of the powers N^j of the nilpotent matrix $N = A - \lambda I$.

Example 11.11 (Number of Jordan Blocks)
Let A be the 5×5 matrix in equation (4), which has one eigenvalue $\lambda = 2$ of multiplicity 5. Verify that a Jordan form of A has two Jordan blocks, one of size 2 and one of size 3, e.g., $J = \mathbf{diag}(B(\lambda, 3), B(\lambda, 2))$.

$$(4) \qquad A = \begin{pmatrix} 3 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ -3 & 3 & 0 & -2 & 3 \end{pmatrix}$$

Details.
First form the nilpotent matrix $N = A - \lambda I$, then compute N^2 and N^3:

$$N = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ -3 & 3 & 0 & -2 & 1 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & -1 & 1 \\ -2 & 2 & 0 & -1 & 1 \\ -2 & 2 & 0 & -1 & 1 \\ 2 & -2 & 0 & 1 & -1 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Computer assist finds $\mathbf{rank}(N) = 3$ and $\mathbf{rank}(N^2) = 2$. Identity $N^3 = 0$ implies nilpotency $p = 3$.

Theorem 11.40 applied to Jordan block $B(\lambda, j)$ provides the equation $M(j) = \mathbf{rank}(N^{j+1}) + \mathbf{rank}(N^{j-1}) - 2\,\mathbf{rank}(N^j)$, $\quad j = 2, \ldots, p$. Then $M(1) = 0, M(2) = 1, M(3) = 1, M(4) = M(5) = 0$. There are only two Jordan blocks, size 2 and 3. One possible Jordan form:

$$J = \mathbf{diag}(B(\lambda, 3), B(\lambda, 2)) = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

```
with(LinearAlgebra):
getBlockCounts:=proc(A,lambda) local m,N,j,r,p,txt;
 m:=RowDimension(A);
 N:=A-lambda*IdentityMatrix(m);
 for j from 1 to m do r[j]:=Rank(N^j); od:
 for p from m to 2 by -1 do
 if(r[p]<>r[p-1])then break;fi:od;
 printf("lambda=%d, nilpotency=p=%d\n",lambda,p);
```

```
 txt:=(j,x)-> printf("Blocks B(%a,%d):%d\n",lambda,j,x):
 for j from p to 2 by -1 do txt(j,-2*r[j]+r[j-1]+r[j+1]):
 od:
end proc:
#
A := Matrix([[3,-1,1,0,0],[2,0,1,1,0],[1,-1,2,1,0],
 [-1,1,0,2,1],[-3,3,0,-2,3]]);
getBlockCounts(A,2);
```

The results: $\lambda = 2$, nilpotency=3, Blocks $B(2,3) : 1$, Blocks $B(2,2) : 1$.

The `maple` answer for J is obtained by the single line `JordanForm(A)`. Also possible is `JordanForm(A,output=['J','Q'])` to print J and Q for identity $AQ = QJ$. The `maple` answers:

$$(5) \qquad J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} 0 & 1 & 2 & -1 & 0 \\ -4 & 2 & 2 & -2 & 2 \\ -4 & 1 & 1 & -1 & 1 \\ -4 & -3 & 1 & -1 & 1 \\ 4 & -5 & -3 & 1 & -3 \end{pmatrix}$$

Example 11.12 (Generalized Eigenvectors)
Let A be the 5×5 matrix in equation (4), which has one eigenvalue $\lambda = 2$ of multiplicity 5. Find the generalized eigenvectors of A as columns of a matrix P, verifying the answer satisfies $AP = PJ$.

Details: Duplicate matrices A, $N = A - 2I$ and J from the preceding example:

$$A = \begin{pmatrix} 3 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ -3 & 3 & 0 & -2 & 3 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ -3 & 3 & 0 & -2 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

Jordan form J shows there is a 3-chain and a 2-chain of generalized eigenvectors for eigenvalue $\lambda = 2$. We will find the two chains.

The 3-chain. The plan is to find a vector $\vec{\mathbf{w}}$ with $N^3\vec{\mathbf{w}} = \vec{\mathbf{0}}$, $N^2\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and $N^2\vec{\mathbf{x}} = \vec{\mathbf{w}}$ has no solution $\vec{\mathbf{x}}$. Then $\vec{\mathbf{v}}_1 = N^2\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_2 = N\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_3 = \vec{\mathbf{w}}$ are the columns of P corresponding to Jordan block $B(\lambda, 3)$, to wit: columns 1,2,3 of P. Computer assist provides

$$N = \begin{pmatrix} 1 & -1 & 1 & 0 & 0 \\ 2 & -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ -3 & 3 & 0 & -2 & 1 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & -1 & 1 \\ -2 & 2 & 0 & -1 & 1 \\ -2 & 2 & 0 & -1 & 1 \\ 2 & -2 & 0 & 1 & -1 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We will choose $\vec{\mathbf{w}}$ to be a basis element for the nullspace of $(N^2)^T$, following Table 2 and Proposition 11.9. This clever choice works because $N^m = 0$. We still have to check $N^2\vec{\mathbf{w}} \ne \vec{\mathbf{0}}$, as in Table 2. Employ `maple` to find the nullspace basis:

$$\mathbf{nullspace}((N^2)^T) = \mathbf{span}\left\{ \begin{pmatrix} 0\\1\\0\\0\\1 \end{pmatrix}, \begin{pmatrix} 0\\-1\\0\\1\\0 \end{pmatrix}, \begin{pmatrix} 0\\-1\\1\\0\\0 \end{pmatrix}, \begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix} \right\}$$

Choose vector $\vec{\mathbf{w}}$ to be the last basis vector above, that is, the vector with components $1,0,0,0,0$. Then (1) equation $N^2\vec{\mathbf{x}} = \vec{\mathbf{w}}$ is insolvable for $\vec{\mathbf{x}}$, (2) $N^2\vec{\mathbf{w}} \ne \vec{\mathbf{0}}$, (3) $N^3\vec{\mathbf{w}} = \vec{\mathbf{0}}$.

Columns 1,2,3 of P will be defined by equations

$$\vec{\mathbf{v}}_1 = N^2\vec{\mathbf{w}} = \begin{pmatrix} 0\\-2\\-2\\-2\\2 \end{pmatrix}, \quad \vec{\mathbf{v}}_2 = N\vec{\mathbf{w}} = \begin{pmatrix} 1\\2\\1\\-1\\-3 \end{pmatrix}, \quad \vec{\mathbf{v}}_3 = \vec{\mathbf{w}} = \begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix}$$

The computation means that $AP = PJ$[9] where

$$P = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 | \vec{\mathbf{0}} | \vec{\mathbf{0}} \right\rangle = \begin{pmatrix} 0 & 1 & 1 & 0 & 0\\ -2 & 2 & 0 & 0 & 0\\ -2 & 1 & 0 & 0 & 0\\ -2 & -1 & 0 & 0 & 0\\ 2 & -3 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{cases} N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}\\ N\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1\\ N\vec{\mathbf{v}}_3 = \vec{\mathbf{v}}_2 \end{cases}$$

The 2-chain. Let $m = 2$ (find a 2-chain). The plan is to find a vector $\vec{\mathbf{w}}$ with $N^2\vec{\mathbf{w}} = \vec{\mathbf{0}}$, $N\vec{\mathbf{w}} \ne \vec{\mathbf{0}}$ and $N\vec{\mathbf{x}} = \vec{\mathbf{w}}$ has no solution $\vec{\mathbf{x}}$. Then $\vec{\mathbf{v}}_4 = N\vec{\mathbf{w}}$, $\vec{\mathbf{v}}_5 = \vec{\mathbf{w}}$ are the columns of P corresponding to Jordan block $B(\lambda, 2)$, to wit: columns 4,5 of P.

We will choose $\vec{\mathbf{w}} \ne \vec{\mathbf{0}}$ to be a vector in the nullspace of N^T, following Table 2 and Proposition 11.9. First, find a basis for the nullspace of N^T, as in Proposition 11.9. Then write $\vec{\mathbf{w}}$ in terms of this basis:

$$\mathbf{nullspace}(N^T) = \mathbf{span}\left\{ \begin{pmatrix} -2\\2\\0\\-1\\1 \end{pmatrix}, \begin{pmatrix} 1\\-1\\1\\0\\0 \end{pmatrix} \right\},$$

$$\vec{\mathbf{w}} = c_1 \begin{pmatrix} -2\\2\\0\\-1\\1 \end{pmatrix} + c_2 \begin{pmatrix} 1\\-1\\1\\0\\0 \end{pmatrix}.$$

[9] Zero columns in P allow rapid testing of $AP = PJ$.

Next, we force $\vec{\mathbf{w}}$ to belong to the nullspace of $N^m = N^2$. Equation

$$N^2\vec{\mathbf{w}} = \begin{pmatrix} 0 \\ 10c_1 - 4c_2 \\ 10c_1 - 4c_2 \\ 10c_1 - 4c_2 \\ -10c_1 + 4c_2 \end{pmatrix} = \vec{\mathbf{0}}$$

holds if and only if $5c_1 - 2c_2 = 0$. Choose $c_1 = 2$, $c_2 = 5$ to make it so, then compute

$$\vec{\mathbf{w}} = 2\begin{pmatrix} -2 \\ 2 \\ 0 \\ -1 \\ 1 \end{pmatrix} + 5\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 5 \\ -2 \\ 2 \end{pmatrix}, \quad N\vec{\mathbf{w}} = \begin{pmatrix} 7 \\ 7 \\ 0 \\ 0 \\ 0 \end{pmatrix} \neq \vec{\mathbf{0}}$$

Conclusions: (1) equation $N\vec{\mathbf{x}} = \vec{\mathbf{w}}$ is insolvable for $\vec{\mathbf{x}}$, (2) $N\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$ and (3) $N^2\vec{\mathbf{w}} = \vec{\mathbf{0}}$. Define

$$\vec{\mathbf{v}}_4 = N\vec{\mathbf{w}} = \begin{pmatrix} 7 \\ 7 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{v}}_5 = \vec{\mathbf{w}} = \begin{pmatrix} 1 \\ -1 \\ 5 \\ -2 \\ 2 \end{pmatrix}$$

Then

$$P = \left\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 | \vec{\mathbf{v}}_4 | \vec{\mathbf{v}}_5 \right\rangle = \begin{pmatrix} 0 & 1 & 1 & 7 & 1 \\ -2 & 2 & 0 & 7 & -1 \\ -2 & 1 & 0 & 0 & 5 \\ -2 & -1 & 0 & 0 & -2 \\ 2 & 3 & 0 & 0 & 2 \end{pmatrix}$$

Matrix multiply verifies $AP = PJ$, which means P is a matrix of generalized eigenvectors for A. The answer for P is not unique, as illustrated by `maple`'s answer in equation (5). ■

Direct Sum Decomposition

The **generalized eigenspace** of eigenvalue λ of an $n \times n$ matrix A is the subspace **kernel**$((A - \lambda I)^p)$ where $p =$ **AlgMult**(λ). We state without proof the main result for generalized eigenspace bases, because details appear in the exercises. A proof is included for the direct sum decomposition, even though Putzer's spectral theory independently produces the same decomposition.

Theorem 11.41 (Generalized Eigenspace Basis)
The subspace **kernel**$((A - \lambda I)^k)$, $k =$ **AlgMult**(λ) has a k-dimensional basis whose vectors are the columns of P corresponding to blocks $B(\lambda, j)$ of J, in Jordan decomposition $A = PJP^{-1}$.

Theorem 11.42 (Direct Sum Decomposition)
Given $n \times n$ matrix A with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, let $n_1 = \textbf{AlgMult}(\lambda_i)$, $\ldots, n_k = \textbf{AlgMult}(\lambda_k)$. Then A induces a direct sum decomposition

$$\mathcal{C}^n = \textbf{kernel}((A - \lambda_1 I)^{n_1} \oplus \cdots \oplus \textbf{kernel}((A - \lambda_k I)^{n_k}.$$

This equation means that each complex vector $\vec{\mathbf{x}}$ in $\mathcal{C}^n$ can be uniquely written as

$$\vec{\mathbf{x}} = \vec{\mathbf{x}}_1 + \cdots + \vec{\mathbf{x}}_k$$

where each $\vec{\mathbf{x}}_i$ belongs to $\textbf{kernel}\left((A - \lambda_i)^{n_i}\right)$, $i = 1, \ldots, k$.

Proof: The previous theorem implies there is a basis of dimension n_i for eigenspace $E_i \equiv \textbf{kernel}((A - \lambda_i I)^{n_i})$, $i = 1, \ldots, k$. Because $n_1 + \cdots + n_k = n$, then there are n vectors in the union of these bases. The independence test for these n vectors amounts to showing that $\vec{\mathbf{x}}_1 + \cdots + \vec{\mathbf{x}}_k = \vec{\mathbf{0}}$ with $\vec{\mathbf{x}}_i$ in E_i, $i = 1, \ldots, k$, implies all $\vec{\mathbf{x}}_i = \vec{\mathbf{0}}$. This will be true provided $E_i \cap E_j = \{\vec{\mathbf{0}}\}$ for $i \neq j$.

Let's assume a Jordan decomposition $A = PJP^{-1}$. If $\vec{\mathbf{x}}$ is common to both E_i and E_j, then basis expansion of $\vec{\mathbf{x}}$ in both subspaces implies a linear combination of the columns of P is zero, which by independence of the columns of P implies $\vec{\mathbf{x}} = \vec{\mathbf{0}}$. ■

Remark. If A is real with real eigenvalues, then generalized eigenspaces have real bases and the decomposition $\vec{\mathbf{x}} = \vec{\mathbf{x}}_1 + \cdots + \vec{\mathbf{x}}_k$ uses real vectors.

The Real Jordan Form of A

Given a real matrix A, generalized eigenanalysis seeks to find a *real* invertible matrix $\mathcal{P}$ and a *real* upper triangular block matrix $\mathcal{J}$ such that $A = \mathcal{P}\mathcal{J}\mathcal{P}^{-1}$.

If λ is a real eigenvalue of A, then a **real Jordan block** is a matrix

$$B = \textbf{diag}(\lambda, \ldots, \lambda) + N, \quad N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

If $\lambda = a + ib$ is a complex eigenvalue of A, then symbols λ, 1 and 0 are replaced respectively by 2×2 real matrices $\Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\mathcal{I} = \textbf{diag}(1, 1)$ and $\mathcal{O} = \textbf{diag}(0, 0)$. The corresponding $2m \times 2m$ real Jordan block matrix is given by the formula

$$B = \textbf{diag}(\Lambda, \ldots, \Lambda) + \mathcal{N}, \quad \mathcal{N} = \begin{pmatrix} \mathcal{O} & \mathcal{I} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \end{pmatrix}.$$

Real Jordan Decomposition

The ideas are best communicated by example. Let

$$A = \begin{pmatrix} -3 & 4 & 1 \\ 0 & -4 & 10 \\ 0 & -5 & 6 \end{pmatrix}.$$

The eigenpairs are

$$\left(-3, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right), \quad \left(1+5i, \begin{pmatrix} -i \\ 1-i \\ 1 \end{pmatrix}\right), \quad \left(1-5i, \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix}\right).$$

The complex Jordan decomposition of matrix A is $A\mathcal{P} = \mathcal{P}\mathcal{J}$ where

$$\mathcal{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+5i & 0 \\ 0 & 0 & 1-5i \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 1 & -i & i \\ 0 & 1-i & 1+i \\ 0 & 1 & 1 \end{pmatrix}$$

The **Real Jordan Decomposition** of matrix A is $AP = PJ$ where

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & -1 & 5 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

The rules:

Replace $\begin{pmatrix} 1+5i & 0 \\ 0 & 1-5i \end{pmatrix}$ by $\begin{pmatrix} 1 & 5 \\ -1 & 5 \end{pmatrix}$

Replace the pair of complex eigenvector columns by the real and imaginary parts of the first eigenvector (the second is not used):

$$\begin{pmatrix} -i \\ 1-i \\ 1 \end{pmatrix}, \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix} \text{ replaced by } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}.$$

The method for $n \times n$ real matrices with n eigenpairs is similar. Each pair of complex conjugate eigenvalues $a + ib$, $a - ib$ produces in J a real Jordan block $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. The corresponding complex eigenvector pair $\vec{\mathbf{u}} + i\vec{\mathbf{v}}$, $\vec{\mathbf{u}} - i\vec{\mathbf{v}}$ is accounted for by inserting into P the real and imaginary parts $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$. A real eigenpair $(\lambda, \vec{\mathbf{x}})$ creates λ on the diagonal of J and real eigenvector $\vec{\mathbf{x}}$ is copied to the corresponding column of P.

Computing Real Exponential Matrices

Discussed here are methods for finding a real exponential matrix e^{At} when A is real. Two formulas are given, one for a real eigenvalue and one for a complex eigenvalue. These formulas supplement the spectral formulas given earlier in the text.

Nilpotent Matrices

A matrix N which satisfies $N^p = 0$ for some integer p is called **nilpotent**. The least integer p for which $N^p = 0$ is called the **nilpotency** of N. A nilpotent matrix N has a finite exponential series:

$$e^{Nt} = I + Nt + N^2\frac{t^2}{2!} + \cdots + N^{p-1}\frac{t^{p-1}}{(p-1)!}.$$

If $N = B(\lambda, p) - \lambda I$, then the finite sum has a splendidly simple expression due to $e^{\lambda t\, I + Nt} = e^{\lambda t}e^{Nt}$. These remarks motivate the following result.

Theorem 11.43 (Exponential of a Jordan Block Matrix)
If λ is real and

$$B = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix} \quad (p \times p \text{ matrix})$$

then

$$e^{Bt} = e^{\lambda t}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{p-2}}{(p-2)!} & \frac{t^{p-1}}{(p-1)!} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The equality also holds if λ is a complex number, in which case both sides of the equation are complex.

Proof: Let matrix $\Phi(t)$ be either the left side or the right side of the matrix equality. A computation shows that $\Phi'(t) = B\Phi(t)$, $\Phi(0) = I$. Apply uniqueness in the Picard-Lindelöf theorem. ■

Real Exponentials for Complex λ

A Jordan decomposition $A = \mathcal{P}J\mathcal{P}^{-1}$ in which A has only real eigenvalues has real generalized eigenvectors appearing as columns in the matrix $\mathcal{P}$, in the order matching Jordan blocks in J. When $\lambda = a + ib$ is complex, $b > 0$, then the real and imaginary parts of each generalized eigenvector are entered pairwise into $\mathcal{P}$; the conjugate eigenvalue $\overline{\lambda} = a - ib$ is skipped. The complex entry along the diagonal of J and the ones on the superdiagonal of J are each changed into a 2×2 matrix under the correspondence

$$a + ib \leftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

The result is a *real* matrix $\mathcal{P}$ and a *real* block upper triangular matrix $\mathcal{J}$ which satisfy $A = \mathcal{P}\mathcal{J}\mathcal{P}^{-1}$.

