Contents

Chapter 10

Phase Plane Methods

Contents

Studied here are planar autonomous systems of differential equations. The topics:

1. Planar Autonomous Systems: Phase Portraits, Stability.
2. Planar Constant Linear Systems: Classification of isolated equilibria, Phase portraits.
3. Planar Almost Linear Systems: Phase portraits, Nonlinear classifications of equilibria.
4. Biological Models: Predator-prey models, Competition models, Survival of one species, Co-existence, Alligators, doomsday and extinction.
5. Mechanical Models: Nonlinear spring-mass system, Soft and hard springs, Energy conservation, Phase plane and scenes.

10.1 Planar Autonomous Systems

A set of two scalar differential equations of the form

(1)
$$\begin{aligned} x'(t) &= f(x(t), y(t)), \\ y'(t) &= g(x(t), y(t)). \end{aligned}$$

is called a **planar autonomous system**. The term **Autonomous** means **Self-Governing**, justified by the absence of the time variable t in the functions $f(x, y)$, $g(x, y)$.

To obtain the vector form, let $\vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $\vec{F}(x, y) = \begin{pmatrix} f(x, y) \\ g(x, y) \end{pmatrix}$ and write (1) as the first order vector-matrix system

(2)
$$\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t)).$$

It is assumed that f, g are continuously differentiable in some region $\mathcal{D}$ in the xy-plane. This assumption makes $\vec{F}$ continuously differentiable in $\mathcal{D}$ and guarantees that Picard's existence-uniqueness theorem for initial value problems applies to the initial value problem $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$, $\vec{u}(0) = \vec{u}_0$. Accordingly, to each $\vec{u}_0 = (x_0, y_0)$ in $\mathcal{D}$ there corresponds a unique solution $\vec{u}(t) = (x(t), y(t))$, represented as a planar curve in the xy-plane, which passes through $\vec{u}_0$ at $t = 0$.

Such a planar curve is called a **Trajectory** or **Orbit** of the system and its parameter interval is some maximal interval of existence $T_1 < t < T_2$, where T_1 and T_2 might be infinite. A graphic of trajectories drawn as parametric curves in the xy-plane is called a **Phase Portrait** and the xy-plane in which it is drawn is called the **Phase Plane**.

Trajectories Don't Cross

Autonomy of the planar system plus uniqueness of initial value problems implies that trajectories $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ cannot touch or cross. Hand-drawn phase portraits are accordingly limited: *you cannot draw a solution trajectory that touches another solution curve*!

Theorem 10.1 (Identical Trajectories)
Assume that Picard's existence-uniqueness theorem applies to initial value problems in $\mathcal{D}$ for the planar system

$$\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t)), \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Let $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ be two trajectories of the system. If times t_1, t_2 exist such that

(3)
$$x_1(t_1) = x_2(t_2), \quad y_1(t_1) = y_2(t_2),$$

then for the value $c = t_1 - t_2$ the equations $x_1(t+c) = x_2(t)$ and $y_1(t+c) = y_2(t)$ are valid for all allowed values of t. This means that the two trajectories are on one and the same planar curve, or in the contrapositive, two different trajectories cannot touch or cross in the phase plane.

Proof: Define $x(t) = x_1(t+c)$, $y(t) = y_1(t+c)$. By the chain rule, $(x(t), y(t))$ is a solution of the planar system, because $x'(t) = x_1'(t+c) = f(x_1(t+c), y_1(t+c)) = f(x(t), y(t))$, and similarly for the second differential equation. Further, (3) implies $x(t_2) = x_2(t_2)$ and $y(t_2) = y_2(t_2)$, therefore Picard's uniqueness theorem implies that $x(t) = x_2(t)$ and $y(t) = y_2(t)$ for all allowed values of t. ■

Equilibria

A trajectory that reduces to a point, or a constant solution $x(t) = x_0$, $y(t) = y_0$, is called an **Equilibrium Solution**. The equilibrium solutions or **Equilibria** are found by solving the nonlinear equations

$$f(x_0, y_0) = 0, \quad g(x_0, y_0) = 0.$$

Each such (x_0, y_0) in $\mathcal{D}$ is a trajectory whose graphic in the phase plane is a single point, called an **Equilibrium Point**. In applied literature, it may be called a **Critical Point**, **Stationary Point** or **Rest Point**. Theorem 10.1 has the following geometrical interpretation.

> Assuming uniqueness, no other trajectory $(x(t), y(t))$ in the phase plane can touch an equilibrium point (x_0, y_0).

Equilibria (x_0, y_0) are often found from linear equations

$$ax_0 + by_0 = e, \quad cx_0 + dy_0 = f,$$

which are solved by linear algebra methods. They constitute an important subclass of algebraic equations which can be solved symbolically. In this special case, symbolic solutions exist for the equilibria.

It is interesting to report that in a practical sense the equilibria may be reported incorrectly, due to the limitations of computer software, even in the case when exact symbolic solutions are available. An example is $x' = x + y$, $y' = \epsilon y - \epsilon$ for small $\epsilon > 0$. The root of the problem is translation of ϵ to a machine constant, which is zero for small enough ϵ. The result is that computer software detects infinitely many equilibria when in fact there is exactly one equilibrium point. This example suggests that symbolic computation be used by default.

Practical Methods for Computing Equilibria

There exists no supporting theory to find equilibria for all choices of F and G. However, there is a rich library of special methods for solving nonlinear algebraic equations, including numerical methods based on celebrated univariate methods, such as **Newton's method** and the **Bisection method**.

Computer algebra systems like `maple`, `maxima` and `mathematica` offer convenient codes to solve the equations, when possible, including symbolic solutions. Applied mathematics depends on the dynamically expanding library of special methods, which grows due to new mathematical discoveries. See the exercises for examples.

Population Biology

Planar autonomous systems have been applied to two-species populations like two species of trout, who compete for food from the same supply, and foxes and rabbits, who compete in a predator-prey situation.

Certain equilibria are significant, because they represent the population sizes for **Cohabitation**. A point in the phase space that is not an equilibrium point corresponds to population sizes that cannot coexist, they must change with time. Some equilibria are consequently **Observable** or **average** population sizes while non-equilibria correspond to snapshot population sizes that are subject to flux. Biologists expect population sizes of such two-species competition models to undergo change until they reach approximately the observable values, on the average.

Rabbit-Fox System

This example is a **Predator-Prey** system, in which the expected observable population sizes are averages, about which the actual populations size oscillate about, periodically over time. Certain equilibria for these systems represent **ideal cohabitation**. Biological experiments suggest that initial population sizes close to the equilibrium values cause populations to stay near the initial sizes, even though the populations oscillate periodically. Observations by field biologists of large population variations seem to verify that individual populations oscillate periodically around the ideal cohabitation sizes.

A typical planar system for predator-prey dynamics of $x(t)$ rabbits and $y(t)$ foxes is the system

$$\begin{aligned} \frac{dx}{dt} &= \frac{1}{200}x(40-y), \\ \frac{dy}{dt} &= \frac{1}{100}y(x-50). \end{aligned}$$

Time variable t is in months. The equilibria are $(0,0), (50,40)$. With initial populations $x(0) = 60$ rabbits and $y(0) = 30$ foxes, both x' and y' are positive near $t = 0$, which implies the populations initially increase in size.

After time, the signs of x' and y' are alternately positive and negative, which reflects the oscillating behavior of the populations about the ideal equilibrium values $x = 50$, $y = 40$. The period of oscillation is about 20 months. This predator-prey model predicts coexistence with average populations of 50 rabbits and 40 foxes.

Trout System

Consider a population of two species of trout who compete for the same food supply. A typical autonomous planar system for the species x and y is

$$\begin{aligned} \frac{dx}{dt} &= x(-2x - y + 180), \\ \frac{dy}{dt} &= y(-x - 2y + 120). \end{aligned}$$

Equilibria. The equilibrium solutions for the trout system are

$$(0,0), \quad (90,0), \quad (0,60), \quad (80,20).$$

Only nonnegative population sizes are physically significant. Units for the population sizes might be in hundreds or thousands of fish. The equilibrium $(0,0)$ corresponds to **Extinction** of both species, while $(0,60)$ and $(90,0)$ correspond to the unusual situation of extinction for one species. The last equilibrium $(80,20)$ corresponds to **Co-Existence** of the two trout species with observable population sizes of 80 and 20.

Phase Portraits

A graphic which contains some equilibria and typical trajectories of a planar autonomous system (1) is called a **Phase Portrait**.

While graphing equilibria is not a challenge, graphing typical trajectories, also called **orbits**, seems to imply that we are going to solve the differential system. This is not the case. Approximations will be used that do not require solution of the differential system.