Theorem 11.44 (Real Block Diagonal Matrix, Eigenvalue $a + ib$)

Let $\Lambda = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $\mathcal{I} = \textbf{diag}(1,1)$ and $\mathcal{O} = \textbf{diag}(0,0)$. Consider a real Jordan block matrix of dimension $2m \times 2m$ given by the formula

$$B = \begin{pmatrix} \Lambda & \mathcal{I} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \Lambda & \mathcal{I} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \Lambda \end{pmatrix}.$$

If $\mathcal{R} = \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}$, then

$$e^{Bt} = e^{at} \begin{pmatrix} \mathcal{R} & t\mathcal{R} & \frac{t^2}{2}\mathcal{R} & \cdots & \frac{t^{m-2}}{(m-2)!}\mathcal{R} & \frac{t^{m-1}}{(m-1)!}\mathcal{R} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{R} & t\mathcal{R} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{O} & \mathcal{R} \end{pmatrix}.$$

Proof: Details are similar to the proof of Theorem 11.43. ■

Solving $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

The solution $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}(0)$ must be real if A is real. The real solution can be expressed as $\vec{\mathbf{x}}(t) = \mathcal{P}\vec{\mathbf{y}}(t)$ where $\vec{\mathbf{y}}'(t) = \mathcal{J}\vec{\mathbf{y}}(t)$ and $\mathcal{J}$ is a real Jordan form of A, containing real Jordan blocks $B_1, \ldots, B_k$ down its diagonal. Theorems above provide explicit formulas for the block matrices $e^{B_i t}$ in the relation

$$e^{\mathcal{J}t} = \textbf{diag}\left(e^{B_1 t}, \ldots, e^{B_k t}\right).$$

The resulting formula

$$\vec{\mathbf{x}}(t) = \mathcal{P}e^{\mathcal{J}t}\mathcal{P}^{-1}\vec{\mathbf{x}}(0)$$

contains only real numbers, real exponentials, plus sine and cosine terms, which are possibly multiplied by polynomials in t.

Numerical Instability

The matrix $A = \begin{pmatrix} 1 & 1 \\ \varepsilon & 1 \end{pmatrix}$ has two possible Jordan forms

$$J(\varepsilon) = \begin{cases} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \varepsilon = 0, \\ \\ \begin{pmatrix} 1+\sqrt{\varepsilon} & 0 \\ 0 & 1-\sqrt{\varepsilon} \end{pmatrix} & \varepsilon > 0. \end{cases}$$

When $\varepsilon \approx 0$, then numerical algorithms become unstable, unable to lock onto the correct Jordan form. Briefly, $\lim_{\varepsilon \to 0} J(\varepsilon) \neq J(0)$.

Details and Proofs

Proof of Theorem 11.38 (Jordan Decomposition) The result holds by default for 1×1 matrices. Assume the result holds for all $k\times k$ matrices, $k<n$. The proof proceeds by induction on n.

The induction assumes, for any $k\times k$ matrix A, that there is a Jordan decomposition $A = PJP^{-1}$. Then the columns of P satisfy Jordan chain relations

$$A\vec{\mathbf{x}}_i^j = \lambda_i\vec{\mathbf{x}}_i^j + \vec{\mathbf{x}}_i^{j-1}, \quad j>1, \quad A\vec{\mathbf{x}}_i^1 = \lambda_i\vec{\mathbf{x}}_i^1.$$

Conversely, if the Jordan chain relations are satisfied for k independent vectors $\{\vec{\mathbf{x}}_i^j\}$, then the vectors form the columns of an invertible matrix P such that $A = PJP^{-1}$ with J in Jordan form. The induction step centers upon producing the chain relations and proving that the n vectors are independent.

Let B be $n\times n$ and λ_0 an eigenvalue of B. The Jordan chain relations hold for $A=B$ if and only if they hold for $A = B-\lambda_0 I$. Without loss of generality, we can assume 0 is an eigenvalue of B.

Because B has 0 as an eigenvalue, then inequalities $p = \dim(\mathbf{kernel}(B)) > 0$ and $k = \dim(\mathbf{Image}(B)) < n$ hold, with $p+k=n$. If $k=0$, then $B=\mathbf{0}$, which is a Jordan form, and there is nothing to prove. Assume henceforth p and k positive. Let $S = \Big\langle\, \mathbf{col}(B,i_1)|\cdots|\,\mathbf{col}(B,i_k)\,\Big\rangle$ denote the matrix of pivot columns $i_1,\ldots,i_k$ of B. The pivot columns are known to span $\mathbf{Image}(B)$. Let A be the $k\times k$ basis representation matrix defined by the equation $BS = SA$, or equivalently, $B\,\mathbf{col}(S,j) = \sum_{i=1}^k a_{ij}\,\mathbf{col}(S,i)$. The induction hypothesis applied to A implies there is a basis of k-vectors satisfying Jordan chain relations

$$A\vec{\mathbf{x}}_i^j = \lambda_i\vec{\mathbf{x}}_i^j + \vec{\mathbf{x}}_i^{j-1}, \quad j>1, \quad A\vec{\mathbf{x}}_i^1 = \lambda_i\vec{\mathbf{x}}_i^1.$$

The values λ_i, $i=1,\ldots,p$, are the distinct eigenvalues of A. Apply S to these equations to obtain for the n-vectors $\vec{\mathbf{y}}_i^j = S\vec{\mathbf{x}}_i^j$ the Jordan chain relations

$$B\vec{\mathbf{y}}_i^j = \lambda_i\vec{\mathbf{y}}_i^j + \vec{\mathbf{y}}_i^{j-1}, \quad j>1, \quad B\vec{\mathbf{y}}_i^1 = \lambda_i\vec{\mathbf{y}}_i^1.$$

Because S has independent columns and the k-vectors $\vec{\mathbf{x}}_i^j$ are independent, then the n-vectors $\vec{\mathbf{y}}_i^j$ are independent.

The **plan** is to isolate the chains for eigenvalue zero, then extend these chains by one vector. Then 1-chains will be constructed from eigenpairs for eigenvalue zero to make n generalized eigenvectors.

Suppose q values of i satisfy $\lambda_i = 0$. We allow $q=0$. For simplicity, assume such values i are $i=1,\ldots,q$. The key formula $\vec{\mathbf{y}}_i^j = S\vec{\mathbf{x}}_i^j$ implies $\vec{\mathbf{y}}_i^j$ is in $\mathbf{Image}(B)$, while $B\vec{\mathbf{y}}_i^1 = \lambda_i\vec{\mathbf{y}}_i^1$ implies $\vec{\mathbf{y}}_1^1,\ldots,\vec{\mathbf{y}}_q^1$ are in $\mathbf{kernel}(B)$. Each eigenvector $\vec{\mathbf{y}}_i^1$ starts a Jordan chain ending in $\vec{\mathbf{y}}_i^{m(i)}$. Then[10] the equation $B\vec{\mathbf{u}} = \vec{\mathbf{y}}_i^{m(i)}$ has an n-vector solution $\vec{\mathbf{u}}$. We label $\vec{\mathbf{u}} = \vec{\mathbf{y}}_i^{m(i)+1}$. Because $\lambda_i = 0$, then $B\vec{\mathbf{u}} = \lambda_i\vec{\mathbf{u}} + \vec{\mathbf{y}}_i^{m(i)}$ results in an extended Jordan chain

$$\begin{array}{lcll}
B\vec{\mathbf{y}}_i^1 & = & \lambda_i\vec{\mathbf{y}}_i^1 & \\
B\vec{\mathbf{y}}_i^2 & = & \lambda_i\vec{\mathbf{y}}_i^2 & +\ \vec{\mathbf{y}}_i^1 \\
 & \vdots & & \\
B\vec{\mathbf{y}}_i^{m(i)} & = & \lambda_i\vec{\mathbf{y}}_i^{m(i)} & +\ \vec{\mathbf{y}}_i^{m(i)-1} \\
B\vec{\mathbf{y}}_i^{m(i)+1} & = & \lambda_i\vec{\mathbf{y}}_i^{m(i)+1} & +\ \vec{\mathbf{y}}_i^{m(i)}
\end{array}$$

[10] The n-vector $\vec{\mathbf{u}}$ is constructed by setting $\vec{\mathbf{u}} = \vec{\mathbf{0}}$, then copy components of k-vector $\vec{\mathbf{x}}_i^{m(i)}$ into pivot locations: $\mathbf{row}(\vec{\mathbf{u}}, i_j) = \mathbf{row}(\vec{\mathbf{x}}_i^{m(i)}, j)$, $j=1,\ldots,k$.

Extend the independent set $\{\vec{\mathbf{y}}_i^1\}_{i=1}^q$ to a basis of **kernel**(B) by adding $s = n - k - q$ additional independent vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_s$. This basis consists of eigenvectors of B for eigenvalue 0. Then the set of n vectors $\vec{\mathbf{v}}_r$, $\vec{\mathbf{y}}_i^j$ for $1 \le r \le s$, $1 \le i \le p$, $1 \le j \le m(i)+1$ consists of eigenvectors of B and vectors that satisfy Jordan chain relations. These vectors are columns of a matrix $\mathcal{P}$ that satisfies $B\mathcal{P} = \mathcal{P}\mathcal{J}$ where $\mathcal{J}$ is a Jordan form.

To prove $\mathcal{P}$ invertible, assume a linear combination of the columns of $\mathcal{P}$ is zero:

$$\sum_{i=q+1}^{p} \sum_{j=1}^{m(i)} b_i^j \vec{\mathbf{y}}_i^j + \sum_{i=1}^{q} \sum_{j=1}^{m(i)+1} b_i^j \vec{\mathbf{y}}_i^j + \sum_{i=1}^{s} c_i \vec{\mathbf{v}}_i = \vec{\mathbf{0}}.$$

Apply B to this equation. Because $B\vec{\mathbf{w}} = \vec{\mathbf{0}}$ for any $\vec{\mathbf{w}}$ in **kernel**(B), then

$$\sum_{i=q+1}^{p} \sum_{j=1}^{m(i)} b_i^j B\vec{\mathbf{y}}_i^j + \sum_{i=1}^{q} \sum_{j=2}^{m(i)+1} b_i^j B\vec{\mathbf{y}}_i^j = \vec{\mathbf{0}}.$$

The Jordan chain relations imply that the k vectors $B\vec{\mathbf{y}}_i^j$ in the linear combination consist of $\lambda_i \vec{\mathbf{y}}_i^j + \vec{\mathbf{y}}_i^{j-1}$, $\lambda_i \vec{\mathbf{y}}_i^1$, $i = q+1, \ldots, p$, $j = 2, \ldots, m(i)$, plus the vectors $\vec{\mathbf{y}}_i^j$, $1 \le i \le q$, $1 \le j \le m(i)$. Independence of the original k vectors $\{\vec{\mathbf{y}}_i^j\}$ plus $\lambda_i \ne 0$ for $i > q$ implies this new set is independent. Then all coefficients in the linear combination are zero.

The first linear combination then reduces to $\sum_{i=1}^q b_i^1 \vec{\mathbf{y}}_i^1 + \sum_{i=1}^s c_i \vec{\mathbf{v}}_i = \vec{\mathbf{0}}$. Independence of the constructed basis for **kernel**(B) implies $b_i^1 = 0$ for $1 \le i \le q$ and $c_i = 0$ for $1 \le i \le s$. Therefore, the columns of $\mathcal{P}$ are independent. The induction is complete. ■

Exercises 11.6

Jordan block definition. Write out the Jordan form matrix explicitly.

1. **diag**$(B(7,2), B(5,3))$

Answer: $\begin{pmatrix} 7&1&0&0&0\\ 0&7&0&0&0\\ 0&0&5&1&0\\ 0&0&0&5&1\\ 0&0&0&0&5 \end{pmatrix}$

2. **diag**$(B(0,2), B(4,3))$

3. **diag**$(B(-1,1), B(-1,2), B(5,3))$

4. **diag**$(B(1,1), B(5,2), B(5,3))$

Jordan form definition. Which are Jordan forms and which are not? Explain.

5. $\begin{pmatrix} 0&1&0&0&0\\ 0&0&0&0&0\\ 0&0&5&1&0\\ 0&0&0&5&1\\ 0&0&0&0&5 \end{pmatrix}$

6. $\begin{pmatrix} 5&1&0&0\\ 0&5&0&0\\ 0&0&5&1\\ 0&0&0&5 \end{pmatrix}$

7. $\begin{pmatrix} 1&0&0&0\\ 0&7&0&0\\ 0&0&1&0\\ 0&0&5&1 \end{pmatrix}$

8. $\begin{pmatrix} 5&1&0&0&0\\ 0&5&0&0&0\\ 0&0&5&1&0\\ 0&0&0&5&0\\ 0&0&0&0&5 \end{pmatrix}$

Decoding $A = PJP^{-1}$. Decode $A = PJP^{-1}$ in each case, displaying explicitly the Jordan chain relations and their solutions.

9. $A = \begin{pmatrix} 4&8&0&0&-8\\ 0&4&0&0&0\\ 2&8&2&0&-8\\ 0&20&0&2&-12\\ 0&8&0&0&-4 \end{pmatrix}$, $J = \mathbf{diag}(-4, 2, 2, 4, 4)$

10. $A = \begin{pmatrix} -4&-4&-12&12&4\\ 0&0&0&0&0\\ -8&4&-12&16&0\\ -8&4&-16&20&0\\ 0&0&-4&4&0 \end{pmatrix}$,

$J = \textbf{diag}(-4, 4, 4, 0, 0)$

Geometric and algebraic multiplicity.

Determine $\textbf{GeoMult}(\lambda)$ and $\textbf{AlgMult}(\lambda)$.

11. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$, $\lambda = 4$

12. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$, $\lambda = 4$

Generalized eigenvectors.

Find all generalized eigenvectors and represent $A = PJP^{-1}$. Check the answer in a computer algebra system.

13. $A = \begin{pmatrix} 4 & 8 & 0 & 0 & -8 \\ 0 & 4 & 0 & 0 & 0 \\ 2 & 8 & 2 & 0 & -8 \\ 0 & 20 & 0 & 2 & -12 \\ 0 & 8 & 0 & 0 & -4 \end{pmatrix}$,

Answer: $J = \textbf{diag}(-4, 4, 4, 2, 2)$,

$P = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

14. $A = \begin{pmatrix} -4 & -4 & -12 & 12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ -8 & 4 & -12 & 16 & 0 \\ -8 & 4 & -16 & 20 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{pmatrix}$,

Answer: $J = \textbf{diag}(-4, 4, 4, 0, 0)$,

$P = \begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 & -1 \\ 1 & -1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \end{pmatrix}$

15. $A = \begin{pmatrix} 0 & 2 & -2 & -2 \\ 2 & 0 & -2 & -4 \\ 2 & 2 & -4 & -2 \\ 0 & 0 & 0 & -4 \end{pmatrix}$,

Ans: $J = \textbf{diag}(0, -4, -2, -2)$,

$P = \begin{pmatrix} 1 & 0 & 1 & -1 \\ 1 & 1 & -4 & 0 \\ 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$

16. $A = \begin{pmatrix} -2 & 2 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(2, 2, B(2, 3))$,

$P = \begin{pmatrix} 1 & 1 & 1 & -2 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

17. $A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(B(2, 3), B(2, 2))$,

$P = \begin{pmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 0 & 2 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

18. $A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 \\ 1 & 3 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$,

Ans: $J = \textbf{diag}(B(2, 4), 2)$,

$P = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Number of Jordan Blocks.

Outlined here is the derivation of

$$s(j) = 2k(j-1) - k(j-2) - k(j).$$

Definitions:

- $s(j)$= number of blocks $B(\lambda, j)$
- $N = A - \lambda I$
- $k(j) = \dim(\textbf{kernel}(N^j))$
- $L_j = \textbf{kernel}(N^{j-1})^\perp$ relative to $\textbf{kernel}(N^j)$
- $\ell(j) = \dim(L_j)$
- p minimizes $\textbf{kernel}(N^p) = \textbf{kernel}(N^{p+1})$

19. Verify $k(j) \le k(j+1)$ from

$$\textbf{kernel}(N^j) \subset \textbf{kernel}(N^{j+1}).$$

20. Verify the direct sum formula

$$\textbf{kernel}(N^j) = \textbf{kernel}(N^{j-1}) \oplus L_j.$$

Then $k(j) = k(j-1) + \ell(j)$.

21. Given $N^m\vec{\mathbf{w}} = \vec{\mathbf{0}}$, $N^{m-1}\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, define $\vec{\mathbf{v}}_i = N^{m-i}\vec{\mathbf{w}}$, $i = 1,\ldots,m$. Prove $\{\vec{\mathbf{v}}_1,\ldots,\vec{\mathbf{v}}_m\}$ is independent and they satisfy Jordan chain relations $N\vec{\mathbf{v}}_1 = \vec{\mathbf{0}}$, $N\vec{\mathbf{v}}_{i+i} = \vec{\mathbf{v}}_i$.

22. A block $B(\lambda,p)$ corresponds to a Jordan chain $\vec{\mathbf{v}}_1$, $\ldots$, $\vec{\mathbf{v}}_p$ constructed from the Jordan decomposition. Use $N^{p-1}\vec{\mathbf{v}}_p = \vec{\mathbf{v}}_1$ and $\textbf{kernel}(N^p) = \textbf{kernel}(N^{p+1})$ to show that the number of such blocks $B(\lambda,p)$ is $\ell(p)$. Then for $p > 1$, $s(p) = k(p) - k(p-1)$.

23. Show that $\ell(j-1) - \ell(j)$ is the number of blocks $B(\lambda,j)$ for $2 < j < p$. Then

$$s(j) = 2k(j-1) - k(j) - k(j-2).$$

24. Test the formulas above on the special matrices

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,1), B(\lambda,1)),$$

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,2), B(\lambda,3)),$$

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,3), B(\lambda,3)),$$

$$A = \textbf{diag}(B(\lambda,1), B(\lambda,1), B(\lambda,3)),$$

Computing Jordan m-chains.

Find the Jordan m-chain formulas for the given eigenvalue. Then solve them to find the generalized eigenvectors.

25. $A = \begin{pmatrix} 1&0&1&0&1\\ 0&1&0&1&1\\ 0&0&1&0&0\\ 0&0&0&1&1\\ 0&0&0&0&1 \end{pmatrix}$

26. $A = \begin{pmatrix} 2&0&0&1&0\\ 1&3&-1&0&0\\ 1&1&1&0&0\\ 0&0&0&2&1\\ 0&0&0&0&2 \end{pmatrix}$, $\lambda = 2$

Generalized Eigenspace Basis.