Equilibria	Plot in the xy-plane all equilibria of (1). See Figure 3.
Window	Select an x-range and a y-range for the graph window which includes all significant equilibria (Figure 3).
Grid	Plot a uniform grid of N grid points ($N \approx 50$ for hand work) within the graph window, to populate the graphical white space (Figure 4). The isocline method might also be used to select grid points.
Field	Draw at each grid point a short tangent vector, a **replacement curve** for a solution curve through a grid point on a small time interval (Figure 5).

Orbits — Draw additional threaded trajectories on long time intervals into the remaining white space of the graphic (Figure 6). This is guesswork, based upon tangents to threaded trajectories matching nearby field tangents drawn in the previous step. See Figures 1 and 2 for details.

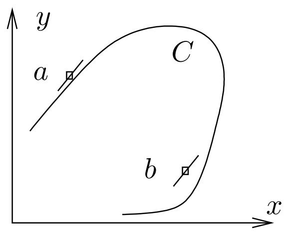


Figure 1. Badly threaded orbit.
Threaded solution curve C correctly matches its tangent to the tangent at nearby grid point a, but it fails to match at grid point b.

Why does a threaded solution curve tangent $\vec{T}_1$ have to *match* [1] a tangent $\vec{T}_2$ at a nearby grid point (see Figure 2)? A tangent vector is given by $\vec{T} = \frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$. Then $\vec{T}_1 = \vec{F}(\vec{u}_1)$, $\vec{T}_2 = \vec{F}(\vec{u}_2)$. However, $\vec{u}_1 \approx \vec{u}_2$ in the graphic, hence by continuity of $\vec{F}$ it follows that $\vec{F}(\vec{u}_1) \approx \vec{F}(\vec{u}_2)$, which implies $\vec{T}_1 \approx \vec{T}_2$.

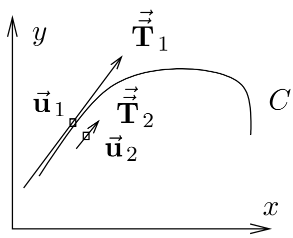


Figure 2. Tangent matching.
Threaded solution curve C matches its tangent $\vec{T}_1$ at $\vec{u}_1$ to direction field tangent $\vec{T}_2$ at nearby grid point $\vec{u}_2$.

It is important to emphasize that solution curves starting at a grid point are defined for a small t-interval about $t = 0$, and therefore their graphics extend on both sides of the grid point. We intend to shorten these curves until they appear to be straight line segments, graphically atop the tangent line, to pixel resolution. Adding an arrowhead pointing in the tangent vector direction is usual. After all this construction, *the shaft of the arrow is graphically atop a short solution curve segment.* In fact, if 50 grid points were used, then 50 short solution curve segments have already been entered onto the graphic! Threaded orbits are added to show what happens to solutions that are plotted on longer and longer t-intervals.

Phase Portrait Illustration

The method outlined above will be applied to the illustration

$$(4)\qquad \begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

The equilibria are $(1, -1)$ and $(-1, 1)$. The graph window is selected as $|x| \le 2$, $|y| \le 2$, in order to include both equilibria. The uniform grid will be 11×11,

[1] *Match* means nearly identical, in an approximate sense: graphics of the two tangents are identical to pixel resolution.

although for hand work 5×5 is normal. Tangents at the grid points are short line segments which do not touch each other – they are graphically the same as short solution curves.

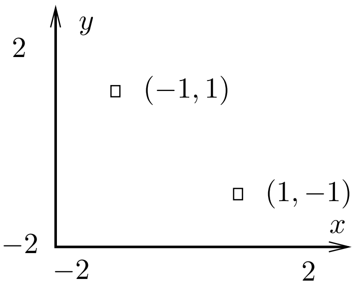


Figure 3. Equilibria $(1,-1)$, $(-1,1)$ with Invented Graph Window.
The equilibria (x,y) are calculated from equations $0 = x + y$, $0 = 1 - x^2$. The graph window $|x| \le 2$, $|y| \le 2$ is invented initially, then updated until Figure 5 reveals sufficiently rich field details.

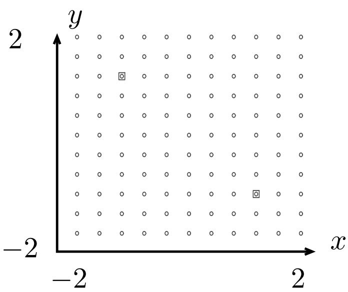


Figure 4. Equilibria $(1,-1)$, $(-1,1)$ and Invented 11×11 Uniform Grid.
The equilibria (squares) happen to cover up two grid points. The invented size 11×11 should fill the white space in the graphic.

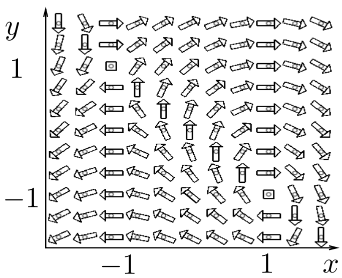


Figure 5. Equilibria, Uniform Grid and Direction Field.
An arrow shaft at a grid point represents a solution curve over a small time interval. Threaded solution curves on long time intervals have tangents matching nearby arrow shaft directions.

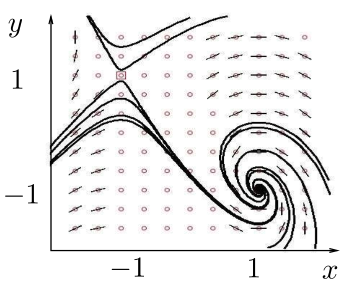


Figure 6. Initial Phase Portrait.
Equilibria $(1,-1)$, $(-1,1)$ and 11×11 uniform grid with threaded solution curves. Arrow shafts included from some direction field arrows. Threaded solution curve tangents are to match nearby direction field arrow shafts. See Figures 1 and 2 for how to match tangents.

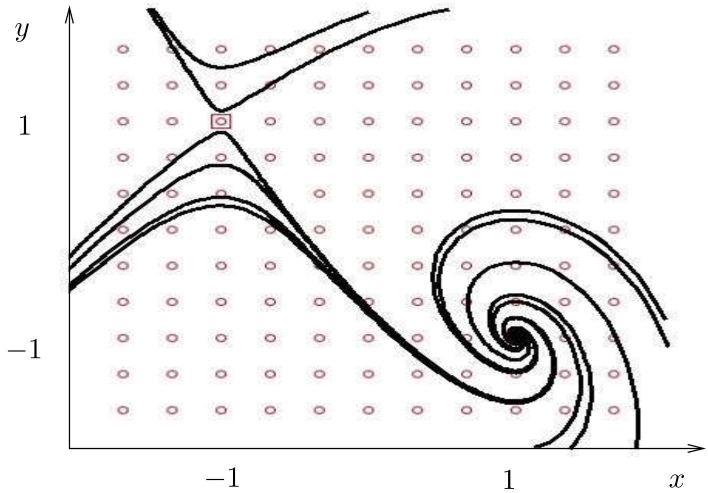


Figure 7. Final Phase portrait.
Shown are some threaded solution curves and an 11×11 grid. The direction field has been removed for clarity. Threaded solution curves do not actually cross, even though graphical resolution might suggest otherwise.

Phase Plot by Computer

Illustrated here is how to make a phase plot like Figure 8 or Figure 9, *infra*, with computer algebra system `maple`, for the system of differential equations

$$(5)\qquad \begin{aligned} x'(t) &= x(t) + y(t), \\ y'(t) &= 1 - x^2(t). \end{aligned}$$

Before the computer work begins, the differential equation is defined and the equilibria are computed. Defaults supplied by `maple` allow an initial phase portrait to be plotted, from which the graph window is invented.

Phase plot tools can simplify initial plot production. To illustrate, `maple` task **Phase Portrait** has this interface:

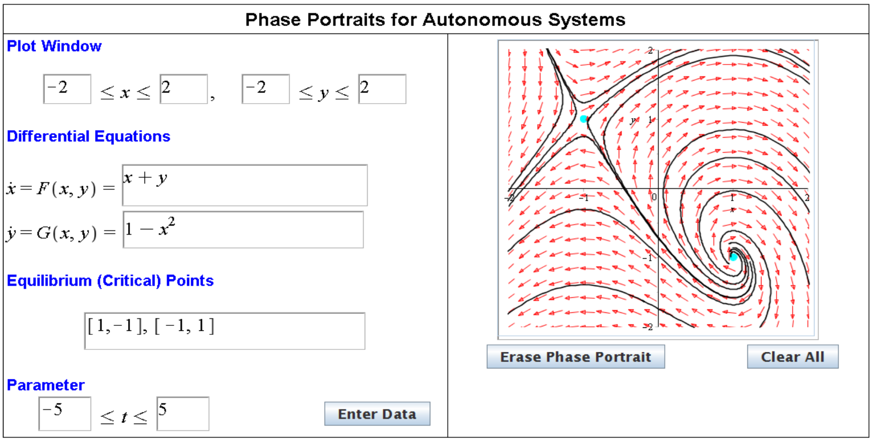


Figure 8. `PhasePortrait` **task in computer algebra system** `Maple` **for equations (5).**

Minimal input requires two differential equations, equilibria, a graph window and time interval for threaded curves. Clicking on the graphic produces threaded solution curves.