Let A be $n\times n$ with distinct eigenvalues λ_i, $n_i = \textbf{AlgMult}(\lambda_i)$ and $E_i = \textbf{kernel}((A-\lambda_i I)^{n_i})$, $i = 1,\ldots,k$. Assume a Jordan decomposition $A = PJP^{-1}$.

27. Let Jordan block $B(\lambda,m)$ appear in J. Prove that a Jordan chain corresponding to this block is a set of m independent columns of P.

28. Let $\mathcal{B}_\lambda$ be the union of all columns of P originating from Jordan chains associated with Jordan blocks $B(\lambda,j)$. Prove that $\mathcal{B}_\lambda$ is an independent set.

29. Verify that $\mathcal{B}_\lambda$ has $\textbf{AlgMult}(\lambda)$ basis elements.

30. Prove that $E_i = \textbf{span}(\mathcal{B}_{\lambda_i})$ and $\dim(E_i) = n_i$, $i = 1,\ldots,k$.

Direct Sum Decomposition.

31. Let $A = \begin{pmatrix} 2&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$. Let $\lambda = 2$. Compute $k = \textbf{AlgMult}(\lambda)$ and a basis of generalized eigenvectors for the subspace $\textbf{kernel}((A-\lambda I)^k)$.

32. Let $A = \begin{pmatrix} 2&1&0&0\\ 0&2&1&0\\ 0&0&2&0\\ 0&0&0&4 \end{pmatrix}$, $\vec{\mathbf{y}} = \begin{pmatrix} 2\\0\\1\\9 \end{pmatrix}$. Find $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2$ in decomposition $\vec{\mathbf{y}} = \vec{\mathbf{x}}_1 + \vec{\mathbf{x}}_2$ in Theorem 11.42.

Exponential Matrices.

Compute the exponential matrix e^{At} on paper. Check the answer using `maple`.

33. $A = \begin{pmatrix} 2&0&0\\ 0&3&0\\ 0&0&0 \end{pmatrix}$

34. $A = \begin{pmatrix} 2&1&0\\ 0&2&0\\ 0&0&4 \end{pmatrix}$

Nilpotent matrices.

Find the nilpotency of N.

35. $N = \begin{pmatrix} 0&1&0\\ 0&0&0\\ 0&0&0 \end{pmatrix}$

36. $N = \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$

Real Jordan Decomposition

Find Jordan decomposition $A = PJP^{-1}$ where J and P are real matrices.

37. $A = \begin{pmatrix} -2 & 6 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}$. Answer:

$\lambda = -2, 4 \pm i$,

$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & -1 & 4 \end{pmatrix}$, $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

38. $A = \begin{pmatrix} -31 & -10 & 18 \\ -15 & -5 & 10 \\ -54 & -20 & 32 \end{pmatrix}$. Answer:

$\lambda = -4, \pm 5i$

$J = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 0 & 5 \\ 0 & -5 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 1 & -1 \\ 3 & 4 & 0 \end{pmatrix}$

Solving $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$

Solve for $\vec{\mathbf{x}}$ in the differential equation.

39. $\vec{\mathbf{x}}' = \begin{pmatrix} -2 & 6 & -1 \\ 0 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} \vec{\mathbf{x}}$.

40. $\vec{\mathbf{x}}' = \begin{pmatrix} -31 & -10 & 18 \\ -15 & -5 & 10 \\ -54 & -20 & 32 \end{pmatrix} \vec{\mathbf{x}}$.

Numerical Instability

Show directly that Jordan form J of A satisfies $\lim_{\epsilon \to 0+} J(\epsilon) \neq J(0)$.

41. $A = \begin{pmatrix} 1 & 1 \\ \epsilon & 1 \end{pmatrix}$

42. $A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & \epsilon & 1 \\ 0 & 0 & 0 \end{pmatrix}$

11.7 Nonhomogeneous Linear Systems

Variation of Parameters

The **Method of Variation of Parameters** solves a linear nonhomogeneous system

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t).$$

Historically, it is substitution method which solves the nonhomogeneous system using a trial solution of the form

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0(t).$$

The vector function $\vec{\mathbf{x}}_0(t)$ is to be determined. The method is imagined to originate by varying $\vec{\mathbf{x}}_0$ in the general solution $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0$ of the linear homogenous system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. The names coined from this idea are *variation of parameters* and *variation of constants*.

Modern use of variation of parameters is through a formula, memorized for routine use.

Theorem 11.45 (Variation of Parameters: Constant Linear System)
Let A be a constant $n \times n$ matrix and $\vec{\mathbf{F}}(t)$ a continuous function near $t = t_0$. The unique solution $\vec{\mathbf{x}}(t)$ of the matrix initial value problem

$$\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0,$$

is given by the **Variation of Parameters formula**

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0 + e^{At}\int_{t_0}^{t} e^{-sA}\vec{\mathbf{F}}(s)ds. \tag{1}$$

Proof of Theorem 11.45. Define

$$\vec{\mathbf{u}}(t) = \vec{\mathbf{x}}_0 + \int_{t_0}^{t} e^{-sA}\vec{\mathbf{F}}(s)ds.$$

To show (1) holds, we must verify $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{u}}(t)$. First, the function $\vec{\mathbf{u}}(t)$ is differentiable with continuous derivative $e^{-tA}\vec{\mathbf{F}}(t)$, by the fundamental theorem of calculus applied to each of its components. The product rule of calculus applies to give

$$\begin{aligned}
\vec{\mathbf{x}}'(t) &= \left(e^{At}\right)'\vec{\mathbf{u}}(t) + e^{At}\vec{\mathbf{u}}'(t) \\
&= Ae^{At}\vec{\mathbf{u}}(t) + e^{At}e^{-At}\vec{\mathbf{F}}(t) \\
&= A\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t).
\end{aligned}$$

Therefore, $\vec{\mathbf{x}}(t)$ satisfies the differential equation $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$. Because $\vec{\mathbf{u}}(t_0) = \vec{\mathbf{x}}_0$, then $\vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$, which shows the initial condition is also satisfied. ■

Theorem 11.46 (Variation of Parameters: General Linear System)
Let $A(t)$ be an $n\times n$ matrix and $\vec{\mathbf{F}}(t)$ a vector function, both with continuous entries near $t=t_0$. Let $\Phi(t)$ be the $n\times n$ matrix solution of $\Phi'(t)=A(t)\Phi(t)$, $\Phi(t_0)=I$, established by the Picard-Lindelöf Theorem.

Then the unique solution $\vec{\mathbf{x}}(t)$ of the matrix initial value problem

$$\vec{\mathbf{x}}'(t)=A(t)\vec{\mathbf{x}}(t)+\vec{\mathbf{F}}(t), \quad \vec{\mathbf{x}}(t_0)=\vec{\mathbf{x}}_0$$

is given by

(2) $$\vec{\mathbf{x}}(t)=\Phi(t)\vec{\mathbf{x}}_0+\Phi(t)\int_{t_0}^{t}\Phi^{-1}(s)\vec{\mathbf{F}}(s)ds.$$

Proof of Theorem 11.46. Define

$$\vec{\mathbf{u}}(t)=\vec{\mathbf{x}}_0+\int_{t_0}^{t}\Phi^{-1}(s)\vec{\mathbf{F}}(s)ds.$$

Equation (2) holds provided $\vec{\mathbf{x}}(t)=\Phi(t)\vec{\mathbf{u}}(t)$. First, the function $\vec{\mathbf{u}}(t)$ is differentiable with continuous derivative $\Phi(t)\vec{\mathbf{F}}(t)$, by the fundamental theorem of calculus applied to each of its components. The product rule of calculus implies

$$\begin{array}{rcl}\vec{\mathbf{x}}'(t) & = & (\Phi(t))'\,\vec{\mathbf{u}}(t)+\Phi(t)\vec{\mathbf{u}}'(t)\\ & = & A(t)\Phi(t)\vec{\mathbf{u}}(t)+\Phi(t)\Phi^{-1}(t)\vec{\mathbf{F}}(t)\\ & = & A(t)\vec{\mathbf{x}}(t)+\vec{\mathbf{F}}(t).\end{array}$$

Therefore, $\vec{\mathbf{x}}(t)$ satisfies the differential equation $\vec{\mathbf{x}}'=A(t)\vec{\mathbf{x}}+\vec{\mathbf{F}}(t)$. Because $\vec{\mathbf{u}}(t_0)=\vec{\mathbf{x}}_0$, then $\vec{\mathbf{x}}(t_0)=\vec{\mathbf{x}}_0$ and the initial condition is satisfied. ■

Example 11.13 (Variation of Parameters: 2×2 System)

Let $A=\begin{pmatrix}4&0\\0&5\end{pmatrix}$ and $\vec{\mathbf{F}}(t)=e^t\begin{pmatrix}2\\1\end{pmatrix}$. Solve $\vec{\mathbf{x}}'=A\vec{\mathbf{x}}+\vec{\mathbf{F}}(t)$ using the formula $\vec{\mathbf{x}}_p=\int_0^t e^{A(t-s)}\vec{\mathbf{F}}(s)ds$ and find the shortest expression

$$\vec{\mathbf{x}}_p(t)=\begin{pmatrix}-\frac{2}{3}\,\mathrm{e}^t\\ -\frac{1}{4}\,\mathrm{e}^t\end{pmatrix}$$

Details for Example 11.13: Because A is diagonal, then $e^{At}=\begin{pmatrix}e^{4t}&0\\0&e^{5t}\end{pmatrix}$. The integration problem:

$$\begin{array}{rcl}\vec{\mathbf{x}}_p(t) & = & \displaystyle\int_0^t \mathrm{e}^{A(t-s)}\vec{\mathbf{F}}(s)ds\\ & = & \displaystyle\int_0^t\begin{pmatrix}\mathrm{e}^{4t-4s}&0\\0&\mathrm{e}^{5t-5s}\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\mathrm{e}^s\,ds\\ & = & \begin{pmatrix}\frac{2}{3}\,\mathrm{e}^{4\,t}-\frac{2}{3}\,\mathrm{e}^t\\ \frac{1}{4}\,\mathrm{e}^{5\,t}-\frac{1}{4}\,\mathrm{e}^t\end{pmatrix}\end{array}$$

The integration was by CAS `maple`:

```
with(LinearAlgebra):A:=Matrix([[4,0],[0,5]]);
V:=t->MatrixExponential(A,t);
F:=t->Vector([2*exp(t),1*exp(t)]);
k:=(t,s)->V(t). V(s)^(-1) . F(s);# integrand=k(t,s)
w:=map(u->int(u,s=0..t),k(t,s));
```

Shortening the expression depends on superposition: $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$. The homogeneous terms for removal have the form

$$\begin{aligned}\vec{\mathbf{x}}_h(t) &= \begin{pmatrix} e^{4t} & 0 \\ 0 & e^{5t}\end{pmatrix}\begin{pmatrix} c_1 \\ c_2\end{pmatrix} \\ &= \begin{pmatrix} c_1e^{4t} \\ c_2e^{5t}\end{pmatrix}\end{aligned}$$

Choose $c_1 = -\frac{2}{3}$, $c_2 = -\frac{1}{4}$, then add this $\vec{\mathbf{x}}_h(t)$ to the integration result:

$$\vec{\mathbf{x}}_p(t) = \begin{pmatrix} -\frac{2}{3}\,\mathrm{e}^t \\ -\frac{1}{4}\,\mathrm{e}^t\end{pmatrix}$$

Theorem 11.47 (Variation of Parameters: Scalar 2nd Order)

Let $a \ne 0$, b, c, f be continuous functions defined near $t = t_0$. Let $x_1(t)$, $x_2(t)$ be two linearly independent solutions of the homogeneous differential equation $a(t)x''(t) + b(t)x'(t) + c(t)x(t) = 0$. Then the unique solution $x_p(t)$ of the second order initial value problem

$$(3) \qquad a(t)x''(t) + b(t)x'(t) + c(t)x(t) = f(t), \quad x(t_0) = 0, \quad x'(t_0) = 0$$

is given by the **Variation of Parameters formula**

$$(4) \qquad x_p(t) = \int_{t_0}^t k(t,s)\frac{f(s)}{a(s)}ds, \quad k(t,r) = \frac{\begin{vmatrix} x_1(s) & x_2(s) \\ x_1(t) & x_2(t)\end{vmatrix}}{\begin{vmatrix} x_1(s) & x_2(s) \\ x_1'(s) & x_2'(s)\end{vmatrix}}$$

Proof of Theorem 11.47. Formula (4) is discovered via Theorem 11.46 using the **companion matrix** for scalar equation (3) on 846, which is $A(t) = \dfrac{1}{a(t)}\begin{pmatrix} 0 & 1 \\ -c(t) & -b(t)\end{pmatrix}$, and $\vec{\mathbf{F}}(t) = \dfrac{1}{a(t)}\begin{pmatrix} 0 \\ f(t)\end{pmatrix}$. This proof path is pursued in the exercises. A direct proof will be given which requires fewer background topics.

Verify Solution. To begin, expand $k(t,s) = u_1(s)x_1(t) + u_2(t)x)2(t)$ where $u_1(s) = -x_2(s)/W(s)$, $u_2(s) = x_1(s)/W(s)$ and $W(s) = \begin{vmatrix} x_1(s) & x_2(s) \\ x_1'(s) & x_2'(s)\end{vmatrix}$. Then

$$x_p(t) = x_1(t)\int_{t_0}^t u_1(s)\frac{f(s)}{a(s)}ds + x_2(t)\int_{t_0}^t u_2(s)\frac{f(s)}{a(s)}ds.$$

Expression $x_p(t)$ is expected to satisfy the differential equation, verified by the following steps.

$\boxed{1}$ LHS $= ax_p'' + bx_p' + cx_p$ — Define x_p by (4)

$\boxed{2}$ $= \int_{t_0}^{t} (a(t)x_1'' + b(t)x_1' + c(t)x_1) \dfrac{f(s)}{a(s)} ds +$

$\int_{t_0}^{t} (a(t)x_2'' + b(t)x_2' + c(t)x_2) \dfrac{f(s)}{a(s)} ds +$

$a(t)\dfrac{f(t)}{a(t)}$

$\boxed{3}$ $= 0 + 0 + f(t)$ — Solution x_p verified.

$\boxed{1}$: Symbol LHS is the left hand side of (3).

$\boxed{2}$: Product rule of calculus and the Fundamental Theorem of Calculus. In particular, due to cancellations:

$$x_p'(t) = x_1'(t) \int_{t_0}^{t} u_1(s) \frac{f(s)}{a(s)} ds + x_2'(t) \int_{t_0}^{t} u_2(s) \frac{f(s)}{a(s)} ds,$$
$$x_p''(t) = x_1''(t) \int_{t_0}^{t} u_1(s) \frac{f(s)}{a(s)} ds + x_2''(t) \int_{t_0}^{t} u_2(s) \frac{f(s)}{a(s)} ds + a(t)\frac{f(t)}{a(t)}.$$

$\boxed{3}$: The homogeneous equation has solutions x_1, x_2.

Initial Conditions. Equation $x_p(t_0) = 0$ follows because the integral is taken over a zero-length interval. Equation $x_p'(t_0) = 0$ follows from $\boxed{2}$ details.

Example 11.14 (Scalar 2nd Order Euler Differential Equation)

Solve for the general solution:

$$x^2y'' + 3xy' + y = \ln(x^2), \quad x > 0$$

Details for Example 11.14:
The answer: $y_p(x) = \ln(x^2) - 4$, $y_h(t) = c_1x^{-1} + c_2x^{-1}\ln|x|$, details below.

Undetermined Coefficients is applied after a change of variables $x = e^t$ into the forced constant equation:

$$\frac{d^2y(e^t)}{dt^2} + 2\frac{dy(e^t)}{dt} + y(e^t) = 2t$$

It's characteristic equation is $r^2 + 2r + 1 = 0$. Then undetermined coefficient solution $2t - 4$ implies

$$y(e^t) = c_1e^{-t} + c_2te^{-t} + 2t - 4$$
$$y(x) = c_1\frac{1}{x} + c_2\frac{\ln|x|}{x} + 2\ln|x| - 4$$

Variation of Parameters directly finds $y(x)$ by integration. To use the formulas, change symbols: $x \to t$ and $y \to x$. Then the original differential equation becomes:

$$t^2x''(t) + 3tx'(t) + x(t) = \ln(t^2)$$

Euler differential equation theory finds a basis $x_1(t) = \frac{1}{t}$, $x_2(t) = \frac{\ln|t|}{t}$ for the homogeneous problem $t^2x''(t) + 3tx'(t) + x(t) = 0$. Then $\dfrac{f(s)}{a(s)} = s^{-2}\ln(s^2)$ and

$$k(t,s) = \frac{\begin{vmatrix} \frac{1}{s} & \frac{\ln|s|}{s} \\ \frac{1}{t} & \frac{\ln|t|}{t} \end{vmatrix}}{\begin{vmatrix} \frac{1}{s} & \frac{\ln|s|}{s} \\ \frac{-1}{s^2} & \frac{1}{s^2} - \frac{\ln|s|}{s^2} \end{vmatrix}} = \frac{s^2 \ln|t/s|}{t}$$

Choose $t_0 = 1$ in the variation of parameters formula. Then for $t > 0$:

$$\begin{aligned} x_p(t) &= \int_1^t k(t,s)\frac{f(s)}{a(s)}ds \\ &= \int_1^t \left(\frac{\ln|t/s|\ln|s^2|}{t}\right) ds \\ &= \ln(t^2) - 4 + \frac{2\ln|t|}{t} + \frac{4}{t} \end{aligned}$$

The last two terms of x_p are homogeneous solutions, discarded to give the shortest particular solution $x_p(t) = \ln(t^2) - 4$.

Example 11.15 (Nonhomogeneous 2×2 System in CAS `maple`)

Solve $x' = 2x + y + t^2$, $y' = 2x + y$, $x(0) = y(0) = 0$ by computer algebra.

Details for Example 11.15:

```
f:=(x,y)->2*x+y; g:=(x,y)->2*x+y;
F:=t->t^2; G:=t->0;
des:=diff(x(t),t)=f(x(t),y(t))+F(t),
     diff(y(t),t)=g(x(t),y(t))+G(t);
dsolve({des,x(0)=0,y(0)=0},[x(t),y(t)]);
```

The reported answer:

$$x(t) = -\frac{2}{9}t^2 - \frac{4t}{27} + \frac{4e^{3t}}{81} - \frac{4}{81} + 1/9\,t^3$$

$$y(t) = -\frac{2}{9}t^3 - 2/9\,t^2 + \frac{4e^{3t}}{81} - \frac{4t}{27} - \frac{4}{81}$$

Undetermined Coefficients

The trial solution method known as the method of undetermined coefficients can be applied to vector-matrix systems $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ when the components of $\vec{\mathbf{F}}$ are linear combinations of terms of the form

$$t^k e^{at}\cos(bt) \quad \text{or} \quad t^k e^{at}\sin(bt),$$

called **Euler solution atoms**. It is efficient for exposition to write $\vec{\mathbf{F}}$ in terms of the columns $\vec{\mathbf{e}}_1, \ldots, \vec{\mathbf{e}}_n$ of the $n \times n$ identity matrix I:

$$\vec{\mathbf{F}}(t) = \sum_{j=1}^{n} F_j(t)\vec{\mathbf{e}}_j.$$

Then a particular solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ is given by

$$\vec{\mathbf{x}}(t) = \sum_{j=1}^{n} \vec{\mathbf{x}}_j(t)$$

where $\vec{\mathbf{x}}_j(t)$ for $1 \le j \le n$ is a particular solution of the simpler equation

$$\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + f(t)\vec{\mathbf{c}}, \quad f = F_j, \quad \vec{\mathbf{c}} = \vec{\mathbf{e}}_j.$$

A trial solution $\vec{\mathbf{x}}(t)$ for non-homogeneous equation $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + f(t)\vec{\mathbf{c}}$ can be determined from the following **Initial Trial Solution Rule**:

> Let $f(t)$ be a linear combination of Euler solution atoms. Identify independent Euler atoms $A_j(t)$ whose linear combinations include all derivatives of $f(t)$. The initial trial solution is expression $\vec{\mathbf{x}}(t) = \sum_j A_j(t)\vec{\mathbf{d}}_j$, a linear combination of atoms with undetermined vector coefficients $\left\{\vec{\mathbf{d}}_j\right\}$.

In the scalar case, the trial solution must be modified if it has an Euler solution atom which is a solution to the homogeneous equation. In the vector case, if $f(t)$ is a polynomial, then this *correction rule* for the initial trial solution is avoided by assuming the matrix A is invertible. This assumption means that $r = 0$ is not a root of $\det(A - rI) = 0$, which prevents the homogenous solution from having any polynomial terms.

The method substitutes the initial vector trial solution into the differential equation to find the undetermined coefficients $\left\{\vec{\mathbf{d}}_j\right\}$. The answers $\left\{\vec{\mathbf{d}}_j\right\}$ replaced in the trial solution determine a particular solution to the non-homogeneous vector differential equation.

Example 11.16 (Undetermined Coefficients: Polynomial Solution)

Solve by undetermined coefficients:

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \vec{\mathbf{x}} + \begin{pmatrix} 1+t \\ t^2 \end{pmatrix}$$

Details Example 11.16:

Solution $\vec{\mathbf{x}}_h$:
Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Find e^{At}:

$$\begin{aligned}
e^{At} &= e^{\lambda_1 t} I + \frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I) && \text{Putzer's formula page 885.} \\
&= e^t I + \frac{e^{-t} - e^t}{-1-1}(A - I) && \text{Because } \lambda_1 = 1,\ \lambda_2 = -1. \\
&= e^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^t - e^{-t}}{2} \begin{pmatrix} 0 & 2 \\ 0 & -2 \end{pmatrix} && \text{Because } A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}. \\
&= \begin{pmatrix} e^t & e^t - e^{-t} \\ 0 & e^{-t} \end{pmatrix} && \text{Verified in maple.}
\end{aligned}$$

Then

$$\vec{\mathbf{x}}_h(t) = e^{At}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (c_1 + c_2)e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The constant vectors in $\vec{\mathbf{x}}_h(t)$ are eigenvectors of A. The eigenanalysis method produces an equivalent formula.

Solution $\vec{\mathbf{x}}_p$:

The desired shortest particular solution is $\vec{\mathbf{x}}_p(t) = \begin{pmatrix} -2t^2 - t - 6 \\ t^2 - 2t + 2 \end{pmatrix}$, obtained by the method of undetermined coefficients.

The forcing term is a vector linear combination of Euler atoms $1, t, t^2$:

$$\vec{\mathbf{F}}(t) = \begin{pmatrix} 1+t \\ t^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Select trial solution[11] $\vec{\mathbf{x}}(t) = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2 + t^2\vec{\mathbf{d}}_3$. Substitute it into $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$:

$$\begin{aligned}
\vec{\mathbf{d}}_2 + 2t\vec{\mathbf{d}}_3 &= A\vec{\mathbf{d}}_1 + tA\vec{\mathbf{d}}_2 + t^2 A\vec{\mathbf{d}}_3 + \vec{\mathbf{F}}(t) \\
\vec{\mathbf{d}}_2 + 2t\vec{\mathbf{d}}_3 &= A\vec{\mathbf{d}}_1 + tA\vec{\mathbf{d}}_2 + t^2 A\vec{\mathbf{d}}_3 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned}$$

Collect left on Euler atoms $1, t, t^2$:

$$\begin{aligned}
(1)\left(\vec{\mathbf{d}}_2 - A\vec{\mathbf{d}}_1 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \quad &+ \quad (t)\left(2\vec{\mathbf{d}}_3 - A\vec{\mathbf{d}}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \\
&+ \quad (t^2)\left(-A\vec{\mathbf{d}}_3 - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \vec{\mathbf{0}}
\end{aligned}$$

Independence of Euler atoms implies the vector coefficients are zero:

$$\begin{aligned}
\vec{\mathbf{d}}_2 - A\vec{\mathbf{d}}_1 - \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \vec{\mathbf{0}} \\
2\vec{\mathbf{d}}_3 - A\vec{\mathbf{d}}_2 - \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \vec{\mathbf{0}} \\
-A\vec{\mathbf{d}}_3 - \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \vec{\mathbf{0}}
\end{aligned}$$

[11]Derivatives of $1, t, t^2$ are spanned by $1, t, t^2$.

Let $B = A^{-1}$. Solve as a triangular system, variables reversed:

$$\begin{aligned}
\vec{\mathbf{d}}_3 &= -B\begin{pmatrix}0\\1\end{pmatrix} &&= \begin{pmatrix}-2\\1\end{pmatrix}\\
\vec{\mathbf{d}}_2 &= B\left(2\vec{\mathbf{d}}_3 - \begin{pmatrix}1\\0\end{pmatrix}\right) &&= \begin{pmatrix}-1\\-2\end{pmatrix}\\
\vec{\mathbf{d}}_1 &= B\left(\vec{\mathbf{d}}_2 - \begin{pmatrix}1\\0\end{pmatrix}\right) &&= \begin{pmatrix}-6\\2\end{pmatrix}
\end{aligned}$$

Replace answers $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3$ in the trial solution to find particular solution:

$$\vec{\mathbf{x}}_p(t) = \begin{pmatrix}-6\\2\end{pmatrix} + t\begin{pmatrix}-1\\-2\end{pmatrix} + t^2\begin{pmatrix}-2\\1\end{pmatrix}$$

Example 11.17 (Undetermined Coefficients: Polynomial-Exponential)

Solve by undetermined coefficients:

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{pmatrix}1 & 2\\0 & -1\end{pmatrix}\vec{\mathbf{x}} + e^{2t}\begin{pmatrix}t\\3\end{pmatrix}$$

Details Example 11.17:

Solution $\vec{\mathbf{x}}_h$:
Let $A = \begin{pmatrix}1 & 2\\0 & -1\end{pmatrix}$. The homogenous solution from Example 11.16:

$$\vec{\mathbf{x}}_h(t) = e^{At}\begin{pmatrix}c_1\\c_2\end{pmatrix} = (c_1+c_2)e^t\begin{pmatrix}1\\0\end{pmatrix} + c_2e^{-t}\begin{pmatrix}-1\\1\end{pmatrix}$$

Solution $\vec{\mathbf{x}}_p$:
The desired shortest particular solution is $\vec{\mathbf{x}}_p(t) = \begin{pmatrix}e^{2t}+te^{2t}\\e^{2t}\end{pmatrix}$, obtained by the method of undetermined coefficients.
The forcing term is a vector linear combination of Euler atoms e^{2t}, te^{2t}:

$$\vec{\mathbf{F}}(t) = \begin{pmatrix}te^{2t}\\3e^{2t}\end{pmatrix} = e^{2t}\begin{pmatrix}0\\3\end{pmatrix} + te^{2t}\begin{pmatrix}1\\0\end{pmatrix}$$

Select trial solution $\vec{\mathbf{x}}(t) = e^{2t}\vec{\mathbf{d}}_1 + te^{2t}\vec{\mathbf{d}}_2$.[12] Substitute it into $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$:

$$2e^{2t}\vec{\mathbf{d}}_1 + e^{2t}\vec{\mathbf{d}}_2 + 2te^{2t}\vec{\mathbf{d}}_2 = e^{2t}A\vec{\mathbf{d}}_1 + te^{2t}A\vec{\mathbf{d}}_2 + e^{2t}\begin{pmatrix}0\\3\end{pmatrix} + te^{2t}\begin{pmatrix}1\\0\end{pmatrix}$$

Cancel e^{2t}. Then collect left on Euler atoms $1, t$:

$$(1)\left(2\vec{\mathbf{d}}_1 - A\vec{\mathbf{d}}_1 - \begin{pmatrix}0\\3\end{pmatrix} + \vec{\mathbf{d}}_2\right) + (t)\left(2\vec{\mathbf{d}}_2 - A\vec{\mathbf{d}}_2 - \begin{pmatrix}1\\0\end{pmatrix}\right) = \vec{\mathbf{0}}$$

[12] Derivatives of e^{2t}, te^{2t} are spanned by e^{2t}, te^{2t}.

Independence of Euler atoms implies the vector coefficients are zero:

$$\begin{aligned} 2\vec{\mathbf{d}}_1 - A\vec{\mathbf{d}}_1 - \begin{pmatrix}0\\3\end{pmatrix} + \vec{\mathbf{d}}_2 &= \vec{\mathbf{0}} \\ 2\vec{\mathbf{d}}_2 - A\vec{\mathbf{d}}_2 - \begin{pmatrix}1\\0\end{pmatrix} &= \vec{\mathbf{0}} \end{aligned}$$

Factor $2I - A$ from each equation. Let $B = (2I - A)^{-1}$. Solve as a triangular system, variables reversed:

$$\begin{aligned} \vec{\mathbf{d}}_2 &= B\begin{pmatrix}1\\0\end{pmatrix} &&= \begin{pmatrix}1\\0\end{pmatrix} \\ \vec{\mathbf{d}}_1 &= B\left(\begin{pmatrix}0\\3\end{pmatrix} - \vec{\mathbf{d}}_2\right) &&= \begin{pmatrix}1\\1\end{pmatrix} \end{aligned}$$

Replace $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ in the trial solution to find particular solution

$$\vec{\mathbf{x}}_p(t) = e^{2t}\begin{pmatrix}1\\1\end{pmatrix} + te^{2t}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^{2t} + te^{2t}\\ e^{2t}\end{pmatrix}$$

There are nuances in the algorithm not revealed in the preceding two examples. Two theorems formalize the methods.

Theorem 11.48 (Polynomial Solutions)
Let $f(t) = \sum_{j=0}^{k} p_j \frac{t^j}{j!}$ be a polynomial of degree k. Assume A is an $n\times n$ constant invertible matrix. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + f(t)\vec{\mathbf{c}}$ has a polynomial solution $\vec{\mathbf{u}}(t) = \sum_{j=0}^{k} \vec{\mathbf{d}}_j \frac{t^j}{j!}$ of degree k with vector coefficients $\left\{\vec{\mathbf{d}}_j\right\}$ given by the relations

$$\vec{\mathbf{d}}_j = -\sum_{i=j}^{k} p_i A^{j-i-1}\vec{\mathbf{c}}, \quad 0 \le j \le k.$$

Theorem 11.49 (Polynomial $\times$ Exponential Solutions)
Let $g(t) = \sum_{j=0}^{k} p_j \frac{t^j}{j!}$ be a polynomial of degree k. Assume A is an $n \times n$ constant matrix and $B = A - aI$ is invertible. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + e^{at}g(t)\vec{\mathbf{c}}$ has a polynomial-exponential solution $\vec{\mathbf{u}}(t) = e^{at}\sum_{j=0}^{k} \vec{\mathbf{d}}_j \frac{t^j}{j!}$ with vector coefficients $\left\{\vec{\mathbf{d}}_j\right\}$ given by the relations

$$\vec{\mathbf{d}}_j = -\sum_{i=j}^{k} p_i B^{j-i-1}\vec{\mathbf{c}}, \quad 0 \le j \le k.$$

Proof of Theorem 11.48. Substitute $\vec{\mathbf{u}}(t) = \sum_{j=0}^{k} \vec{\mathbf{d}}_j \frac{t^j}{j!}$ into the differential equation, then

$$\sum_{j=0}^{k-1} \vec{\mathbf{d}}_{j+1}\frac{t^j}{j!} = A\sum_{j=0}^{k} \vec{\mathbf{d}}_j \frac{t^j}{j!} + \sum_{j=0}^{k} p_j \frac{t^j}{j!}\vec{\mathbf{c}}.$$

Terms on the right for $j = k$ must add to zero and the others must match the left side coefficients of $t^j/j!$, giving the relations

$$A\vec{\mathbf{d}}_k + p_k\vec{\mathbf{c}} = \vec{\mathbf{0}}, \quad \vec{\mathbf{d}}_{j+1} = A\vec{\mathbf{d}}_j + p_j\vec{\mathbf{c}}.$$

Solve the relations recursively to give the formulas

$$\begin{aligned}\vec{\mathbf{d}}_k &= -p_k A^{-1}\vec{\mathbf{c}},\\ \vec{\mathbf{d}}_{k-1} &= -\left(p_{k-1}A^{-1} + p_k A^{-2}\right)\vec{\mathbf{c}},\\ &\vdots\\ \vec{\mathbf{d}}_0 &= -\left(p_0 A^{-1} + \cdots + p_k A^{-k-1}\right)\vec{\mathbf{c}}.\end{aligned}$$

The relations above can be summarized by the formula

$$\vec{\mathbf{d}}_j = -\sum_{i=j}^{k} p_i A^{j-i-1}\vec{\mathbf{c}}, \quad 0 \le j \le k.$$

The calculation shows that if $\vec{\mathbf{u}}(t) = \sum_{j=0}^{k} \vec{\mathbf{d}}_j \frac{t^j}{j!}$ and $\vec{\mathbf{d}}_j$ is given by the last formula, then $\vec{\mathbf{u}}(t)$ substituted into the differential equation gives matching LHS and RHS. ■

Proof of Theorem 11.49. Let $\vec{\mathbf{u}}(t) = e^{at}\vec{\mathbf{v}}(t)$. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + e^{at}g(t)\vec{\mathbf{c}}$ implies $\vec{\mathbf{v}}' = (A - aI)\vec{\mathbf{v}} + g(t)\vec{\mathbf{c}}$. Apply Theorem 11.48 to $\vec{\mathbf{v}}' = B\vec{\mathbf{v}} + g(t)\vec{\mathbf{c}}$. ■

Exercises 11.7

Variation of Parameters

Let $A(t) = \begin{pmatrix} 0 & 1 \\ -c(t)/a(t) & -b(t)/a(t)\end{pmatrix}$, $\vec{\mathbf{F}}(t) = \dfrac{1}{a(t)}\begin{pmatrix} 0 \\ f(t)\end{pmatrix}$, $\vec{\mathbf{x}} = \begin{pmatrix} u(t) \\ u'(t)\end{pmatrix}$.

1. Verify equivalence of $a(t)u'' + b(t)u' + c(t)u = f(t)$ and $\vec{\mathbf{x}}' = A(t)\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$.

2. For $u'' + 100u = \sin(t)$, find $A(t)$ and $\vec{\mathbf{F}}(t)$.

3. For $u'' = f(t)$, find $A(t)$ and $\vec{\mathbf{F}}(t)$.

4. For $u'' = f(t)$, let $u_1 = 1$, $u_2 = t$, $\Phi(t) = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2'\end{pmatrix}$. Verify $|\Phi(t)| = 1$, then find $A(t) = \Phi'(t)\Phi^{-1}(t)$.

5. State Theorem 11.46 for $n = 2$, then explain how it applies to this special case.

6. Prove Theorem 11.47 using the previous exercise.

Variation of Parameters: Scalar 2nd Order

Let $a(t)u'' + b(t)u' + c(t)u = 0$ have two independent solutions u_1, u_2.

Define $\Psi(t) = \begin{pmatrix} u_1 & u_2 \\ u_1' & u_2'\end{pmatrix}$. Then:

7. Matrix $\Psi(t)$ has an inverse.

8. Matrix $\Phi(t) = \Psi(t)\Psi^{-1}(t_0)$ is invertible and $\Phi(t_0) = I$.

9. Let $\Psi(t) = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}$. Define $\begin{pmatrix} u \\ v\end{pmatrix} = \Psi(t)\int_0^t \Psi^{-1}(s)f(s)ds$. Then u is a particular solution of $u'' = f(t)$.

10. Let $\Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & -e^{-t}\end{pmatrix}$. Define $\begin{pmatrix} u \\ v\end{pmatrix} = \Psi(t)\int_0^t \Psi^{-1}(s)f(s)ds$. Then u is a particular solution of $u'' - u = f(t)$.

Variation of Parameters

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3\end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ using $\vec{\mathbf{x}}_p = \int_0^t e^{A(t-s)}\vec{\mathbf{F}}(s)ds$ and computer assist.

11. $\vec{\mathbf{F}}(t) = e^t\begin{pmatrix} 1 \\ 2\end{pmatrix}$, $\vec{\mathbf{x}}_p = \begin{pmatrix} e^{2t} - e^t \\ e^{3t} - e^t\end{pmatrix}$

12. $\vec{\mathbf{F}}(t) = \begin{pmatrix} e^t \\ e^{-t}\end{pmatrix}$, $\vec{\mathbf{x}}_p = \begin{pmatrix} e^{2t} - e^t \\ \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}\end{pmatrix}$

Undetermined Coefficients

Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Solve $\vec{\mathbf{x}}'=A\vec{\mathbf{x}}+\vec{\mathbf{F}}(t)$ by undetermined coefficients. Assume $\vec{\mathbf{x}}_h(t)=c_1e^t\begin{pmatrix} 1 \\ 0 \end{pmatrix}+c_2e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

13. $\vec{\mathbf{F}}(t) = e^t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,
$\vec{\mathbf{x}}_p=\begin{pmatrix} e^{-t}+3te^t-e^t \\ e^t-e^{-t} \end{pmatrix}$

14. $\vec{\mathbf{F}}(t) = 2\begin{pmatrix} \cos t \\ e^t \end{pmatrix}$,
$\vec{\mathbf{x}}_p = \begin{pmatrix} 2te^t+\sin(t)-\cos(t)+e^{-t} \\ e^t-e^{-t} \end{pmatrix}$

Undetermined Coefficients

Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Solve $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ by undetermined coefficients. Assume $\vec{\mathbf{x}}_h(t) = \begin{pmatrix} c_1e^{2t} \\ c_2e^{3t} \end{pmatrix}$.