The Phase Portrait Task is unlikely to be able to produce a final, production figure. Other tools are normally used afterwards, to make the final figure.

The initial plot code:

```
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:Times:=t=-20..20:
DEtools[DEplot]([des],[x(t),y(t)],Times,wind);
```

The initial plot suggests which initial conditions near the equilibria should be selected in order to create typical orbits on the graphic. The final code with initial data and options:

```
des:=diff(x(t),t)=x(t)+y(t),diff(y(t),t)=1-x(t)^2:
wind:=x=-2..2,y=-2..2:Times:=t=-20..20:
opts:=stepsize=0.05,dirgrid=[13,13],
axes=none,thickness=3,arrows=small:
ics:=[[x(0)=-1,y(0)=1.1],[x(0)=-1,y(0)=1.5],
[x(0)=-1,y(0)=.9],[x(0)=-1,y(0)=.6],[x(0)=-1,y(0)=.3],
[x(0)=1,y(0)=-0.9],[x(0)=1,y(0)=-0.6],[x(0)=1,y(0)=-0.6],
[x(0)=1,y(0)=-0.3],[x(0)=1,y(0)=-1.6],[x(0)=1,y(0)=-1.3],
[x(0)=1,y(0)=-1.1]]:
DEtools[DEplot]([des],[x(t),y(t)],Times,wind,ics,opts);
```

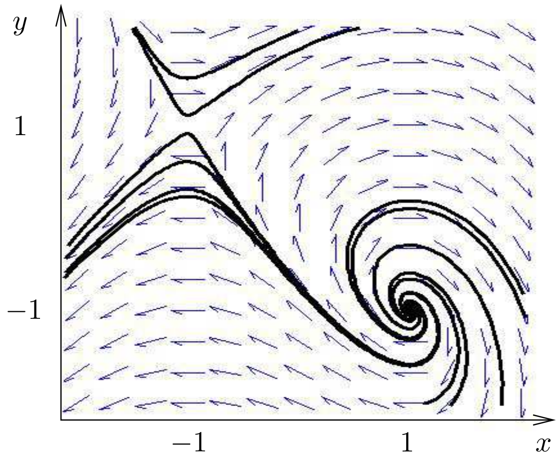


Figure 9. Phase Portrait for (5).
The graphic shows typical solution curves and a direction field. The graphic was produced in `maple` using a 13×13 grid.

Stability

Consider an autonomous system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ with $\vec{F}$ continuously differentiable in a region $\mathcal{D}$ in the plane.

Stable equilibrium. An equilibrium point $\vec{u}_0$ in $\mathcal{D}$ is said to be **Stable** provided for each $\epsilon > 0$ there corresponds $\delta > 0$ such that

(a) given $\vec{u}(0)$ in $\mathcal{D}$ with $\|\vec{u}(0) - \vec{u}_0\| < \delta$, then the solution $\vec{u}(t)$ exists on $0 \le t < \infty$ and

(b) $\|\vec{u}(t) - \vec{u}_0\| < \epsilon$ for $0 \le t < \infty$.

Unstable equilibrium. The equilibrium point $\vec{u}_0$ is called **Unstable** provided it is **not stable**, meaning at least one of **(a)** or **(b)** fails.

Asymptotically stable equilibrium. The equilibrium point $\vec{u}_0$ is said to be **Asymptotically Stable** provided **(a)** and **(b)** hold (it is **stable**), and additionally

(c) $\lim_{t\to\infty} \|\vec{u}(t) - \vec{u}_0\| = 0$ for $\|\vec{u}(0) - \vec{u}_0\| < \delta$.

Applied accounts of stability tend to emphasize item **(b)**. Careful application of stability theory requires attention to **(a)**, which is the question of extension of solutions of initial value problems to the half-axis.

Basic extension theory for solutions of autonomous equations says that **(a)** will be satisfied provided **(b)** holds for those values of t for which $\vec{u}(t)$ is already defined. Stability verifications in mathematical and applied literature often implicitly use extension theory, in order to present details compactly. The reader is advised to adopt the same predisposition as researchers, who assume the reader to be equally clever as they.

Physical stability. In the model $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$, physical stability addresses changes in $\vec{F}$ as well as changes in $\vec{u}(0)$. The meaning is this: physical parameters of the model, e.g., the mass $m > 0$, damping constant $c > 0$ and Hooke's constant $k > 0$ in a damped spring-mass system

$$\begin{aligned} x' &= y, \\ y' &= -\frac{c}{m}\, y - \frac{k}{m}\, x, \end{aligned}$$

may undergo small changes without significantly affecting the solution.

In physical stability, stable equilibria correspond to **Physically Observed** data whereas other solutions correspond to **Transient Observations** that disappear over time.

A typical instance is the trout system

$$(6) \qquad \begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

Physically observed data in the trout system (6) corresponds to the **carrying capacity**, represented by the **Stable Equilibrium** point $(80, 20)$, whereas transient observations are snapshot population sizes that are subject to change over time. The strange extinction equilibria $(90, 0)$ and $(0, 60)$ are **unstable equilibria**, which disagrees with intuition about zero births for less than two individuals, but agrees with graphical representations of the trout system in Figure 10. Changing $\vec{F}(\vec{u})$ for a trout system adjusts the physical constants which describe the birth and death rates, whereas changing $\vec{u}(0)$ alters the initial population sizes of the two trout species.

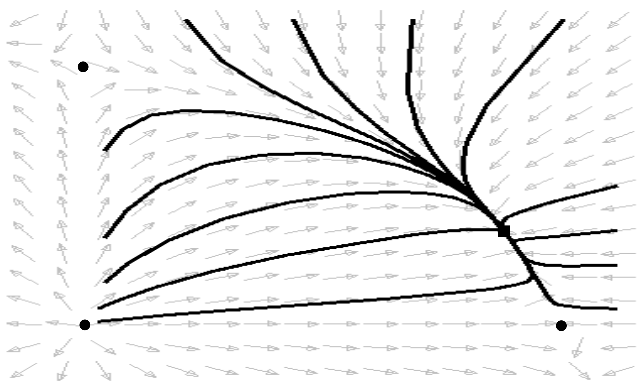
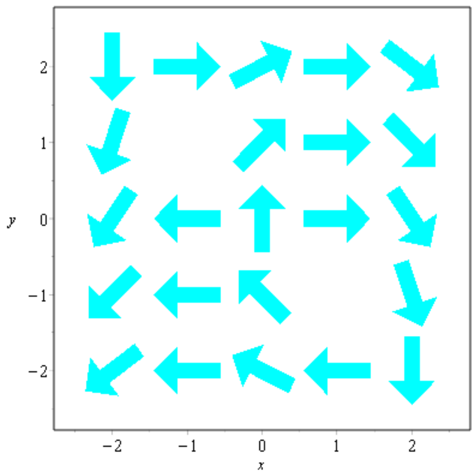


Figure 10. Phase Portrait for Trout System (6).
Shown are typical solution curves and a direction field. Equilibrium $(80, 20)$ is asymptotically stable (a square). Equilibria $(0, 0)$, $(90, 0)$, $(0, 60)$ are unstable (circles).

Direction Fields by Computer

Direction fields are produced by `Maple` with tool `DEtools[dfieldplot]` or with interactive graphical task `PhasePortrait`. Basic code that produces a direction field can be written with minimal effort:



Maple code:

```
de1:=diff(x(t),t)=x(t)+y(t);
de2:=diff(y(t),t)=1-x(t)*x(t);vars:=[x(t),y(t)];
trange:=t=-10..10:xrange:=x=-2..2:yrange:=y=-2..2:
opts1:=trange,xrange,yrange:
opts2:=arrows=large,color=cyan,dirfield=[5,5]:
DEtools[dfieldplot]([de1,de2],vars,opts1,opts2);
```

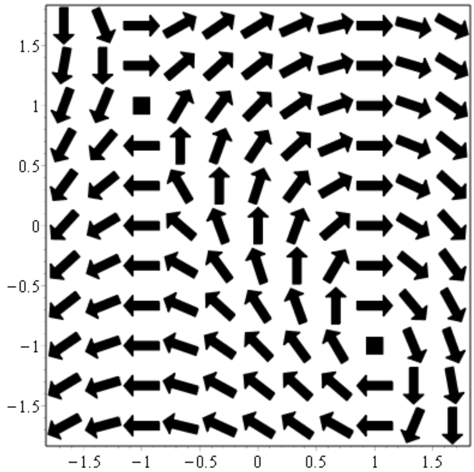
A Direction Field Procedure

The ideas discussed below for `maple` apply to other programming languages, such as `Maxima, Mathematica, Ruby, Python` and Microsoft developer languages. `Maple` code below considers the system

$$x' = F_1(x,y), \quad y' = F_2(x,y)$$

with example $x' = F_1 = x + y, y' = F_2 = 1 - x^2$, which was treated above.

```
F1:=(x,y)->evalf(x+y):F2:=(x,y)->evalf(1-x^2):
P:=directionField(F1,F2):plots[display](P);# proc below
```



Maple function plottools[rectangle] requires two arguments ul, lr, which are the upper left (ul) and lower right (lr) vertices of the rectangle.

Maple function plottools[arrow] requires five arguments P, Q, sw, aw, af: the two points P, Q which define the arrow shaft and direction, plus the shaft width sw, arrowhead width aw and arrowhead length fraction af (fraction of the shaft length).

The two functions rectangle, arrow plot a polygon from its vertices. Function rectangle computes four vertices and function arrow computes seven vertices. Maple function plots[display] plots the vertices.

```
 # 2D phase plane direction field with uniform nxm grid.
 # Tangent length is 9/10 the grid box width W0.
directionField:=
 proc(F1,F2,a:=-2,b:=2,c:=-2,d:=2,n:=11,m:=11)
description "Custom direction field for F1,F2\
Window: a <= x <= b, c <= y <= d, Grid: n by m\
Tangent length = 9/10 grid box width W0.";
local x,y,X,Y,V,H,K,i,j,M1,M2,W0,h,p1,p2,q1,q2; global P;
H:=evalf((b-a)/(n+1)):K:=evalf((d-c)/(m+1)):W0:=min(H,K):
X:=t->a+H*(t):Y:=t->c+K*(t):P:=[]:
for i from 1 to n do
for j from 1 to m do
x:=X(i):y:=Y(j):M1:=F1(x,y): M2:=F2(x,y):
if (M1 =0 and M2 =0) then # no tangent, make a box
h:=W0/5:V:=plottools[rectangle]([x-h,y+h],[x+h,y-h]):
else
h:=evalf(((1/2)*9*W0/10)/sqrt(M1^2+M2^2)):
p1:=x-h*M1:p2:=y-h*M2:q1:=x+h*M1:q2:=y+h*M2:
V:=plottools[arrow]([p1,p2],[q1,q2],0.2*W0,0.5*W0,1/4):
fi:if (P = []) then P:=V: else P:=P,V: fi:
od:od:
```

```
RETURN (P);
end proc:
```

Exercises 10.1

Autonomous Planar Systems.

Consider

$$(7)\qquad \begin{aligned} x'(t) &= x(t)+y(t), \\ y'(t) &= 1-x^2(t). \end{aligned}$$

1. (Vector-Matrix Form) System (7) can be written in vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

2. (Picard's Theorem) Picard's vector existence-uniqueness theorem applies to system (7) with initial data $x(0) = x_0$, $y(0) = y_0$. Show the details.

Trajectories Don't Cross.

3. (Theorem 10.1 Details) Show $\frac{dy}{dt} = g(x_1(t+c), y_1(t+c))$, then show that $y'(t) = g(x(t), y(t))$ in the proof of Theorem 10.1.

4. (Orbits Can Cross) The example

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3y^{2/3}$$

has infinitely many orbits crossing at $x = y = 0$. Exhibit two distinct orbits which cross at $x = y = 0$. Does this example contradict Theorem 10.1?

Equilibria. A point (x_0, y_0) is called an **Equilibrium** provided $x(t) = x_0$, $y(t) = y_0$ is a solution of the dynamical system.

5. Justify that $(1,-1), (-1,1)$ are the only equilibria for the system $x' = x+y$, $y' = 1-x^2$.

6. Display the details which justify that $(0,0), (90,0), (0,60), (80,20)$ are all equilibria for the system $x'(t) = x(-2x-y+180)$, $y'(t) = y(-x-2y+120)$.

Practical Methods for Computing Equilibria.

7. (Murray System) The biological system

$$x' = x(6-2x-y), y' = y(4-x-y)$$

has equilibria $(0,0)$, $(3,0)$, $(0,4)$, $(2,2)$. Justify the four answers.

8. (Nullclines) Curves along which either $x' = 0$ or $y' = 0$ are called **nullclines**. The biological system

$$x' = x(6-2x-y), y' = y(4-x-y)$$

has nullclines $x = 0$, $y = 0$, $6-2x-y = 0$, $4-x-y = 0$. Justify the four answers.

9. (Nullclines by Computer) Produce a graphical display of the nullclines of the Murray System above. Maple code below makes a plot from equations $x(6-2x-y) = 0$, $y(4-x-y) = 0$.

```
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=-5..5,y=-10..10;
opts:=wind,contours=[0];
plots[contourplot](eqns,opts);
```

10. (Isoclines by Computer) Level curves $f(x,y) = c$ are called **Isoclines**.

`Maple` will plot level curves $f(x,y) = -2$, $f(x,y) = 0$, $f(x,y) = 2$ using the nullcline code above, with replacement `contours=[-2,0,2]`. Produce an isocline plot for the Murray System above with these same contours.

11. (Implicit Plot) Equilibria can be found graphically by an implicit plot.

```
# MAPLE implicit plot
eqns:={x*(6-2*x-y),y*(4-x-y)};
wind:=x=-5..5,y=-10..10;
plots[implicitplot](eqns,wind);
```

Produce the implicit plot. Is it the same as the nullcline plot?

12. (Implicit Plot) Find the equilibria graphically by an implicit plot. Then find the equilibria exactly.

$$\left\{ \begin{aligned} x'(t) &= x(t)+y(t), \\ y'(t) &= 4-x^2(t). \end{aligned} \right.$$

Rabbit-Fox System.

13. (Predator-Prey) Consider a rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

Argue why extinction of the rabbits ($x = 0$) implies extinction of the foxes ($y = 0$).

14. (Predator-Prey) The rabbit and fox system

$$\begin{aligned} x' &= \frac{1}{200}x(40-y), \\ y' &= \frac{1}{100}y(x-40), \end{aligned}$$

has extinction of the foxes ($y = 0$) implying Malthusian population explosion of the rabbits ($\lim_{t=\infty} x(t) = \infty$). Explain.

Trout System. Consider

$$\begin{aligned} x'(t) &= x(-2x-y+180), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

15. (Carrying Capacity) Show details for calculation of the equilibrium $x = 80$, $y = 20$, which is **co-existence**.

16. (Stability) Equilibrium point $x = 80$, $y = 20$ is stable. Explain this statement using geometry from Figure 10 and the definition of stability.

Phase Portraits. Consider

$$\begin{aligned} x'(t) &= x(t)+y(t), \\ y'(t) &= 1-x^2(t). \end{aligned}$$

17. (Equilibria) Solve for x, y in the system

$$\begin{aligned} 0 &= x+y, \\ 0 &= 1-x^2, \end{aligned}$$

for equilibria $(1,-1)$, $(-1,1)$. Explain why $|x| \le 2$, $|y| \le 2$ is a suitable graph window.

18. (Grid Points) Draw a 5×5 grid on the graph window $|x| \le 2, |y| \le 2$. Label the equilibria.

19. (Direction Field) Draw direction field arrows on the 5×5 grid of the previous exercise. They coincide with the tangent direction $\vec{v} = x'\vec{\imath} + y'\vec{\jmath} = (x+y)\vec{\imath} + (1-x^2)\vec{\jmath}$, where (x, y) is the grid point. The arrows may not touch.

20. (Threaded Orbits) On the direction field of the previous exercise, draw orbits (*threaded solution curves*), using the rules:

1. Orbits don't cross.
2. Orbits pass direction field arrows with nearly matching tangent.

Phase Plot by Computer. Use a computer algebra system or a numerical workbench to produce phase portraits for the given dynamical system. A graph window should contain all equilibria.

21. (Rabbit-Fox System I)

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

22. (Rabbit-Fox System II)

$$\begin{aligned} x' &= \frac{1}{100}x(50-y), \\ y' &= \frac{1}{200}y(x-40). \end{aligned}$$

23. (Trout System I)

$$\begin{aligned} x'(t) &= x(-2x-y+180), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

24. (Trout System II)

$$\begin{aligned} x'(t) &= x(-2x-y+200), \\ y'(t) &= y(-x-2y+120). \end{aligned}$$

Stability Conditions. Consider equilibrium point $(0,0)$ and nearby solution curves $x(t), y(t)$ with $(x(0), y(0))$ near $(0,0)$.