15. $\vec{\mathbf{F}}(t) = e^t\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{\mathbf{x}}_p = e^t\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

16. $\vec{\mathbf{F}}(t) = 4\begin{pmatrix} e^t \\ e^{-t} \end{pmatrix}$, $\vec{\mathbf{x}}_p = e^{-t}\begin{pmatrix} -4 \\ -1 \end{pmatrix}$

17. $\vec{\mathbf{F}}(t) = 10\begin{pmatrix} \cos t \\ e^t \end{pmatrix}$,
$\vec{\mathbf{x}}_p = \begin{pmatrix} -4\cos(t)+2\sin(t) \\ -5e^t \end{pmatrix}$

18. $\vec{\mathbf{F}}(t) = 2e^t\begin{pmatrix} \cos t \\ 1 \end{pmatrix}$,
$\vec{\mathbf{x}}_p = e^t\begin{pmatrix} -\cos(t)+\sin(t) \\ -1 \end{pmatrix}$

11.8 Second-order Systems

A model problem for second order systems is the system of three masses coupled by springs studied in section 11.1, equation (6):

$$
\begin{aligned}
m_1 x_1''(t) &= -k_1 x_1(t) + k_2[x_2(t) - x_1(t)], \\
m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\
m_3 x_3''(t) &= -k_3[x_3(t) - x_2(t)] - k_4 x_3(t).
\end{aligned} \tag{1}
$$

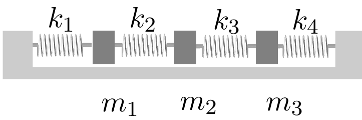

Figure 22. Three masses connected by springs. The masses slide on a frictionless surface.

In vector-matrix form, this system is a **second order system**

$$M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t)$$

where the **displacement** $\vec{\mathbf{x}}$, **mass matrix** M and **stiffness matrix** K are defined by the formulas

$$\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.$$

Because M is invertible, the system can always be written as

$$\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}, \quad A = M^{-1}K.$$

Euler's Substitution $\vec{x} = e^{\lambda t}\vec{v}$

Fundamental substitution $\vec{\mathbf{x}} = e^{\lambda t}\vec{\mathbf{v}}$ due to L. Euler applies to any vector-matrix differential system.

Euler's substitution $\vec{\mathbf{x}} = e^{\lambda t}\vec{\mathbf{v}}$ is perhaps the premier method for remembering the identities

$\lvert A - \lambda^2 I\rvert = 0$	**Characteristic equation** of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$
$\left(A - r^2 I\right)\vec{\mathbf{v}} = \vec{\mathbf{0}}, \quad \vec{\mathbf{v}} \neq \vec{\mathbf{0}}$	**Eigenpair equation**

Theorem 11.50 (Properties of Euler's Substitution $\vec{\mathbf{x}} = e^{\lambda t}\,\vec{\mathbf{v}}$)
Equation $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{v}}$ defines a nonzero solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ if and only if $(r^2, \vec{\mathbf{v}})$ is an eigenpair of matrix A.

Proof: Assume $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{v}}$ is a solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$. Substitution gives $r^2 e^{rt}\vec{\mathbf{v}} = A\vec{\mathbf{v}}e^{rt}$. Cancel the exponential, then $r^2\vec{\mathbf{v}} = A\vec{\mathbf{v}}$. Linear algebraic homogeneous system $\left(A - r^2 I\right)\vec{\mathbf{v}} = \vec{\mathbf{0}}$ has a nonzero solution $\vec{\mathbf{v}}$ if and only if the determinant of coefficients vanishes: $|A - r^2 I| = 0$.

Assume $(r^2, \vec{\mathbf{v}})$ is an eigenpair of A. The eigenpair equation: $r^2\vec{\mathbf{v}} = A\vec{\mathbf{v}}$. Multiply by e^{rt}: $r^2 e^{rt}\vec{\mathbf{v}} = A\vec{\mathbf{v}}e^{rt}$. Then $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{v}} \neq \vec{\mathbf{0}}$ is a solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

Negative Eigenvalues of A

Suppose $(\lambda^2, \vec{\mathbf{v}})$ is an eigenpair of real $n \times n$ matrix A but λ^2 is negative or zero. What is the Euler solution $\vec{\mathbf{x}} = e^{\lambda t}\vec{\mathbf{v}}$ in this case?

For instance, if $\lambda^2 = -4$, then $\lambda = \pm 2i$. Nonzero eigenvector $\vec{\mathbf{v}}$ has real components, therefore Euler solution $\vec{\mathbf{x}}(t) = e^{\lambda t}\vec{\mathbf{v}}$ is a vector with complex entries: $\vec{\mathbf{x}}(t) = e^{2it}\vec{\mathbf{v}} = \cos(2t)\vec{\mathbf{v}} + i\sin(2t)\vec{\mathbf{v}}$. If A is real, then $\cos(2t)\vec{\mathbf{v}}$ and $\sin(2t)\vec{\mathbf{v}}$ are independent real solutions of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$. Formally, they are n-vectors times Euler solution atoms.

> To each negative root $\lambda = -\omega^2$ of $|A - \lambda I| = 0$ with associated eigenpair $(\lambda, \vec{\mathbf{v}})$ corresponds two independent real solutions $\cos(\omega t)\vec{\mathbf{v}}$ and $\sin(\omega t)\vec{\mathbf{v}}$ to the equation $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

Cayley-Hamilton-Ziebur Method for $\vec{x}'' = A\vec{x}$

The theory of Euler solution atoms impacts intuition for second order systems in an essential way. Acronym **CHZ** abbreviates *Cayley-Hamilton-Ziebur*. See page 841 for the history.

Theorem 11.51 (Cayley-Hamilton-Ziebur Structure for $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$)
The solution $\vec{\mathbf{x}}(t)$ of second order equation $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$ is a vector linear combination of Euler solution atoms corresponding to roots of the equation $\det(A - r^2 I) = 0$.

Remarks. The equation $|A - r^2 I| = 0$ is formed by substitution of $\lambda = r^2$ into the eigenanalysis characteristic equation $|A - \lambda I| = 0$. In symbols, the structure theorem says $\vec{\mathbf{x}} = \vec{\mathbf{d}}_1 A_1 + \cdots + \vec{\mathbf{d}}_k A_{2n}$, where $A_1, \ldots, A_{2n}$ are Euler solution atoms corresponding to roots r of the determining equation $|A - r^2 I| = 0$. Because Euler solution atoms are real, then all vectors in the relation have real entries. However, only $2n$ arbitrary real constants appear in the $2n^2$ components of $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_{2n}$, the remaining components being dependent on them.

Proof of the CHZ Structure Theorem. Consider the case when A is 2×2 ($n = 2$), because the proof details are similar in higher dimensions. Expand $|A - \lambda I| = 0$ to find the characteristic equation $\lambda^2 + c\lambda + d = 0$, for some constants c, d. The Cayley-Hamilton theorem says that $A^2 + cA + d\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\vec{\mathbf{x}}$ be a solution of $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$. Multiply the Cayley-Hamilton identity by vector $\vec{\mathbf{x}}$ and simplify to obtain

$$A^2\vec{\mathbf{x}} + cA\vec{\mathbf{x}} + d\vec{\mathbf{x}} = \vec{\mathbf{0}}.$$

Using equation $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$ backwards, we compute $A^2\vec{\mathbf{x}} = A\vec{\mathbf{x}}'' = \vec{\mathbf{x}}''''$. Replace the terms of the displayed equation to obtain the relation

$$\vec{\mathbf{x}}'''' + c\vec{\mathbf{x}}'' + d\vec{\mathbf{x}} = \vec{\mathbf{0}}.$$

Each component y of vector $\vec{\mathbf{x}}(t)$ then satisfies the 4th order linear homogeneous equation $y^{(4)} + cy^{(2)} + dy = 0$, which has characteristic equation $r^4 + cr^2 + d = 0$. This equation is the expansion of determinant equation $|A - r^2 I| = 0$. Therefore y is a linear combination of the Euler solution atoms found from the roots of this equation. It follows then that $\vec{\mathbf{x}}(t)$ is a vector linear combination of the Euler solution atoms so identified. ■

Theorem 11.52 (CHZ Method and Negative Eigenvalues)
Assume $n \times n$ matrix A has only negative eigenvalues. Then solution $\vec{\mathbf{x}}(t)$ of second order equation $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$ is a vector linear combination of Euler solution atoms of the form $\cos(\omega t)$, $\sin(\omega t)$, where $|A - \omega^2 I| = 0$.

Proof: The result follows from Theorem 11.51, because negative roots of equation $|A - rI| = 0$ have the form $r = -\omega^2$ for some positive number ω, which implies the Euler solution atoms for A are of the form $\cos(\omega t)$, $\sin(\omega t)$. ■

Euler Substitution and Solution Atoms

Euler's substitution $\vec{\mathbf{x}} = e^{kt}\vec{\mathbf{v}}$ has limited use for solving $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$. Advantages of the CHZ method will be illustrated.

Illustration 1. Assume A is 2×2 and $|A - \lambda I| = 0$ has roots $\lambda = -4, -16$. Then $|A - r^2 I| = 0$ has four complex roots $\pm 2i, \pm 4i$ and Euler solution atom list $\cos(2t), \cos(4t), \sin(2t), \sin(4t)$. Because eigenvectors $\vec{\mathbf{v}}$ are real, then Euler substitutions are complex: $e^{2it}\vec{\mathbf{v}}$, $e^{-2it}\vec{\mathbf{v}}$, $e^{4it}\vec{\mathbf{v}}$ and $e^{-4it}\vec{\mathbf{v}}$.

The CHZ method is free of complex numbers. In the 2×2 example we have $\vec{\mathbf{x}} = \vec{\mathbf{d}}_1 \cos(2t) + \vec{\mathbf{d}}_2 \cos(4t) + \vec{\mathbf{d}}_3 \sin(2t) + \vec{\mathbf{d}}_4 \sin(4t)$, where $\vec{\mathbf{d}}_1$ to $\vec{\mathbf{d}}_4$ are *real* vectors.

Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$ allows the switch between complex solutions and real solutions. Euler's substitution $\vec{\mathbf{x}} = e^{2it}\vec{\mathbf{v}}$ is a solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ provided $((2i)^2, \vec{\mathbf{v}})$ is an eigenpair of A. This means $\vec{\mathbf{v}}$ is a real eigenvector for eigenvalue -4 ($A\vec{\mathbf{v}} = -4\vec{\mathbf{v}}$ is required) and therefore $\vec{\mathbf{x}} = e^{2it}\vec{\mathbf{v}}$ is a *complex* solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

Illustration 2. Assume A is 2×2 and $|A - \lambda I| = 0$ has roots $\lambda = 4, 16$. Then $|A - r^2 I| = 0$ has four real roots $2, 2, 4, 4$ and Euler atom list $e^{2t}, te^{2t}, e^{4t}, te^{4t}$.

The CHZ method implies the general solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ has the real form $\vec{\mathbf{d}}_1 e^{2t} + \vec{\mathbf{d}}_2 te^{2t} + \vec{\mathbf{d}}_3 e^{4t} + \vec{\mathbf{d}}_4 te^{4t}$.

Euler's substitution produces only two atoms e^{2t}, e^{4t} and we are left with the mystery of how atoms te^{2t}, te^{4t} were discovered to be part of the solution.

Converting $\vec{x}'' = A\vec{x}$ to $\vec{u}' = C\vec{u}$

Given a second order $n \times n$ system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, define the variable $\vec{\mathbf{u}}$ and the $2n \times 2n$ block matrix C as follows.

$$\text{(2)} \qquad \vec{\mathbf{u}} = \begin{pmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{pmatrix}, \quad C = \left(\begin{array}{c|c} \mathbf{0} & I \\ \hline A & \mathbf{0} \end{array}\right).$$

Then each solution $\vec{\mathbf{x}}$ of the second order system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ produces a corresponding solution $\vec{\mathbf{u}}$ of the first order system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$. Similarly, each solution $\vec{\mathbf{u}}$ of $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ gives a solution $\vec{\mathbf{x}}$ of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ by the formula $\vec{\mathbf{x}} = \langle I|\mathbf{0}\rangle\vec{\mathbf{u}}$.

Characteristic Equation for $\vec{x}'' = A\vec{x}$

The characteristic equation for the $n \times n$ second order system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ will be derived anew from the corresponding $2n \times 2n$ first order system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$.

Theorem 11.53 (Characteristic Equation)
Let $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ be given with $n \times n$ constant matrix A. Let

$$\vec{\mathbf{u}} = \begin{pmatrix} \vec{\mathbf{x}}s \\ \vec{\mathbf{x}}' \end{pmatrix}, \quad C = \left(\begin{array}{c|c} \mathbf{0} & I \\ \hline A & \mathbf{0} \end{array}\right).$$

The first order system for $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ is $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$. Then:

$$\det(C - \lambda I) = (-1)^n \det(A - \lambda^2 I). \tag{3}$$

Proof: The method of proof is to verify the product formula

$$\left(\begin{array}{c|c} -\lambda I & I \\ \hline A & -\lambda I \end{array}\right)\left(\begin{array}{c|c} I & \mathbf{0} \\ \hline \lambda I & I \end{array}\right) = \left(\begin{array}{c|c} \mathbf{0} & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right).$$

Then the determinant product formula applies to give

$$\det(C - \lambda I)\det\left(\begin{array}{c|c} I & \mathbf{0} \\ \hline \lambda I & I \end{array}\right) = \det\left(\begin{array}{c|c} \mathbf{0} & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right). \tag{4}$$

Cofactor expansion is applied to give the two identities

$$\det\left(\begin{array}{c|c} I & \mathbf{0} \\ \hline \lambda I & I \end{array}\right) = 1, \quad \det\left(\begin{array}{c|c} \mathbf{0} & I \\ \hline A - \lambda^2 I & -\lambda I \end{array}\right) = (-1)^n \det(A - \lambda^2 I).$$

Then (4) implies (3). ■

Solving $\vec{u}' = C\vec{u}$ and $\vec{x}'' = A\vec{x}$

Theorem 11.54 (Eigenanalysis of A and C)
Consider the $n \times n$ second order system $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ and its corresponding $2n \times 2n$ first order system $\vec{\mathbf{u}}' = C\vec{\mathbf{u}}$ defined by

$$C = \left(\begin{array}{c|c} \mathbf{0} & I \\ \hline A & \mathbf{0} \end{array}\right), \quad \vec{\mathbf{u}} = \begin{pmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}' \end{pmatrix}. \tag{5}$$

Then $(\lambda, \vec{\mathbf{y}})$ is an eigenpair of C if and only if $(\lambda^2, \vec{\mathbf{w}})$ is an eigenpair of A and $\vec{\mathbf{y}} = \begin{pmatrix} \vec{\mathbf{w}} \\ \lambda\vec{\mathbf{w}} \end{pmatrix}$.

Proof: The equivalent statement

$$(6)\qquad (C-\lambda I)\begin{pmatrix}\vec{\mathbf{w}}\\ \vec{\mathbf{z}}\end{pmatrix}=\vec{\mathbf{0}}\quad\text{if and only if}\quad\left\{\begin{array}{rcl}A\vec{\mathbf{w}}&=&\lambda^2\vec{\mathbf{w}},\\ \vec{\mathbf{z}}&=&\lambda\vec{\mathbf{w}}.\end{array}\right.$$

is proved from $C-\lambda I=\left(\begin{array}{c|c}-\lambda I & I\\ \hline A & -\lambda I\end{array}\right)$ and block multiply. ■

Theorem 11.55 (General Solutions of $\vec{\mathbf{u}}\,'=C\,\vec{\mathbf{u}}$ and $\vec{\mathbf{x}}\,''=A\,\vec{\mathbf{x}}$)
Let A be a given $n\times n$ constant matrix and define the corresponding $2n\times 2n$ system by

$$\vec{\mathbf{u}}'=C\vec{\mathbf{u}},\quad C=\left(\begin{array}{c|c}\mathbf{0} & I\\ \hline A & \mathbf{0}\end{array}\right),\quad \vec{\mathbf{u}}=\begin{pmatrix}\vec{\mathbf{x}}\\ \vec{\mathbf{x}}'\end{pmatrix}.$$

Assume C has eigenpairs $\{(\lambda_j,\vec{\mathbf{y}}_j)\}_{j=1}^{2n}$ and $\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{2n}$ are independent. Let I and $\mathbf{0}$ denote the $n\times n$ identity and zero matrix. Define $\vec{\mathbf{w}}_j=\big\langle I|\mathbf{0}\big\rangle\vec{\mathbf{y}}_j$, $j=1,\ldots,2n$. Then $\vec{\mathbf{u}}'=C\vec{\mathbf{u}}$ and $\vec{\mathbf{x}}''=A\vec{\mathbf{x}}$ have general solutions

$$\begin{array}{rcll}\vec{\mathbf{u}}(t)&=&c_1e^{\lambda_1 t}\vec{\mathbf{y}}_1+\cdots+c_{2n}e^{\lambda_{2n}t}\vec{\mathbf{y}}_{2n} & (2n\times 1),\\ \vec{\mathbf{x}}(t)&=&c_1e^{\lambda_1 t}\vec{\mathbf{w}}_1+\cdots+c_{2n}e^{\lambda_{2n}t}\vec{\mathbf{w}}_{2n} & (n\times 1).\end{array}$$

Proof:
General solution of $\vec{\mathbf{u}}'=C\vec{\mathbf{u}}$. Independence of vector Euler solutions $e^{\lambda_1 t}\vec{\mathbf{y}}_1,\ldots,e^{\lambda_{2n}t}\vec{\mathbf{y}}_{2n}$ will be verified. Assume a linear combination of these solutions is zero, then at $t=0$ the exponentials equal 1, which reduces to a linear combination of $\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{2n}$. By independence of the latter, then all weights are zero: the Euler solutions are independent. Hence $\vec{\mathbf{u}}(t)$ is a general solution of $\vec{\mathbf{u}}'=C\vec{\mathbf{u}}$.

General solution of $\vec{\mathbf{x}}''=A\vec{\mathbf{x}}$. Independence of vector Euler solution $e^{\lambda_1 t}\vec{\mathbf{w}}_1,\ldots,e^{\lambda_{2n}t}\vec{\mathbf{w}}_{2n}$ will be verified. Suppose constants $a_1,\ldots,a_{2n}$ are given with $\sum_{j=1}^{2n}a_je^{\lambda_j t}\vec{\mathbf{w}}_j=\vec{\mathbf{0}}$. Replace $t=0$ in this relation to give (1) $\sum_{j=1}^{2n}a_j\vec{\mathbf{w}}_j=\vec{\mathbf{0}}$. Differentiate this relation on variable t to give $\sum_{j=1}^{2n}a_j\lambda_j\,e^{\lambda_j t}\vec{\mathbf{w}}_j=\vec{\mathbf{0}}$ for all t, then set $t=0$ to obtain (2) $\sum_{j=1}^{2n}a_j\lambda_j\,\vec{\mathbf{w}}_j=\vec{\mathbf{0}}$. Combine (1) and (2) using $\vec{\mathbf{y}}_j=\begin{pmatrix}\vec{\mathbf{w}}_j\\ \lambda_j\vec{\mathbf{w}}_j\end{pmatrix}$ from Theorem 11.54 into the vector equation $\sum_{j=1}^{2n}a_j\vec{\mathbf{y}}_j=\vec{\mathbf{0}}$. Independence of $\vec{\mathbf{y}}_1,\ldots,\vec{\mathbf{y}}_{2n}$ implies that the weights are zero: $a_1=\cdots=a_{2n}=0$. ■

Eigenanalysis for Non-positive Eigenvalues

Assume all eigenvalues μ of A are negative or zero. Eigenvalue μ of A is related to an eigenvalue λ of C by the relation $\mu=-\omega^2=\lambda^2$ for some real $\omega\ge 0$. Then $\lambda=\pm\omega i$ and $\omega=\sqrt{|\mu|}$.

Lemma 11.2 (Cosine and Sine Solutions)
Let $(-\omega^2,\vec{\mathbf{v}})$ be an eigenpair of the real $n\times n$ matrix A with $\omega\ge 0$. Define

$$u(t)=\left\{\begin{array}{ll}c_1\cos\omega t+c_2\sin\omega t & \omega>0,\\ c_1+c_2t & \omega=0.\end{array}\right.$$

Then $\vec{\mathbf{x}}(t)=u(t)\vec{\mathbf{v}}$ satisfies $\vec{\mathbf{x}}''(t)=A\vec{\mathbf{x}}(t)$.

Proof:
Then $u''(t) = -\omega^2 u(t)$ (both sides are zero for $\omega = 0$). Vector function $\vec{\mathbf{x}}(t) = u(t)\vec{\mathbf{v}}$ satisfies $\vec{\mathbf{x}}''(t) = -\omega^2\vec{\mathbf{x}}(t)$. Also, $A\vec{\mathbf{x}}(t) = u(t)A\vec{\mathbf{v}} = -\omega^2\vec{\mathbf{x}}(t)$. This proves $\vec{\mathbf{x}}(t) = u(t)\vec{\mathbf{v}}$ satisfies $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$. ■

Theorem 11.56 (Eigenanalysis Solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$)
Let real $n \times n$ matrix A have eigenpairs $\{(\mu_j, \vec{\mathbf{v}}_j)\}_{j=1}^n$. Assume A has distinct eigenvalues $\mu_j = -\omega_j^2$ with $\omega_j \geq 0$, $j = 1, \ldots, n$ and that $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$ are linearly independent. Then the general solution of $\vec{\mathbf{x}}''(t) = A\vec{\mathbf{x}}(t)$ is given in terms of $2n$ arbitrary constants $a_1, \ldots, a_n, b_1, \ldots, b_n$ by the formula

$$\vec{\mathbf{x}}(t) = \sum_{j=1}^{n}\left(a_j \cos\omega_j t + b_j \frac{\sin\omega_j t}{\omega_j}\right)\vec{\mathbf{v}}_j \tag{7}$$

This expression uses the limit convention $\left.\dfrac{\sin\omega t}{\omega}\right|_{\omega=0} = t$.

Proof:
Lemma 11.2 and superposition establish that $\vec{\mathbf{x}}(t)$ is a solution. It only remains to prove that it is the general solution, meaning that the arbitrary constants can be assigned to allow any possible initial condition $\vec{\mathbf{x}}(0) = \vec{\mathbf{x}}_0$, $\vec{\mathbf{x}}'(0) = \vec{\mathbf{y}}_0$. Define the constants uniquely by the relations

$$\begin{aligned} \vec{\mathbf{x}}_0 &= \textstyle\sum_{j=1}^n a_j\vec{\mathbf{v}}_j, \\ \vec{\mathbf{y}}_0 &= \textstyle\sum_{j=1}^n b_j\vec{\mathbf{v}}_j, \end{aligned}$$

which is possible by the assumed independence of the vectors $\{\vec{\mathbf{v}}_j\}_{j=1}^n$. Then equation (7) implies $\vec{\mathbf{x}}(0) = \sum_{j=1}^n a_j\vec{\mathbf{v}}_j = \vec{\mathbf{x}}_0$ and $\vec{\mathbf{x}}'(0) = \sum_{j=1}^n b_j\vec{\mathbf{v}}_j = \vec{\mathbf{y}}_0$. ■

Why doesn't equation (7) work for duplicate eigenvalues?

Consider $A = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ for which the characteristic equation $|A - r^2 I| = 0$ has duplicate complex roots $\pm 2i, \pm 2i$. Then CHZ predicts real solution $\vec{\mathbf{x}} = \vec{\mathbf{d}}_1\cos(2t) + \vec{\mathbf{d}}_2 t\cos(2t) + \vec{\mathbf{d}}_3\sin(2t) + \vec{\mathbf{d}}_4 t\sin(2t)$ whereas incorrect application of equation (7) would report $\vec{\mathbf{x}} = a_1\vec{\mathbf{v}}_1\cos(2t) + a_2\vec{\mathbf{v}}_2\cos(2t) + b_1\vec{\mathbf{v}}_1\sin(2t) + b_2\vec{\mathbf{v}}_2\sin(2t)$, the symbols $\vec{\mathbf{v}}_j$ being real eigenvectors of A for eigenvalues $-4, -4$.
Euler solution atoms $t\cos(2t), t\sin(2t)$ are missing in equation (7), but maybe the equation is correct anyway? The answer is **NO**, because differentiation across equation (7) on symbols a_1, a_2, b_1, b_2 reveals there are only two independent vector solutions represented, instead of the required four. The conclusion: equation (7) doesn't work for multiple eigenvalues.

Theorem 11.57 (CHZ and Eigenvectors: $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$)
If the hypothesis of Theorem 11.56 holds, then in CHZ solution $\vec{\mathbf{x}} = \sum_{j=1}^{2n}\vec{\mathbf{d}}_j A_j(t)$ each $\vec{\mathbf{d}}_j$ is a scalar multiple of an eigenvector of A.[13]

Proof. Let $\vec{\mathbf{x}}$ be a solution of $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ and represent it in two ways, first by CHZ and second by eigenanalysis:

$$\vec{\mathbf{x}} = \sum_{j=1}^{2n}\vec{\mathbf{d}}_j A_j(t) = \sum_{j=1}^{n}\left(a_j\cos\omega_j t + b_j\frac{\sin\omega_j t}{\omega_j}\right)\vec{\mathbf{v}}_j$$

[13] **Warning**: A vector $\vec{\mathbf{d}}_j$ can be zero: $0\vec{\mathbf{v}}$ is a linear combination of eigenvector $\vec{\mathbf{v}}$.

Assume by re-labeling that the Euler atoms are $A_j(t) = \cos(\omega_j t)$ and $A_{j+n}(t) = \frac{\sin \omega_j t}{\omega_j}$, $1 \le j \le n$. Then $\sum_{j=1}^{2n} \vec{\mathbf{d}}_j A_j(t) = \sum_{j=1}^{n} a_j \vec{\mathbf{v}}_j A_j(t) + b_j \vec{\mathbf{v}}_j A_{j+n}(t)$. Independence of $\{A_j\}_{j=1}^{2n}$ implies vector coefficients of the atoms on each side of the equation must match: each $\vec{\mathbf{d}}_j$ is a scalar multiple of an eigenvector of A. ■

Earthquakes

Reproduced here are earthquake modeling formulas from page 833. The formulas are applied to 5-story buildings using the solution methods of this section.

A horizontal earthquake oscillation $F(t) = F_0 \cos \omega t$ affects each floor of a 5-floor building; see Figure 23. The effect of the earthquake depends upon the natural frequencies of oscillation of the floors.

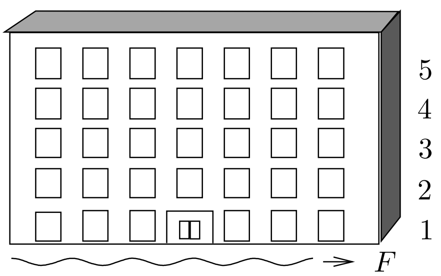

Figure 23. A 5-Floor Building. A horizontal earthquake wave F affects every floor. A typical wave has wavelength many times larger than the illustration.

Assumptions and Symbols for a 5-Floor Building

- Each floor is considered a point mass located at its center-of-mass. The floors have masses $m_1, \ldots, m_5$.
- Each floor is restored to its equilibrium position by a linear restoring force or Hooke's force $-k$(elongation). The Hooke's constants are $k_1, \ldots, k_5$.
- The locations of masses representing the 5 floors are $x_1, \ldots, x_5$. The equilibrium position is $x_1 = \cdots = x_5 = 0$.
- Damping effects of the floors are ignored: it is a *frictionless* system.

Derivation Details

The differential equations for the model are obtained by **competition**: the Newton's second law force is set equal to the sum of the Hooke's forces and the external force due to the earthquake wave. This results in the following system, where $k_6 = 0$, $E_j = m_j F''$ for $j = 1, 2, 3, 4, 5$ and $F = F_0 \cos \omega t$.

$$\begin{aligned}
m_1 x_1'' &= -(k_1 + k_2)x_1 + k_2 x_2 + E_1, \\
m_2 x_2'' &= k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3 + E_2, \\
m_3 x_3'' &= k_3 x_2 - (k_3 + k_4)x_3 + k_4 x_4 + E_3, \\
m_4 x_4'' &= k_4 x_3 - (k_4 + k_5)x_4 + k_5 x_5 + E_4, \\
m_5 x_5'' &= k_5 x_4 - (k_5 + k_6)x_5 + E_5.
\end{aligned}$$

In particular, the equations for a floor depend only upon the neighboring floors. The bottom floor and the top floor are exceptions: they have just one neighboring floor.

Vector-Matrix 2nd Order System
Let:

$$M = \begin{pmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \vec{\mathbf{H}} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{pmatrix},$$

$$K = \begin{pmatrix} -k_1-k_2 & k_2 & 0 & 0 & 0 \\ k_2 & -k_2-k_3 & k_3 & 0 & 0 \\ 0 & k_3 & -k_3-k_4 & k_4 & 0 \\ 0 & 0 & k_4 & -k_4-k_5 & k_5 \\ 0 & 0 & 0 & k_5 & -k_5-k_6 \end{pmatrix}$$

In the last row, $k_6 = 0$ reflects the absence of a floor above the fifth floor. The second order system:

$$M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t) + \vec{\mathbf{H}}(t)$$

Matrix M is called the **mass matrix** and matrix K is called the **Hooke's matrix**. The **external force** $\vec{\mathbf{H}}(t)$ can be written as a scalar function $E(t) = -F''(t)$ times a constant vector:

$$\vec{\mathbf{H}}(t) = -\omega^2 F_0 \cos\omega t \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \end{pmatrix}.$$

Identical Floors
Assume that all floors have the same mass m and the same Hooke's constant k. Then $M = mI$ and $M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t) + \vec{\mathbf{H}}(t)$ becomes:

$$(8) \qquad \vec{\mathbf{x}}'' = \frac{1}{m}\begin{pmatrix} -2k & k & 0 & 0 & 0 \\ k & -2k & k & 0 & 0 \\ 0 & k & -2k & k & 0 \\ 0 & 0 & k & -2k & k \\ 0 & 0 & 0 & k & -k \end{pmatrix}\vec{\mathbf{x}} - F_0\omega^2\cos(\omega t)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Hooke's matrix K is symmetric ($K^T = K$) with negative entries only on the diagonal. The last diagonal entry is $-k$ (a error to write $-2k$).

Particular Solution: Identical Floors
The method of undetermined coefficients predicts a trial solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{c}}\, \cos\omega t$. Terms $\sin\omega t$ cannot appear in the trial solution because the $\vec{\mathbf{x}}'$ term is absent in equation (8).

Constant vector $\vec{\mathbf{c}}$ will be found by trial solution substitution. Let $\vec{\mathbf{b}}$ equal the column vector of ones in equation (8). Substitute the trial solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{c}}\cos\omega t$ into (8). Cancel the common factor $\cos\omega t$. Then $\left(m^{-1}K + \omega^2 I\right)\vec{\mathbf{c}} = F_0\omega^2\vec{\mathbf{b}}$. Let $B = m^{-1}K + \omega^2 I$. Determinant formula $B^{-1} = \dfrac{\mathbf{adj}(B)}{\det(B)}$ gives:

$$\vec{\mathbf{c}} = F_0\omega^2\frac{\mathbf{adj}(B)}{\det(B)}\vec{\mathbf{b}}$$

Homogeneous Solution

Theorem 11.56 provides:

$$\vec{\mathbf{x}}_h(t) = \sum_{j=1}^{5}(a_j\cos\omega_j t + b_j\sin\omega_j t)\vec{\mathbf{v}}_j$$

where $r = \omega_j$ and $\vec{\mathbf{v}} = \vec{\mathbf{v}}_j \neq \vec{\mathbf{0}}$ satisfy the **eigenpair equation**:

$$\left(\frac{1}{m}K + r^2 I\right)\vec{\mathbf{v}} = \vec{\mathbf{0}}$$

Identical Floors $k/m = 10$

Then:

$$\frac{1}{m}K = \begin{pmatrix} -20 & 10 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 10 & -10 \end{pmatrix}$$

Let $B(\omega, k/m) = (1/m)K + \omega^2 I$. Natural frequency values $\omega_1, \ldots, \omega_5$ are found by solving for ω in determinant equation $|B(\omega, 10)| = 0$ to obtain Table 3.

Table 3. Natural Frequencies ω for the Special Case $k/m = 10$.

Frequency	Value
ω_1	0.900078068
ω_2	2.627315231
ω_3	4.141702938
ω_4	5.320554507
ω_5	6.068366391

Identical Floors: General Solution

Superposition provides the general solution $\vec{\mathbf{x}}(t) = \vec{\mathbf{x}}_h(t) + \vec{\mathbf{x}}_p(t)$. If the floors are at rest, then $\vec{\mathbf{x}}_h = \vec{\mathbf{0}}$. Term $\vec{\mathbf{x}}_p$ measures bounded oscillations of the center of mass of each floor due to the incoming earthquake wave.

Identical Floors: Resonance Effects for $k/m = 10$
Special solution $\vec{\mathbf{x}}_p(t)$ can be used to obtain some insight into practical resonance effects between the incoming earthquake wave and movement of the building floors.

Let ω be the incoming wave natural frequency. Solution $\vec{\mathbf{x}}_p$ has components $A_1\cos(\omega t), \ldots, A_5\cos(\omega t)$. Let I have columns $e_1, \ldots, e_5$. The amplitude formula for $1 \le j \le 5$:

$$A_j = e_j^T \vec{\mathbf{c}} \cos(0) = \frac{F_0\omega^2}{|B(\omega, 10)|} e_j^T \,\mathbf{adj}(B(\omega, 10))\vec{\mathbf{b}}$$

The fraction has bounded numerator. Determinant $|B(\omega, 10)|$ in the denominator can be near zero when ω is close to one of the natural frequencies $\omega_1, \ldots, \omega_5$. Then the amplitude of a component of $\vec{\mathbf{x}}_p$ can be very large, which means the floor takes an excursion that is too large to maintain structural integrity.

Physical Interpretation: An earthquake wave of proper frequency, lasting sufficiently long, can demolish a floor and hence demolish the entire building. Small amplitude earthquake waves can initiate destructive oscillation of structures having unlucky natural frequencies.

Coupled Spring-Mass Systems: Derivations

Reproduced here from page 813 are notation and assumptions for three masses attached to each other by four springs as in Figure 14.

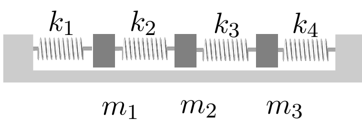

Figure 24. Three masses connected by springs. The masses slide along a frictionless track.

The analysis uses the following constants, variables and assumptions.

Mass Constants	The boxcar masses m_1, m_2, m_3 are assumed to be point masses concentrated at their center of gravity.
Spring Constants	The mass of each spring is negligible. The springs obey Hooke's law: *Force = k(elongation)*. The Hooke's constants are denoted k_1, k_2, k_3, k_4. The springs restore after compression and extension.
Position Variables	Symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the mass positions along the horizontal surface, measured from their equilibrium positions, plus right and minus left.
Fixed Ends	The first and last spring are attached to fixed walls.

The **competition method** is used to derive the equations of motion, using:

$$\text{Newton's Second Law Force} = \text{Sum of the Hooke's Forces}.$$

The model equations are

$$
\begin{aligned}
m_1 x_1''(t) &= -k_1 x_1(t) + k_2[x_2(t) - x_1(t)], \\
m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)] + k_3[x_3(t) - x_2(t)], \\
m_3 x_3''(t) &= -k_3[x_3(t) - x_2(t)] - k_4 x_3(t).
\end{aligned} \tag{9}
$$

The equations are justified in the case of all positive variables by observing that the first three springs are elongated by x_1, $x_2 - x_1$, $x_3 - x_2$, respectively. The last spring is compressed by x_3, which accounts for the minus sign.

Another way to justify the equations is through mirror-image symmetry: interchange $k_1 \longleftrightarrow k_4$, $k_2 \longleftrightarrow k_3$, $x_1 \longleftrightarrow x_3$, then equation 2 should be unchanged and equation 3 should become equation 1.

Matrix Formulation. System (9) can be written as a second order vector-matrix system

$$
\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}
\begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix}
=
\begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
$$

More succinctly, the system is written as

$$
M\vec{\mathbf{x}}''(t) = K\vec{\mathbf{x}}(t)
$$

where the **displacement** $\vec{\mathbf{x}}$, **mass matrix** M and **stiffness matrix** K are defined by the formulas

$$
\vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad
M = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}, \quad
K = \begin{pmatrix} -k_1 - k_2 & k_2 & 0 \\ k_2 & -k_2 - k_3 & k_3 \\ 0 & k_3 & -k_3 - k_4 \end{pmatrix}.
$$

Two Masses

Modeling of two masses connected by springs uses ideas and methods from three-mass modeling equation (9).