25. (Instability: Repeller) Prove: If for every $\delta > 0$ there is one solution with $|x(0)^2 + y(0)^2| < \delta^2$ such that $\lim_{t\to\infty} |x(t)| + |y(t)| = \infty$ then equilibrium $(0,0)$ is unstable.

26. (Stability: Attractor) Prove that $x'(t) < 0$ and $y'(t) < 0$ for all nearby solutions implies stability at $(0,0)$, but not asymptotic stability.

27. (Instability in x) Prove that $\lim_{t\to\infty} |x(t)| = \infty$ implies instability at $(0,0)$.

28. (Instability in y) Prove that $\lim_{t\to\infty} |y(t)| = \infty$ implies instability at $(0,0)$.

Geometric Stability.

29. (Attractor) Imagine a dust particle in a fluid draining down a funnel, whose trace is a space curve. Assume fluid drains at $x = 0$, $y = 0$ and the funnel centerline is along the z-axis. Project the space curve onto the xy-plane. Is this planar orbit stable at $(0,0)$ in the sense of the definition?

30. (Repeller) Imagine a paint droplet from a paint spray can, pointed downward, which traces a space curve. Project the space curve onto the xy-plane orthogonal to the spray nozzle direction, centerline along the z-axis. Is this planar orbit stable at $(0,0)$ in the sense of the definition?

Geometric Stability: Phase Portrait.

31. (Rabbit–Fox I Stability) Plot a phase portrait for system

$$\begin{aligned} x' &= \frac{1}{200}x(30-y), \\ y' &= \frac{1}{100}y(x-40). \end{aligned}$$

Provide geometric evidence for stability of equilibrium $x = 40$, $y = 30$.

32. (Rabbit–Fox II Instability) Plot a phase portrait for system

$$\begin{aligned} x' &= \frac{1}{100}x(50-y), \\ y' &= \frac{1}{200}y(x-40). \end{aligned}$$

Provide geometric evidence for instability of equilibrium $x = 0$, $y = 0$ and stability of equilibrium $x = 40$, $y = 50$.

10.2 Planar Constant Linear Systems

A **constant linear** planar system is a set of two scalar differential equations of the form

(1) $$\begin{aligned} x'(t) &= ax(t) + by(t)), \\ y'(t) &= cx(t) + dy(t)), \end{aligned}$$

where a, b, c and d are constants. In matrix form,

$$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t), \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \vec{u}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Solutions drawn in phase portraits don't cross, because of Picard's theorem. The system is autonomous. The origin is always an equilibrium solution. There can be infinitely many equilibria, found by solving $A\vec{u} = \vec{0}$ for the constant vector $\vec{u}$, when A is not invertible.

Formula. System (1) can be solved by a formula which parallels the theorem for second order constant coefficient equations $Ay'' + By' + Cy = 0$. You are invited to learn Putzer's spectral method, page **??**, which is used to derive the formulas. For now, let's accept the formulas displayed in the next theorem. Putzer's result depends only on the Cayley-Hamilton theorem, which says that a matrix A satisfies the characteristic equation $|A - \lambda I| = 0$ under substitution $\lambda = A$.

Theorem 10.2 (Planar Constant Linear System: Putzer's Formula)
Consider the real planar system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$. Let λ_1, λ_2 be the roots of the characteristic equation $\det(A - \lambda I) = 0$. The real general solution $\vec{u}(t)$ is given by the formula

$$\vec{u}(t) = \Phi(t)\vec{u}(0)$$

where the 2×2 real invertible matrix $\Phi(t)$ is defined as follows.

Real $\lambda_1 \neq \lambda_2$	$\Phi(t) = e^{\lambda_1 t} I + \dfrac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1}(A - \lambda_1 I)$.
Real $\lambda_1 = \lambda_2$	$\Phi(t) = e^{\lambda_1 t} I + te^{\lambda_1 t}(A - \lambda_1 I)$.
Complex $\lambda_1 = \overline{\lambda}_2$, $\lambda_1 = a + bi$, $b > 0$	$\Phi(t) = e^{at}\left(\cos(bt)\, I + (A - aI)\dfrac{\sin(bt)}{b}\right)$.

Continuity and Redundancy

The formulas are continuous in the sense that limiting $\lambda_1 \to \lambda_2$ in the first formula or $b \to 0$ in the last formula produces the middle formula for real equal roots. The first formula is also valid for complex conjugate roots λ_1, $\lambda_2 = \overline{\lambda}_1$ and it reduces to the third when $\lambda_1 = a + ib$, therefore the third formula is technically redundant, but nevertheless useful, because it contains no complex numbers.

Recommended: Memorize the first formula, derive the other two.

About the Newton Quotient. The Newton quotient $\frac{g(x)-g(x_0)}{x-x_0}$ in the first formula of the theorem uses $g(x) = e^{xt}$, $x = \lambda_2$, $x_0 = \lambda_1$, $x - x_0 = \lambda_2 - \lambda_1$. Calculus defines $g'(x_0)$ as the Newton quotient limit as $x \to x_0$.

Illustrations

Typical cases are represented by the following 2×2 matrices A. The two roots λ_1, λ_2 of the characteristic equation must fall into one of the three possibilities: real distinct, real equal or complex conjugate.

$\lambda_1 = 5$, $\lambda_2 = 2$ — Real distinct roots.

$$A = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix} \qquad \vec{u}(t) = \left(e^{5t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{e^{2t} - e^{5t}}{2-5} \begin{pmatrix} -6 & 3 \\ -6 & 3 \end{pmatrix} \right) \vec{u}(0).$$

$\lambda_1 = \lambda_2 = 3$ — Real equal roots.

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \qquad \vec{u}(t) = e^{3t} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix} \vec{u}(0).$$

$\lambda_1 = \overline{\lambda_2} = 2 + 3i$ — Complex conjugate roots.

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \qquad \vec{u}(t) = e^{2t} \begin{pmatrix} \cos 3t & \sin 3t \\ -\sin 3t & \cos 3t \end{pmatrix} \vec{u}(0).$$

Isolated Equilibria

An autonomous system is said to have an **isolated equilibrium** at $\vec{u} = \vec{u}_0$ provided $\vec{u}_0$ is the only constant solution of the system in $|\vec{u} - \vec{u}_0| < r$, for $r > 0$ sufficiently small.

Theorem 10.3 (Isolated Equilibrium)
The following are equivalent for a constant planar system $\frac{d}{dt}\vec{u}(t) = A\vec{u}(t)$:

1. The system has an isolated equilibrium at $\vec{u} = \vec{0}$.

2. $\det(A) \neq 0$.

3. The roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ satisfy $\lambda_1\lambda_2 \neq 0$.

Proof: The expansion $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$ shows that $\det(A) = \lambda_1\lambda_2$. Hence **2** $\equiv$ **3**. We prove now **1** $\equiv$ **2**. If $\det(A) = 0$, then $A\vec{u} = \vec{0}$ has infinitely many solutions $\vec{u}$ on a line through $\vec{0}$, therefore $\vec{u} = \vec{0}$ is not an isolated equilibrium. If $\det(A) \neq 0$, then $A\vec{u} = \vec{0}$ has exactly one solution $\vec{u} = \vec{0}$, so the system has an isolated equilibrium at $\vec{u} = \vec{0}$.

Classification of Isolated Equilibria

For linear equations

$$\frac{d}{dt}\vec{u}(t) = A\vec{u}(t),$$

we explain the phase portrait classifications

spiral, center, saddle, node

near the isolated equilibrium point $\vec{u} = \vec{0}$, and how to detect them when they occur. Below, λ_1, λ_2 are the roots of $\det(A - \lambda I) = 0$.

Figures 13–12 illustrate the classifications. See also duplicate Figures 16–19, which are organized by geometry.

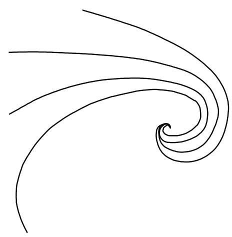


Figure 11. Spiral

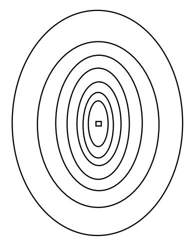


Figure 12. Center

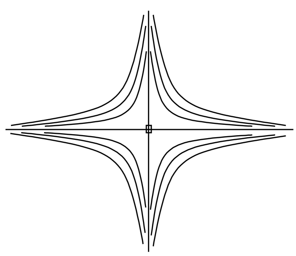


Figure 13. Saddle

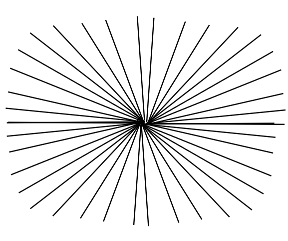


Figure 14. Proper node

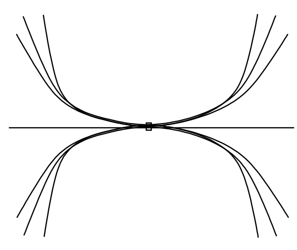


Figure 15. Improper node

Spiral $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a \neq 0$, $b > 0$.

A **Spiral** has solution formula

$$\begin{aligned}\vec{u}(t) &= e^{at}\cos(bt)\,\vec{c}_1 + e^{at}\sin(bt)\,\vec{c}_2,\\ \vec{c}_1 &= \vec{u}(0), \quad \vec{c}_2 = \frac{A - aI}{b}\,\vec{u}(0).\end{aligned}$$

All solutions are bounded harmonic oscillations of natural frequency b times an exponential amplitude which grows if $a > 0$ and decays if $a < 0$. An orbit in the phase plane **spirals out** if $a > 0$ and **spirals in** if $a < 0$.

Center $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a = 0$, $b > 0$

A **center** has solution formula

$$\begin{aligned}\vec{u}(t) &= \cos(bt)\,\vec{c}_1 + \sin(bt)\,\vec{c}_2,\\ \vec{c}_1 &= \vec{u}(0), \quad \vec{c}_2 = \frac{1}{b}\,A\vec{u}(0).\end{aligned}$$

All solutions are bounded harmonic oscillations of natural frequency b. Orbits in the phase plane are periodic closed curves of period $2\pi/b$ which encircle the origin.

Saddle λ_1, λ_2 real, $\lambda_1\lambda_2 < 0$

A **saddle** has solution formula

$$\begin{aligned}\vec{u}(t) &= e^{\lambda_1 t}\vec{c}_1 + e^{\lambda_2 t}\vec{c}_2,\\ \vec{c}_1 &= \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\,\vec{u}(0), \quad \vec{c}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\,\vec{u}(0).\end{aligned}$$

The phase portrait shows two lines through the origin which are tangents at $t = \pm\infty$ for all orbits.
The line directions are given by the eigenvectors of matrix A. See Figure 13.

Node λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$

The solution formulas are

$$\begin{aligned}\vec{u}(t) &= e^{\lambda_1 t}\left(\vec{a}_1 + t\vec{a}_2\right), \quad \text{when} \quad \lambda_1 = \lambda_2,\\ \vec{a}_1 &= \vec{u}(0), \quad \vec{a}_2 = (A - \lambda_1 I)\vec{u}(0),\end{aligned}$$

$$\begin{aligned}\vec{u}(t) &= e^{\lambda_1 t}\vec{b}_1 + e^{\lambda_2 t}\vec{b}_2, \quad \text{when} \quad \lambda_1 \neq \lambda_2,\\ \vec{b}_1 &= \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\,\vec{u}(0), \quad \vec{b}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\,\vec{u}(0).\end{aligned}$$

Node subclassifications **proper** and **improper** are discussed below.

Definition 10.1 (Node)
A **node** is defined to be an equilibrium point (x_0, y_0) such that

1. Either $\lim_{t\to\infty}(x(t), y(t)) = (x_0, y_0)$ or else $\lim_{t\to-\infty}(x(t), y(t)) = (x_0, y_0)$, for all initial conditions $(x(0), y(0)$ close to (x_0, y_0).
2. For each initial condition $(x(0), y(0))$ near (x_0, y_0), there exists a straight line L through (x_0, y_0) such that $(x(t), y(t))$ is **tangent** at $t = \infty$ to L. More precisely, line L has a tangent vector $\vec{v}$ and $\lim_{t\to\infty}(x'(t), y'(t)) = c\vec{v}$ for some constant c.

Proper Node. Also called a **Star Node**. Matrix A is required to have two eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$ with $\lambda_1 = \lambda_2$. Then $\vec{u}(0)$ in $\mathcal{R}^2 = \textbf{span}(\vec{v}_1, \vec{v}_2)$ implies $\vec{u}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$ and $\vec{a}_2 = (A - \lambda_1 I)\vec{u}(0) = \vec{0}$. Therefore, $\vec{u}(t) = e^{\lambda_1 t}\vec{a}_1$ implies trajectories are tangent to the line through $(0, 0)$ in direction $\vec{v} = \vec{a}_1/|\vec{a}_1|$. Because $\vec{u}(0) = \vec{a}_1$ is arbitrary, $\vec{v}$ can be any direction, which explains the star-like phase portrait in Figure 14.

Improper Node with One Eigenpair. The non-diagonalizable case is also called a **Degenerate Node**. Matrix A is required to have just one eigenpair $(\lambda_1, \vec{v}_1)$ and $\lambda_1 = \lambda_2$. Then $\vec{u}'(t) = (\vec{a}_2 + \lambda_1\vec{a}_1 + t\lambda_1\vec{a}_2)e^{\lambda_1 t}$ implies $\vec{u}'(t)/|\vec{u}'(t)| \approx \vec{a}_2/|\vec{a}_2|$ at $|t| = \infty$. Matrix $A - \lambda_1 I$ has rank 1, hence $\textbf{Image}(A - \lambda_1 I) = \textbf{span}(\vec{v})$ for some nonzero vector $\vec{v}$. Then $\vec{a}_2 = (A - \lambda_1 I)\vec{u}(0)$ is a multiple of $\vec{v}$. Trajectory $\vec{u}(t)$ is tangent to the line through $(0, 0)$ with direction $\vec{v}$, as in Figure 15.

Improper Node with Distinct Eigenvalues. Discussed here is the first possibility when matrix A has real eigenvalues with $\lambda_2 < \lambda_1 < 0$. Not discussed is the second possibility $\lambda_2 > \lambda_1 > 0$, which has similar details. Then $\vec{u}'(t) = \lambda_1\vec{b}_1 e^{\lambda_1 t} + \lambda_2\vec{b}_2 e^{\lambda_2 t}$ implies $\vec{u}'(t)/|\vec{u}'(t)| \approx \vec{b}_1/|\vec{b}_1|$ at $t = \infty$. In terms of eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$, we compute $\vec{b}_1 = c_1\vec{v}_1$ and $\vec{b}_2 = c_2\vec{v}_2$ where $\vec{u}(0) = c_1\vec{v}_1 + c_2\vec{v}_2$. Trajectory $\vec{u}(t)$ is tangent to the line through $(0, 0)$ with direction $\vec{v}_1$. See Figure 15.

Attractor and Repeller

An equilibrium point is called an **Attractor** provided orbits starting nearby limit to the point as $t \to \infty$. A **Repeller** is an equilibrium point such that orbits starting nearby limit to the point as $t \to -\infty$. Terms like **Attracting node** and **Repelling spiral** are defined analogously.

Linear Classification Shortcut for $\frac{d}{dt}\vec{u} = A\vec{u}$

Presented here is a practical method for deciding the classification of center, spiral, saddle or node for a linear system $\frac{d}{dt}\vec{u} = A\vec{u}$. The method uses just the eigenvalues of A and the corresponding Euler atoms.

Cayley-Hamilton Basis.

A system $\frac{d}{dt}\vec{u} = A\vec{u}$ will have general solution

$$\vec{u} = \vec{d}_1(\text{Euler Atom 1}) + \vec{d}_2(\text{Euler Atom 2}).$$

The vectors $\vec{d}_1, \vec{d}_2$ depend on A and $\vec{u}(0)$. They are never explicitly used in the shortcut, hence never computed.

The two Euler solution atoms are found from roots λ of the characteristic equation $|A - \lambda I| = 0$. There are two kinds of atoms:

> No sine or cosine appear in the atoms, making a **non-rotating** phase portrait, which is either a node or a saddle.
>
> Sine and cosine appear in the atoms, which make a **rotating** phase portrait, which is either a center or a spiral.

Table 1. Non-Rotating Phase Portraits

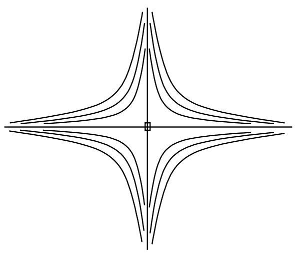


Figure 16. Saddle

Euler solution atoms for a saddle or node have form e^{at}, e^{bt} or else e^{at}, te^{at}. There are no sine or cosine terms.