Two Masses, Right End Free

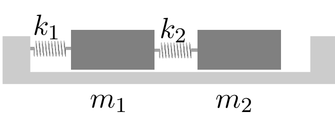

Figure 25. Two masses anchored left and connected by springs.

The model equations:

$$
\begin{aligned}
m_1 x_1''(t) &= -k_1 x_1(t) + k_2[x_2(t) - x_1(t)] \\
m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)]
\end{aligned} \tag{10}
$$

Two Masses, Both Ends Free

Equations (10) modified with $k_1 = 0$ gives model equations:

(11)
$$\begin{aligned} m_1 x_1''(t) &= k_2[x_2(t) - x_1(t)] \\ m_2 x_2''(t) &= -k_2[x_2(t) - x_1(t)] \end{aligned}$$

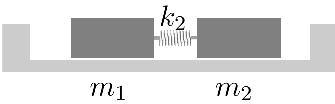

Figure 26. Two masses connected by one spring.

Example 11.18 (Two Masses with Free Right End)

Consider equation (10) with $m_1 = 2m_2$, $\dfrac{k_1}{m_1} = \dfrac{k_2}{m_2} = 50$:

$$\vec{\mathbf{x}}'' = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix} \vec{\mathbf{x}}$$

Then the vector solution in terms of arbitrary constants a_1, a_2, b_1, b_2 is given by:

$$\vec{\mathbf{x}} \quad = \quad (a_1 \cos 5t + b_1 \sin 5t) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (a_2 \cos 10t + b_2 \sin 10t) \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Details Example 11.18:
Eigenpairs of $A = \begin{pmatrix} -75 & 25 \\ 50 & -50 \end{pmatrix}$ are $\left(-25, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, $\left(-100, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right)$. The example is completed by Theorem 11.56.

Three Rail Cars

A special case of the coupled spring-mass system is three rail cars on a level frictionless track connected by springs, as in Figure 28.[14]

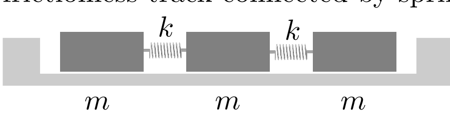

Figure 28. Three identical flatbed cars connected by identical springs.

Except for the springs on fixed ends, this problem is the same as the one in Figure 22. Let $k_1 = k_4 = 0$, $k_2 = k_3 = k$, $m_1 = m_2 = m_3 = m$ to give the system

(12)
$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \\ x_3'' \end{pmatrix} = \begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

[14]The cars are custom flatbed utility cars, **not boxcars**. Railway cars such as tankers, hoppers and boxcars are equipped with automatic Janney couplers, compression only dash-pots/bumpers and safety lanyards.

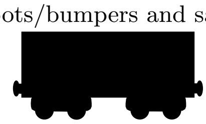

Figure 27. Railroad Boxcar Silhouette.

Example 11.19 (Identical Cars with $k = m$)

Consider equation (12) for $k = m$:

$$\vec{\mathbf{x}}'' = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \vec{\mathbf{x}}$$

Then the vector solution in terms of arbitrary constants a_1, a_2, a_3, b_1, b_2, b_3 is given by:

$$\begin{aligned} \vec{\mathbf{x}} &= (a_1 + b_1 t) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (a_2 \cos t + b_2 \sin t) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &\quad + \left(a_3 \cos \sqrt{3}t + b_3 \sin \sqrt{3}t\right) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \end{aligned} \tag{13}$$

Boxcars and Buffer Springs. Boxcars have buffer-spring shock absorbers which exert a force only under compression. Suppose one car moves along the track, then contacts two stationary cars, then transfers its momentum to the other cars, followed by disengagement. This situation could have a matrix model $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}} + B\vec{\mathbf{x}}'$. Matrix A contains Hooke's constants depending on $\vec{\mathbf{x}}$. Matrix B contains dashpot constants depending on $\vec{\mathbf{x}}$ and $\vec{\mathbf{x}}'$. The complexity seems suited for computer simulation.

Assume the dashpot constants are zero. The shock absorber springs act normally upon compression; the cars disengage upon full spring expansion. Model $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ has Hooke's constants in 3×3 matrix A. Solution expression (13) applies until a car disengages, measured by the first time $t = t_1 > 0$ at which $x_2(t) = x_1(t)$ or $x_3(t) = x_2(t)$. When a car contacts another car then the shock assembly compresses slightly but does not engage: the car making contact transfers momentum.

Analysis of one car moving into contact with two stationary cars uses equation (13) on $0 \le t \le t_1$. For $t > t_1$, model $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$ is discarded. One example is the first car transfers momentum and stops, while the other two cars travel at fixed speed. The model applies to determine both the time t_1 and the speed of the other two cars after $t = t_1$.

Dynamic Dashpot

A **dynamic dashpot** is a variable shock absorber, a component of **active suspension**.

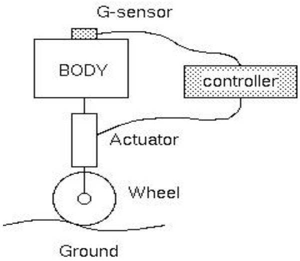

Figure 29. Active Suspension Components.

Bose Corporation (1980) designed variable shock absorbers for truck seats, especially 18-wheelers. Active suspension solutions for vehicles were designed by Toyota (1994), General Motors (2002), Volvo (2002), Range-Rover (2004) and Mercedes-Benz (2013). Camera and road-sensor devices have been implemented by Mercedes-Benz (2014).

An instance of Figure 29 is one wheel suspension with spring and shock absorber. Assumptions will fit the system to a damped spring-mass model.

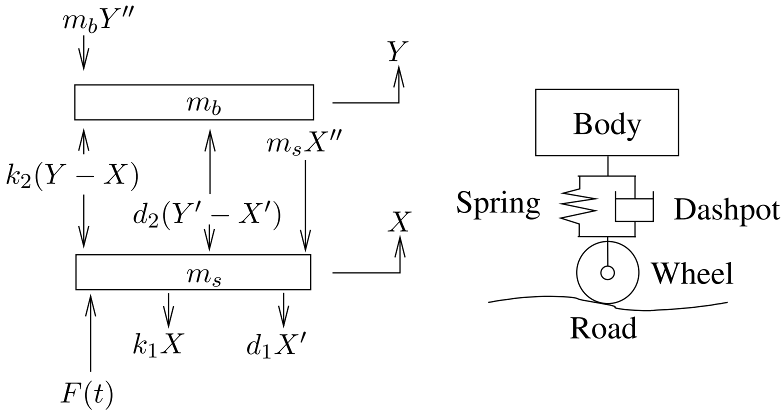

Figure 30. One wheel suspension with spring and shock absorber.

Assumptions for Figure 30.

Y = body mass displacement from equilibrium $Y = 0$
m_b = body mass
m_s = suspension system mass
X = suspension system mass displacement from equilibrium $X = 0$
k_1 = wheel and tire Hooke's constant
k_2 = suspension Hooke's constant
d_1 = wheel and tire dashpot constant
d_2 = suspension dashpot constant
$F(t)$ = roadway force on the wheel-suspension-body unit

$$(14)\qquad \begin{cases} m_sX'' = -k_1X - d_1X' - k_2(Y-X) - d_2(Y'-X') + F(t), \\ m_bY'' = k_2(Y-X) + d_2(Y'-X') \end{cases}$$

Ideal Suspension

Industrial solutions have used a tunable shock absorber, which means $c_2(t)$ is a function of time t defined in response to road data $F(t)$ and the current states $X(t), Y(t)$. **Is it realistic** to expect nearly motionless body vibration $Y \approx 0$ with suitable real-time changes in suspension dashpot constant $c_2(t)$? Manufacturers report yes, given suitably benign roadway data. Simulation uses the electrical-mechanical analogy to design an electrical circuit for model (14). Roadway data $F(t)$ from cameras and sensors is modeled by a variable input (emf) in the electrical circuit while mechanical displacements X, Y appear as electrical currents. Figure 31 shows an equivalent electrical network for computer simulation, extracted from a 2009 undergraduate Bachelor's Thesis project at Worcester Polytechnic Institute.[15]

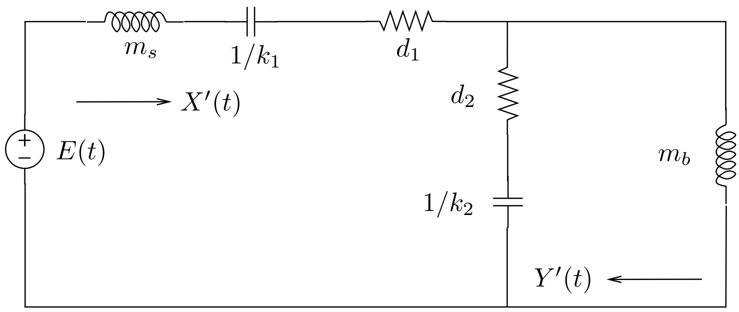

Figure 31. Dynamic shock absorber simulator circuit (2009)

Active Suspension Regulator

Added to Figure 30 is a regulator, which can be imagined as a linear electromagnetic motor that turns a shaft, one voltage input providing an upward force and the other input a downward force. Electrical supply voltages adjust the forces dynamically with sensor feedback. Standard suspension is $F(t) = 0$. Symbol $F(t)$ in Figures 30, 31 is a force, but each instance has a different meaning.

[15] Pashaj, B., Bermejo Calle, M. J., and Sebuwufu, P. (2009), **Dynamic Shock Absorber**, `https://digitalcommons.wpi.edu/mqp-all/2634`.

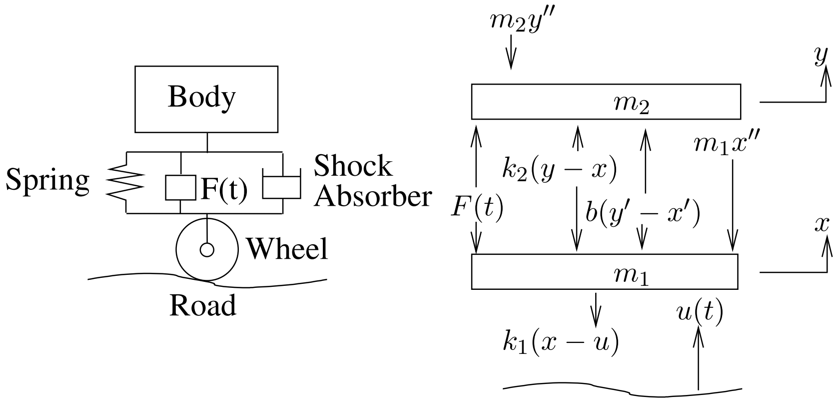

Figure 32. One wheel suspension with spring, shock absorber and regulator $F(t)$. Variables and forces are defined in the force diagram on the right. All units MKS.

Assumptions for Figure 32.

m_2 = body mass
y = body mass displacement from equilibrium $y = 0$
m_1 = suspension system mass
x = suspension system mass displacement from equilibrium $x = 0$
k_1 = wheel and tire Hooke's constant
k_2 = suspension Hooke's constant
b = shock absorber dashpot constant
$F(t)$ = regulator force between body and suspension system
$u(t)$ = roadway vertical displacement on the wheel-suspension-body unit

Equations (15) are derived from the force diagram in Figure 32.

$$(15) \qquad \begin{cases} m_1x'' = k_2(y-x) + b(y'-x') - k_1(x-u) - F(t), \\ m_2y'' = -k_2(y-x) - b(y'-x') + F(t) \end{cases}$$

Regulator. Assume system parameters in MKS units:

$$\begin{array}{ll} k_1 = 135000 & k_2 = 5700 \\ m_1 = 50 & m_2 = 465 \\ b = 290 & u(t) = 0.015\sin(t), \\ x(0) = x'(0) = 0 & \text{(suspension } m_1 \text{ at rest)} \\ y(t) = 0 & \text{(body } m_2 \text{ motionless)} \end{array}$$

The vertical roadway displacement $u(t) = 0.015\sin(t)$ fits a railroad track, zero to 1.5 cm deviation from perfectly flat. It is not suited for a highway. Period 2π is selected for simplicity. Equation (15) with values inserted implies equation (16):

$$(16) \qquad \begin{cases} 50\,x'' = 5700y - 140700x + 290y' - 290x' + 2025\sin(t) - F(t), \\ 465\,y'' = -5700y + 5700x - 290y' + 290x' + F(t), \\ x(0) = x'(0) = y(0) = y'(0) = 0 \end{cases}$$

Let's verify the **ideal suspension** regulator force:

$$
\text{(17)} \qquad \begin{cases} F(t) & = \dfrac{2565\sqrt{3}}{2699}\sin\left(30\sqrt{3}t\right) - \dfrac{230850}{2699}\sin(t) \\ & +\dfrac{11745}{2699}\cos\left(30\sqrt{3}t\right) - \dfrac{11745}{2699}\cos(t) \end{cases}
$$

if $y(t) = 0$, then $x(t)$ and $F(t)$ are determined by:

$$
\text{(18)} \qquad \begin{cases} 50x'' = -140700x - 290x' + 2025\sin(t) - F(t), \\ 0 = 5700x + 290x' + F(t), \\ x(0) = x'(0) = 0 \end{cases}
$$

Add equations (18):

$$
50x'' = -135000x + 2025\sin(t), \quad x(0) = x'(0) = 0.
$$

Then $x(t) = -\dfrac{9\sqrt{3}}{53980}\sin(30\sqrt{3}t) + \dfrac{81}{5398}\sin(t)$. Solve for $F(t)$ from the second equation in (18). Then equation (17) holds with approximation

$$
F(t) \approx 4.35\cos(51.9\,t) + 1.64\sin(51.9\,t) - 4.35\cos(t) - 85.5\sin(t)
$$

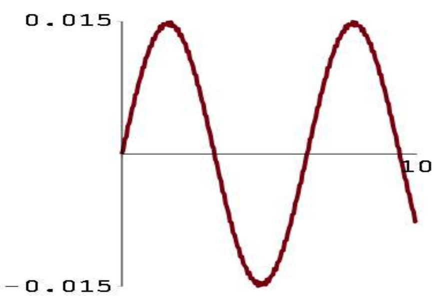

Figure 33. Suspension displacement

Solution $x(t)$ for a motionless body $y(t) = 0$ with roadway displacement $u(t) = 0.015\sin(t)$.

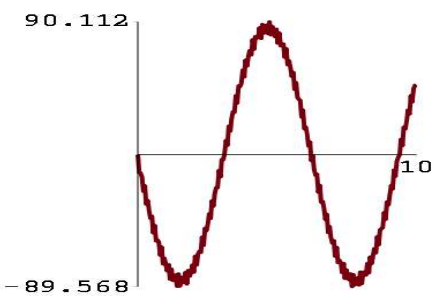

Figure 34. Regulator force

Force $F(t)$ for a motionless body $y(t) = 0$ with roadway displacement $u(t) = 0.015\sin(t)$.

Jagged edges in Figure 33 are caused by the high frequency term in $x(t) = -\dfrac{9\sqrt{3}}{53980}\sin(30\sqrt{3}t) + \dfrac{81}{5398}\sin(t)$. Similarly for Figure 34.

Exercises 11.8

Euler's Substitution: $\vec{\mathbf{u}}\,' = C\vec{\mathbf{u}}$

1. Change variables: $\vec{\mathbf{u}} = e^{rt}\vec{\mathbf{w}}$. Answer: $\vec{\mathbf{w}}\,' = (C - rI)\vec{\mathbf{w}}$

2. Prove: $(\lambda, \vec{\mathbf{v}})$ is an eigenpair of C if and only if $(0, \vec{\mathbf{v}})$ is an eigenpair of $C - \lambda I$.

3. Let $|C - \lambda I|$ have factor λ^2. Let $\vec{\mathbf{u}}\,' = C\vec{\mathbf{u}}$ have solution $\vec{\mathbf{u}} = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2$. Prove: $C\vec{\mathbf{d}}_2 = \vec{\mathbf{0}}$, $C\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_2$. Are $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ eigenvectors of C? Discuss.

4. Let $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\vec{\mathbf{u}} = \vec{\mathbf{d}}_1 + t\vec{\mathbf{d}}_2$. Let $\vec{\mathbf{u}}$ solve $\vec{\mathbf{u}}\,' = C\vec{\mathbf{u}}$. Find $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$ in terms of arbitrary constants c_1, c_2.

Euler's Substitution: $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$

5. Change variables: $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{y}}$. Answer: $\vec{\mathbf{y}}\,'' + 2r\vec{\mathbf{y}}\,' = (A - r^2 I)\vec{\mathbf{y}}$

6. Prove: $\vec{\mathbf{x}} = e^{rt}\vec{\mathbf{v}}$ is a nonzero solution of $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$ if and only if $(r^2, \vec{\mathbf{v}})$ is an eigenpair of A.

Repeated Root: $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$

Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, eigenvalues $0, 0$.

7. Verify: Matrix A is a Jordan block with generalized eigenvectors the columns of I.

8. Prove: $x_1 = c_1 + c_2 t + c_3 \frac{t^2}{2} + c_4 \frac{t^3}{6}$, $x_2 = c_3 + c_4 t$ for arbitrary constants c_1 to c_4.

9. Prove: The solution of $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$ is a vector linear combination of atoms $1, t, t^2, t^3$.

10. Let $\vec{\mathbf{x}} = \vec{\mathbf{d}}_1 + \vec{\mathbf{d}}_2 t + \vec{\mathbf{d}}_3 \frac{t^2}{2} + \vec{\mathbf{d}}_4 \frac{t^3}{6}$. Assume $\vec{\mathbf{x}}$ solves $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$. Prove: $A\vec{\mathbf{d}}_3 = A\vec{\mathbf{d}}_4 = \vec{\mathbf{0}}$, $A\vec{\mathbf{d}}_1 = \vec{\mathbf{d}}_3$, $A\vec{\mathbf{d}}_2 = \vec{\mathbf{d}}_4$. These are generalized eigenvector chains for eigenvalue zero.

CHZ Method

11. Given a 3×3 matrix A, supply proof details for the Cayley-Hamilton-Ziebur structure theorem.

12. Invent a non-diagonal 3×3 example $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$ and solve it by CHZ.

13. Solve $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$ by CHZ for any 2×2 diagonal matrix with negative diagonal elements.

14. Solve $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$ by CHZ for any 3×3 diagonal matrix with negative diagonal elements.

Conversion

Given $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$, let $\vec{\mathbf{u}} = \begin{pmatrix} \vec{\mathbf{x}} \\ \vec{\mathbf{x}}\,' \end{pmatrix}$. Display system $\vec{\mathbf{u}}\,' = C\vec{\mathbf{u}}$.

15. $A = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$

16. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{pmatrix}$

Eigenanalysis $\lambda \le 0$

Display the general solution of $\vec{\mathbf{x}}\,'' = A\vec{\mathbf{x}}$.