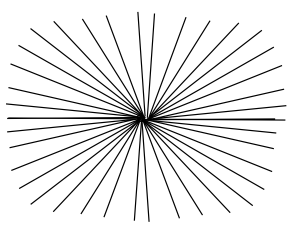


Figure 17. Proper node

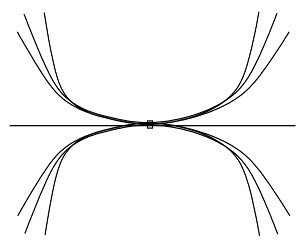


Figure 18. Improper node

Table 2. Rotating Phase Portraits

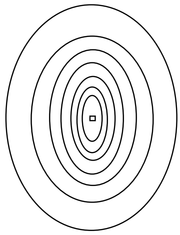


Figure 19. Center

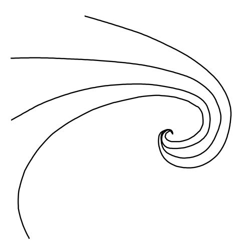


Figure 20. Spiral

Divide and Conquer. Given 2×2 matrix A with $|A| \neq 0$, find the roots of the characteristic equation $|A - \lambda I| = 0$ and construct the two Euler solution atoms. The classification figure, selected from center, spiral, node, saddle, depends only on the atoms. Examine the atoms for sines and cosines. If present, then it will be a rotating figure (center, spiral), otherwise it will be a non-rotating figure (node, saddle). One more divide and conquer decides the figure, because within each figure group, rotating or non-rotating, there is only one possible choice.

> **Rotation Test**. Suppose sines and cosines appear in the Euler atoms. If the Euler atoms are pure sine and cosine, then $(0,0)$ is a **center**, otherwise $(0,0)$ is a **spiral**.
>
> **Non-Rotation Test**. Suppose no sines or cosines appear in the Euler atoms. If at $t = \infty$ one Euler atom limits to zero and the other Euler atom limits to infinity, then $(0,0)$ is a **saddle**, otherwise it is a **node**.

Stability Classification by Euler Atoms.

> A center is always stable, characterized by Euler atoms being pure sine and cosine.
>
> If $(0,0)$ is not a center, then $(0,0)$ is stable at $t = \infty$ if and only if both Euler atoms limit to zero at $t = \infty$.

Divide and conquer via Euler atoms requires no table to decide upon the basic phase portrait classification: spiral, center, saddle, node. Stability is likewise decided by Euler atoms.

Node Sub-classifications

If finer geometric sub-classifications of a node are useful to you, then eigenanalysis is required. Assumed below are λ_1, λ_2 real and $\lambda_1\lambda_2 > 0$. *Diagonalizable* means there are two eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$.

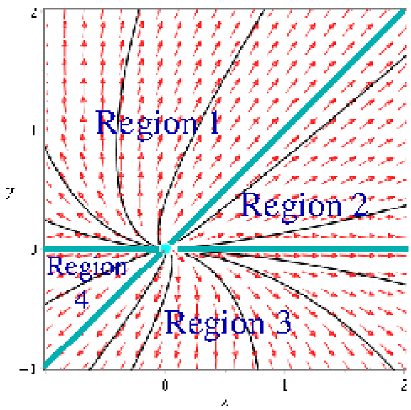
Let $(x_0, y_0) \neq (0,0)$ denote an arbitrary initial point. Start at this point a trajectory $(x(t), y(t))$. Think of (x_0, y_0) as click point on the graphic in a computer phase portrait plotter: the threaded curve goes through (x_0, y_0).

Separatrix

A **separatrix** is a union S of equilibria and special trajectories. Separatrices are graphing tools. The possible separatrices include every solution curve, so there is art involved to construct a useful separatrix.

Literature may try to describe the phase portrait geometry of linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ using **eigenvector directions**. The terminology assumes you know how to construct a separatrix S from the eigenvectors. A separatrix for a nonlinear system $\vec{\mathbf{u}}' = \vec{\mathbf{F}}(\vec{\mathbf{u}})$ is not constructed from eigenvectors but from experimentally found trajectories in a phase portrait plotter.

For nodes, a separatrix S is constructed which divides the plane into two regions or four regions. A trajectory from (x_0, y_0) stays in the region where it starts: **trajectories do not cross** S. If (x_0, y_0) is in S then the trajectory remains in S: crossing means the trajectory changed regions.



Four regions are separated by four cyan lines each of which is a trajectory, their union a separatrix S. The linear system is

$$x' = 2x + y, \quad y' = 3y$$

with eigenpairs

$$\left(2, \begin{pmatrix}1\\0\end{pmatrix}\right), \quad \left(3, \begin{pmatrix}1\\1\end{pmatrix}\right)$$

The construction for nodes uses eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$ with real nonzero eigenvalues λ_1, λ_2. Let $\vec{\mathbf{v}}_2 = \vec{\mathbf{v}}_1$ if $\lambda_1 = \lambda_2$ and there is only one eigenpair.

Lemma 10.1 A separatrix for a node is $S = \mathbf{span}(\vec{\mathbf{v}}_1) \cup \mathbf{span}(\vec{\mathbf{v}}_2)$.

Proof. Euler's method provides trajectories of $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$:

$$\vec{\mathbf{u}}_1(t) = e^{\lambda_1 t}\vec{\mathbf{v}}_1, \; \vec{\mathbf{u}}_2(t) = -e^{\lambda_1 t}\vec{\mathbf{v}}_1, \vec{\mathbf{u}}_3(t) = e^{\lambda_2 t}\vec{\mathbf{v}}_2, \vec{\mathbf{u}}_4(t) = -e^{\lambda_2 t}\vec{\mathbf{v}}_2$$

The separatrix is constructed as the union of equilibrium $(0, 0)$ and the four trajectories, it being understood that $\vec{\mathbf{v}}_1 = \vec{\mathbf{v}}_2$ causes there to be only two trajectories. Then

$$S = (0, 0) \cup \vec{\mathbf{u}}_1 \cup \vec{\mathbf{u}}_2 \cup \vec{\mathbf{u}}_3 \cup \vec{\mathbf{u}}_4 = \mathbf{span}(\vec{\mathbf{v}}_1) \cup \mathbf{span}(\vec{\mathbf{v}}_2)$$

■

The **exceptional case** where the Lemma is not used as a graphing tool is equal eigenvalues $\lambda_1 = \lambda_2$ and independent eigenvectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$. The general solution is $\vec{\mathbf{u}}(t) = (c_1\vec{\mathbf{v}}_1 + c_2\vec{\mathbf{v}}_2)\,e^{\lambda_1 t} = \vec{\mathbf{u}}(0)e^{\lambda_1 t}$. Geometrically, a trajectory starting at (x_0, y_0) traverses for $-\infty < t < \infty$ the ray determined by vector $\vec{\mathbf{u}}(0)$, which is the vector joining $(0, 0)$ to (x_0, y_0). Each such ray is a separatrix in the sense that trajectories cannot cross it. The Lemma is correct: S is a separatrix, but it is not useful for phase plotting. The phase portrait is a **star node**.

Node with Equal Eigenvalues

There are two sub-classifications for a matrix A with real equal eigenvalues $\lambda_1 = \lambda_2$.

Star Node: Matrix A is diagonalizable with $\lambda_1 = \lambda_2 \neq 0$. Trajectories are rays from the origin. Equilibrium $(0, 0)$ is an attractor (or a repeller) from all points (x_0, y_0). Separatrix not used.

Degenerate Node: Matrix A is not diagonalizable with $\lambda_1 = \lambda_2 \neq 0$ and one eigenpair $(\lambda_1, \vec{v}_1)$. Equilibrium $(0, 0)$ is an attractor (or a repeller) from all points (x_0, y_0). A threaded trajectory from (x_0, y_0) does not cross separatrix $S = \mathbf{span}(\vec{\mathbf{v}}_1)$, which is the union of $(0, 0)$ and two trajectories.

Node with Unequal Eigenvalues

Matrix A has two eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$, because $\lambda_1 \neq \lambda_2$. Define separatrix $S = \mathbf{span}(\vec{\mathbf{v}}_1) \cup \mathbf{span}(\vec{\mathbf{v}}_2)$, which is a union of two lines through the origin separating the plane into four regions. Equilibrium $(0, 0)$ is an attractor (or a repeller) from all (x_0, y_0), the trajectory not crossing separatrix S.

Proper Node and Improper Node Classifications

The classifications **proper** and **improper** organize the possible node phase portraits. This terminology may appear in dynamical system literature.

Proper Node: The equilibrium is an attractor (or repeller) from all (x_0, y_0)s. Phase portrait: *star node*. Separatrix not used.