17. $A = \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}$

18. $A = \begin{pmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \\ 5 & 0 & -1 \end{pmatrix}$

Earthquakes

Apply formulas from the *Earthquakes subsection* page 929 to find particular solution $\vec{\mathbf{x}}_p$, the natural frequencies ω_j and the amplitudes of $\vec{\mathbf{x}}_p(t)$ near the largest natural frequency. Assume $F(t) = F_0 \cos(\omega t)$.

19. Three-floor problem, $k/m = 10$.

20. Four-floor problem, $k/m = 10$.

Two Masses

Assume MKS units. Let $m_1 = 2$, $m_2 = 0.5$, $k_1 = 75$, $k_2 = 25$ in system:

$$\begin{aligned} m_1 x_1'' &= -k_1 x_1 + k_2[x_2 - x_1] \\ m_2 x_2'' &= -k_2[x_2 - x_1] \end{aligned}$$

21. Convert the system to the form $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$.

22. Show details for finding the vector solution $\vec{\mathbf{x}}(t)$.

Three Rail Cars: $k{=}2m$

Assume MKS units. Consider

$$\vec{\mathbf{x}}'' = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -4 & 2 \\ 0 & 2 & -2 \end{pmatrix} \vec{\mathbf{x}}$$

23. Show eigenpair details for the 3×3 matrix.

24. Find the vector solution $\vec{\mathbf{x}}(t)$.

Three Rail Cars: Disengagement

For $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$, assume FPS units and

$$A = \begin{pmatrix} -4 & 4 & 0 \\ 6 & -12 & 6 \\ 0 & 4 & -4 \end{pmatrix}$$

Suppose the springs disengage upon full expansion. Let the cars engage at $t = 0$ with $x_1 = x_2 = x_3 = 0$.

25. Verify A has eigenvalues $\lambda = -16, 0, -4$ and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

26. For $x_1{=}x_2{=}x_3{=}0$ at $t{=}0$, verify:
$x_1(t){=}c_1t{+}c_2\sin(2t){-}c_3\sin(4t)$,
$x_2(t) = c_1t + 3c_3\sin(4t)$,
$x_3(t){=}c_1t{-}c_2\sin(2t){-}c_3\sin(4t)$

27. Let $x_1' = 48$, $x_2' = 0$, $x_3' = 0$ at $t = 0$. Verify disengagement time $t_1 = \pi/2$ and determine the car velocities thereafter.

28. Let $x_1'(0) = 144$, $x_2'(0) = 48$, $x_3'(0) = 48$. Verify disengagement time $t_1 = \pi/2$ and determine the car velocities thereafter.
Answer: Velocities 144, 48, 48 at $t = t_1$.

Dynamic Dashpot

Assume conventions for Figure 26 and dynamic dashpot system

$$\begin{aligned} m_sX'' &= -k_1X - d_1X' - k_2(Y - X) \\ &\quad - d_2(Y' - X') + F(t), \\ m_bY'' &= k_2(Y - X) + d_2(Y' - X') \end{aligned}$$

See page 936.

29. Assume $Y = 0$, ideal suspension. Derive:

$$\begin{aligned} &m_sX'' = -k_1X - d_1X' + F(t), \\ &d_2X' + k_2X = 0 \end{aligned}$$

30. Assume $Y = 0$, ideal suspension and $X(0) = 0.015$ meters. Find $X(t)$ and $F(t)$.

11.9 Numerical Methods for Systems

An initial value problem for a system of two differential equations is given by the equations

$$
\begin{aligned}
x'(t) &= f(t, x(t), y(t)), \\
y'(t) &= g(t, x(t), y(t)), \\
x(t_0) &= x_0, \\
y(t_0) &= y_0.
\end{aligned} \tag{1}
$$

A **numerical method** for (1) is an algorithm that computes an approximation table with first line t_0, x_0, y_0. Generally, the table has equally spaced t-values, two consecutive t-values differing by a constant value $h \neq 0$, called the **step size**. To illustrate, if $t_0 = 2$, $x_0 = 5$, $y_0 = 100$, then a typical approximation table for step size $h = 0.1$ might look like

t	x	y
2.0	5.00	100.00
2.1	5.57	103.07
2.2	5.62	104.10
2.3	5.77	102.15
2.4	5.82	101.88
2.5	5.96	100.55

Graphics

The approximation table represents the data needed to plot a solution curve to system (1) in three dimensions (t, x, y) or in two dimensions, using a tx-scene or a ty-scene. In all cases, the plot is a simple connect-the-dots graphic.

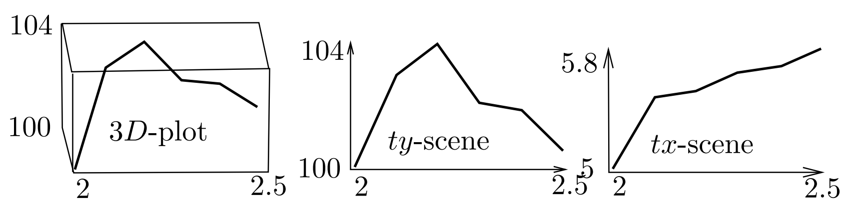

Figure 35. Dot table plots.
The three dimensional plot is a space curve made directly from the dot table. The tx-scene and the ty-scene are made from the same approximation table using corresponding data columns.

Near-Sighted Algorithms

All of the popular algorithms for numerical generation of an approximation table for system (1) are **near-sighted algorithm**, because they predict the next line in the table from the current table line, ignoring effects and errors for all other preceding table lines. Among such algorithms are **Euler's method**, **Heun's**

method and the **RK4 method**, which are showcased here for learning purposes. Computer production algorithms are available in `maple`, `mathematica` and `matlab`.

Numerical Algorithms: Planar Case

Stated here without proof are three numerical algorithms for solving planar initial value problems (1). Justification of the formulas is obtained from the vector relations in the next subsection.

Notation. Let t_0, x_0, y_0 denote the entries of the approximation table on a particular line. Let h be the increment for the table and let $t_0 + h$, x, y denote the table entries on the next line.

Planar Euler Method

$$\begin{aligned} x &= x_0 + hf(t_0, x_0, y_0), \\ y &= y_0 + hg(t_0, x_0, y_0). \end{aligned}$$

Planar Heun Method

$$\begin{aligned} x_1 &= x_0 + hf(t_0, x_0, y_0), \\ y_1 &= y_0 + hg(t_0, x_0, y_0), \\ x &= x_0 + h(f(t_0, x_0, y_0) + f(t_0 + h, x_1, y_1))/2 \\ y &= y_0 + h(g(t_0, x_0, y_0) + g(t_0 + h, x_1, y_1))/2. \end{aligned}$$

Planar RK4 Method

$$\begin{aligned} k_1 &= hf(t_0, x_0, y_0), \\ m_1 &= hg(t_0, x_0, y_0), \\ k_2 &= hf(t_0 + h/2, x_0 + k_1/2, y_0 + m_1/2), \\ m_2 &= hg(t_0 + h/2, x_0 + k_1/2, y_0 + m_1/2), \\ k_3 &= hf(t_0 + h/2, x_0 + k_2/2, y_0 + m_2/2), \\ m_3 &= hg(t_0 + h/2, x_0 + k_2/2, y_0 + m_2/2), \\ k_4 &= hf(t_0 + h, x_0 + k_3, y_0 + m_3), \\ m_4 &= hg(t_0 + h, x_0 + k_3, y_0 + m_3), \\ x &= x_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ y &= y_0 + \frac{1}{6}\left(m_1 + 2m_2 + 2m_3 + m_4\right). \end{aligned}$$

Example 11.20 (Planar Methods)
Solve $x' = x, y' = -2y$, $x(0) = y(0) = 2$ with step size $h = 0.1$ for 10 steps, using methods Euler, Heun and RK4 in computer algebra system `MAPLE`.

Details
Computer code for the three algorithms can be found in the solution to Exercise 1. Newer `MAPLE` versions have the algorithms available as documented below.

```
des:=diff(x(t),t)=x(t),diff(y(t),t)=-2*y(t);ics:=x(0)=2,y(0)=2;
args:=[des,ics],numeric,stepsize=0.1,output=listprocedure;
p:=dsolve(args,method=classical[foreuler]);# or: heunform, rk4
X:=eval(x(t),p); Y:=eval(y(t),p);
printf("Euler\n  t       X(t)      Y(t)\n");
seq(printf("%f  %f  %f\n",0.1*j,X(0.1*j),Y(0.1*j)),j=0..10);
```

The expected results are 1, 2, 4 digits of accuracy respectively for the computed values. At $t = 1$ the `maple` code for step size 0.1 computes $y(t)$ for Euler, Heun, RK4 as 0.214748, 0.274896, 0.270679 compared to exact value $y(1) = 2e^{-2} = 0.2706705664$.

Numerical Algorithms: General Case

Consider a vector initial value problem

$$\vec{\mathbf{u}}'(t) = \vec{\mathbf{F}}(t, \vec{\mathbf{u}}(t)), \quad \vec{\mathbf{u}}(t_0) = \vec{\mathbf{u}}_0.$$

Stated here are the vector formulas for Euler, Heun and RK4 methods. These myopic algorithms predict the next table entry $t_0 + h$, $\vec{\mathbf{u}}$ from the current entry t_0, $\vec{\mathbf{u}}_0$. The number of scalar values in a table row is $1 + n$, where n is the dimension of the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{F}}$.

Vector Euler Method

$$\vec{\mathbf{u}} = \vec{\mathbf{u}}_0 + h\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0)$$

Vector Heun Method

$$\vec{\mathbf{w}} = \vec{\mathbf{u}}_0 + h\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0), \quad \vec{\mathbf{u}} = \vec{\mathbf{u}}_0 + \frac{h}{2}\left(\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0) + \vec{\mathbf{F}}(t_0 + h, \vec{\mathbf{w}})\right)$$

Vector RK4 Method

$$
\begin{aligned}
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0),\\
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0 + h/2, \vec{\mathbf{u}}_0 + \vec{\mathbf{k}}_1/2),\\
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0 + h/2, \vec{\mathbf{u}}_0 + \vec{\mathbf{k}}_2/2),\\
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0 + h, \vec{\mathbf{u}}_0 + \vec{\mathbf{k}}_3),\\
\vec{\mathbf{u}} &= \vec{\mathbf{u}}_0 + \frac{1}{6}\left(\vec{\mathbf{k}}_1 + 2\vec{\mathbf{k}}_2 + 2\vec{\mathbf{k}}_3 + \vec{\mathbf{k}}_4\right).
\end{aligned}
$$

Example 11.21 (Exact Solution $\vec{\mathbf{u}}\,' = A\,\vec{\mathbf{u}} + \vec{\mathbf{F}}\,(t)$)

Let $A = \begin{pmatrix} 1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$, $\vec{\mathbf{F}}(t) = \begin{pmatrix}1\\1\\0\end{pmatrix}$. Solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$.

Details

Handwritten method: find a fundamental matrix $\Phi(t)$ and then $e^{At} = \Phi(t)\Phi(0)^{-1}$. The homogeneous solution is $u_h(t) = e^{At}\vec{\mathbf{c}}$ for constant vector $\vec{\mathbf{c}}$. A particular solution $\vec{\mathbf{u}}_p(t)$ is computed from the variation of parameters formula page 912.

CAS method: One possible method uses `MAPLE` library `DEtools`:

```
A:=Matrix([[1, -1 , 0],[1 , 1 , 0],[0 , 0 , 2]]);
F:=Vector([1,1,0]);Sol:=DEtools[matrixDE](A,F,t);
Xh:=Sol[1].Vector([c1,c2,c3]);Xp:=Vector(convert(Sol[2],list));
U:=unapply(Xh+Xp,t);U(t);# General solution of u'=Au+F(t)
simplify(A.U(t)+F-map(diff,U(t),t));# Answer check
```

$$
\vec{\mathbf{u}}(t) = \begin{bmatrix} e^t\cos(t)\, c_1 + e^t \sin(t)\, c_2 - 1\\ e^t\sin(t)\, c_1 - e^t\cos(t)\, c_2\\ e^{2t} c_3\end{bmatrix}
$$

Example 11.22 (Vector Euler Method)

Let $A = \begin{pmatrix} 1 & -1 & 0\\ 1 & 1 & 0\\ 0 & 0 & 2\end{pmatrix}$. $\vec{\mathbf{F}}(t) = \begin{pmatrix}e^t\\1\\0\end{pmatrix}$. Solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$, $\vec{\mathbf{u}}(0) = \begin{pmatrix}1\\0\\0\end{pmatrix}$ with step size $h = 0.1$ for 10 steps, using the vector Euler method implemented in computer algebra system `MAPLE`.

Details

The vector algorithm uses `MAPLE` functions and basic vector-matrix algebra.

```
# Euler's method with vector notation
A:=Matrix([[1, -1 , 0],[1 , 1 , 0],[0 , 0 , 2]]);
F0:=unapply(A.<x,y,z>+Vector([exp(t),1,0]),(t,x,y,z)):
F0(t,x,y,z);# Scalar variables
F:=(t,X)->F0(t,X[1],X[2],X[3]);# Vector variables
U0:=<1,0,0>;n:=10;h:=0.1;t0:=0;Vals:=U0; # Initialize
for j from 1 to n do
U:=U0+h*F(t0,U0);U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsEuler:=Vals[n+1];
```

$$\texttt{ValsEuler} = \begin{bmatrix} 3.1116983042 \\ 4.4649291918 \\ 0.0 \end{bmatrix}$$

Example 11.23 (Vector Heun Method)

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. $\vec{\mathbf{F}}(t) = \begin{pmatrix} e^t \\ 1 \\ 0 \end{pmatrix}$. Solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with step size $h = 0.1$ for 10 steps, using the vector Heun method implemented in computer algebra system MAPLE.

Details

```
# Heun's method with vector notation
A:=Matrix([[1, -1 , 0],[1 , 1 , 0],[0 , 0 , 2]]);
F0:=unapply(A.<x,y,z>+Vector([exp(t),1,0]),(t,x,y,z)):
F0(t,x,y,z);# Scalar variables
F:=(t,X)->F0(t,X[1],X[2],X[3]);# Vector variables
U0:=<1,0,0>;n:=10;h:=0.1;t0:=0:Vals:=U0; # Initialize
for j from 1 to n do
w:=U0+h*F(t0,U0);
U:=U0+0.5*h*(F(t0,U0)+F(t0+h,w));U0:=U;t0:=t0+h;Vals:=Vals,U0;
od:
ValsHeun:=Vals[n+1];
```

$$\texttt{ValsHeun} = \begin{bmatrix} 2.8724813157 \\ 4.9105494201 \\ 0.0 \end{bmatrix}$$

Example 11.24 (Vector RK4 Method)

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. $\vec{\mathbf{F}}(t) = \begin{pmatrix} e^t \\ 1 \\ 0 \end{pmatrix}$. Solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with step size $h = 0.1$ for 10 steps, using the vector RK4 method implemented in computer algebra system MAPLE.

Details

```
# RK4 method with vector notation
A:=Matrix([[1, -1 , 0],[1 , 1 , 0],[0 , 0 , 2]]);
F0:=unapply(A.<x,y,z>+Vector([exp(t),1,0]),(t,x,y,z)):
F0(t,x,y,z);# Scalar variables
F:=(t,X)->F0(t,X[1],X[2],X[3]);# Vector variables
U0:=<1,0,0>;n:=10;h:=0.1;t0:=0:Vals:=U0; # Initialize
for j from 1 to n do
k1:=h*F(t0,U0);
k2:=h*F(t0+h/2,U0+k1/2);
k3:=h*F(t0+h/2,U0+k2/2);
k4:=h*F(t0+h,U0+k3);
U:=U0+(k1+2*k2+2*k3+k4)/6;U0:=U;t0:=t0+h;Vals:=Vals,U0;od:
ValsRK4:=Vals[n+1];
```

$$\texttt{ValsRK4} = \begin{bmatrix} 2.8467234249 \\ 4.9149919169 \\ 0.0 \end{bmatrix}$$

Example 11.25 (Compare Vector Methods Euler, Heun and RK4)

Let $A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $\vec{\mathbf{F}}(t) = \begin{pmatrix} e^t \\ 1 \\ 0 \end{pmatrix}$. Solve $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with step size $h = 0.1$ for 10 steps, using the vector methods Euler, Heun and RK4 in computer algebra system MAPLE. Compare to 6 digits computed values at $t = 1$ for the three methods.

Details

Refer to the previous three examples for `maple` values `ValsEuler, ValsHeun, ValsRK4, Exact`.

$$\begin{bmatrix} 2.872481 \\ 4.910549 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 2.872481 \\ 4.910549 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 2.846723 \\ 4.914992 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 2.846719 \\ 4.914968 \\ 0.0 \end{bmatrix}.$$

Exercises 11.9

Planar Methods

Apply the Euler, Heun and RK4 methods. Compare with the exact solution in a table.

1. $x' = x$, $y' = -y$, $x(0) = 2$, $y(0) = 2$. $h = 0.1$, 10 steps

2. $x' = -3x + y$, $y' = x - 3y$, $x(0) = 2$, $y(0) = 0$, $h = 0.1$, 10 steps

3. $x' = -x + y$, $y' = -x - y$, $x(0) = 0$, $y(0) = 3$, $h = 0.2$, 5 steps

4. $x' = 2x - 4y$, $y' = x - 3y$, $x(0) = 4$, $y(0) = 0$, $h = 0.1$, 10 steps

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, 2×2

Apply vector Euler, Heun and RK4 methods for 10 steps with $h = 0.1$.

5. $\vec{\mathbf{u}}' = \begin{pmatrix} u_1 + u_2 \\ -u_1 + u_2 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

6. $\vec{\mathbf{u}}' = \begin{pmatrix} -3u_1 + u_2 \\ u_1 - 3u_2 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$

Apply vector Euler, Heun and RK4 methods for 10 steps with $t_0 = 0$, $h = 0.1$. Compare results for the last step.

7. $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, $\vec{\mathbf{F}} = \begin{pmatrix} e^t \\ 0 \end{pmatrix}$,

$\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Ans Euler: $3.81, -5.33$

8. $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{F}} = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}$,

$\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans RK4: $2.576, -5.528, 0.0$

Vector Methods $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$, 3×3

Apply vector Euler, Heun and RK4 methods for 10 steps with $h = 0.1$.

9. $A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans Heun: $1.36, -3.67, 0.00$

10. $A = \begin{pmatrix} 1 & 3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{\mathbf{u}}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Ans RK4: $-2.307, -3.075, 0.00$

PDF Sources

Text, Solutions and Corrections

Author: Grant B. Gustafson, University of Utah, Salt Lake City 84112.
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