Improper Node: The equilibrium is an attractor (or repeller) from all (x_0, y_0). Separatrix: $S = \mathbf{span}(\vec{\mathbf{v}}_1)$ for one eigenpair $(\lambda_1, \vec{\mathbf{v}}_1)$ and $S = \mathbf{span}(\vec{\mathbf{v}}_1) \cup \mathbf{span}(\vec{\mathbf{v}}_2)$ for two eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$. Trajectories do not cross S. Phase portraits: *degenerate node* and *node with unequal eigenvalues*.

How to sort out the terminology? The rule is: **proper** = **star**. Every non-star node is **improper**. It may help to associate the terminology with phase portrait plots in Figures 17 and 18 on page 772.

Examples and Methods

Example 10.1 (Spiral)
Show the classification details for the spirals represented by the matrices

$$\begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix}, \quad \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}.$$

Solution: Matrix $\begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix}$ has characteristic equation $(\lambda - 5)^2 + 4 = 0$. Then $\lambda = 5 \pm 2i$ and the Euler atoms are $e^{5t}\cos(2t), e^{5t}\sin(2t)$. The atoms have sines and cosines, which limits the classification to a center or a spiral. The presence of the exponential factor e^{5t} implies it is not a center, therefore it is a spiral. Because the atoms limit to zero at $t = -\infty$, then $(0, 0)$ is a repeller. Classification: unstable spiral.

Matrix $\begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$ has characteristic equation $(\lambda+1)^2+9 = 0$. Then $\lambda = -1\pm 3i$ and the Euler atoms are $e^{-t}\cos(3t), e^{-t}\sin(3t)$. The atoms have sines and cosines, which implies rotation, either a center or a spiral. The presence of the exponential factor e^{-t} implies it is not a center, therefore it is a spiral. Because the atoms limit to zero at $t = \infty$, then $(0, 0)$ is an attractor. Classification: stable spiral.

Example 10.2 (Center)
Show the classification details for matrix $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$, which represents a center.

Solution: The characteristic equation $\lambda^2+4 = 0$ has complex roots $\lambda = \pm 2i$. The Euler atoms are $\cos(2t), \sin(2t)$, therefore a rotating figure is expected. Because of pure sines and cosines and no exponentials, the initial classification of spiral or center reduces to a center. Always a center is stable. Classification: stable center.

Example 10.3 (Saddle)
Show the classification details for the saddles represented by the matrices

$$\begin{pmatrix} 5 & 4 \\ 10 & 1 \end{pmatrix}, \quad \begin{pmatrix} -5 & 4 \\ 2 & 1 \end{pmatrix}$$

Solution: We'll use the theorem $|A - \lambda I| = \lambda^2 + \textbf{trace}(A)(-\lambda) + |A|$ to find the characteristic equation. Symbol $\textbf{trace}(A)$ is the sum of the diagonal elements of A and symbol $|A|$ is the determinant of A, evaluated by Sarrus's rule.

The characteristic equations are

$$\lambda^2 - 6\lambda - 35 = 0, \quad \lambda^2 + 4\lambda - 13 = 0.$$

The roots are $3 \pm 2\sqrt{11}$ $(9.6, -3.6)$ and $-2 \pm \sqrt{17}$ $(2.1, -6.1)$, respectively. Therefore, the roots a, b are real with $a > 0$ and $b < 0$. Euler atoms are e^{at}, e^{bt}. The absence of sines and cosines implies the equilibrium $(0, 0)$ is non-rotating, either a saddle or a node. Because one atom limits to ∞ and the other to zero, at $t = \pm\infty$, then $(0, 0)$ is a saddle. A saddle is always unstable. Classifications: $(0, 0)$ is an unstable saddle for both matrices.

Saddles have a separatrix $S = \textbf{span}(\vec{\mathbf{v}}_1) \cup \textbf{span}(\vec{\mathbf{v}}_2)$ that divides the plane into four regions. The analysis follows the node case, $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ being the eigenvectors. Calculus uses the terminology **asymptotes** to describe S and the limit of a point (x, y) on a saddle graphic as $x^2 + y^2 \to \infty$. For instance, the second matrix has separatrix $S = \textbf{span}\left(\begin{pmatrix} 0.56 \\ 1 \end{pmatrix}\right) \cup \textbf{span}\left(\begin{pmatrix} -3.56 \\ 1 \end{pmatrix}\right)$, the column vectors defining the calculus asymptotes.

Example 10.4 (Node Sub-Classification: Equal Eigenvalues)
Show the node classification details for the matrices $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$.

Solution: A 2×2 matrix is called **diagonalizable** provided it has 2 eigenpairs. Then $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ is diagonalizable whereas $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ is not diagonalizable.

The eigenvalues of both matrices are $5, 5$. Euler atoms are the same for both matrices: e^{5t}, te^{5t}. The absence of sines and cosines limits the classification to saddle or node. Because these atoms limit to zero at $t = -\infty$, then $(0, 0)$ is a node. For both, $(0, 0)$ is a repeller.

Classifications: $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$ is an **unstable proper node** (*star node*) and $\begin{pmatrix} 5 & 1 \\ 0 & 5 \end{pmatrix}$ is an **unstable improper node** (*degenerate node*). See page 773. The star node does not use a separatrix as a graphing tool. A separatrix S for the degenerate node is the line through $(0, 0)$ with direction $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, making for two regions separated by S: the upper half-plane and the lower half-plane. Expect orbits to be tangent to S at $t = -\infty$.

Example 10.5 (Node Sub-Classification: Unequal Eigenvalues)

Show the node classification details for the matrices $\begin{pmatrix} -5 & 0 \\ 0 & -7 \end{pmatrix}$, $\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$.

Solution: Both matrices are diagonal. Each has two independent eigenvectors $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, the columns of the identity matrix. Eigenvalues are the diagonal elements.

Matrix $\begin{pmatrix} -5 & 0 \\ 0 & -7 \end{pmatrix}$ has unequal eigenvalues $-5, -7$ with Euler atoms e^{-5t}, e^{-7t}. Absence of sines and cosines limits the classification to saddle or node. The atoms have limit zero at $t = \infty$, which eliminates the saddle classification and classifies $(0, 0)$ as an attractor, a stable improper node. Orbits are tangent at $t = \infty$ to $\pm\vec{v}_1$, eigenvector for $\lambda_1 = -5$. A separatrix S constructed from eigenvectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ has four regions: the usual 4 quadrants in the plane.

Matrix $\begin{pmatrix} 5 & 0 \\ 0 & 7 \end{pmatrix}$ has unequal eigenvalues $5, 7$ with Euler atoms e^{5t}, e^{7t}. Absence of sines and cosines limits the classification to saddle or node. The atoms have limit zero at $t = -\infty$, which eliminates the saddle classification. Therefore, $(0, 0)$ is a repeller, an unstable improper node. Orbits are tangent to eigenvector $\pm\vec{v}_1$ at $t = -\infty$. A separatrix S is identical to the separatrix for the first matrix, because of identical eigenvectors. ■

Computer Phase Portraits. In computer **node** plots for unequal eigenvalues, an eigenvector direction can be detected from orbit limits at $t = \pm\infty$. Attractors will have the eigenvector direction for eigenvalue λ with $|\lambda|$ smallest. Repellers will have the eigenvector direction for eigenvalue λ with $|\lambda|$ largest.

Exercises 10.2

Planar Constant Linear Systems

1. (Picard's Theorem) Explain why planar solutions don't cross, by appeal to Picard's existence-uniqueness theorem for $\frac{d}{dt}\vec{u}=A\vec{u}$.

2. (Equilibria) System $\frac{d\vec{u}}{dt} = A\vec{u}$ always has solution $\vec{u}(t) = \vec{0}$, so there is always one equilibrium point. Give an example of a matrix A for which there are infinitely many equilibria.

Putzer's Formula

3. (Cayley-Hamilton) Define matrices $\vec{\mathbf{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\vec{\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Given matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, expand left and right sides to verify the **Cayley-Hamilton identity** $A^2-(a+d)A + (ad-bc)\vec{\mathbf{I}} = \vec{\mathbf{0}}$.

4. (Complex Roots) Verify the Putzer solution $\vec{u} = \Phi(t)\vec{u}(0)$ of $\vec{u}' = A\vec{u}$ for complex roots $\lambda_1 = \overline{\lambda_2} = a+bi$, $b > 0$, where $\Phi(t)$ is

$$e^{at}\left(\cos(bt)\, I + (A - aI)\frac{\sin(bt)}{b}\right).$$

5. (Distinct Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\vec{u}.$$

6. (Real Equal Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 6 & -4 \\ 4 & -2 \end{pmatrix}\vec{u}.$$

7. (Complex Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\vec{u}.$$

8. (Purely Complex Eigenvalues) Solve

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}\vec{u}.$$

Continuity and Redundancy

9. (Real Equal Eigenvalues) Show that limiting $\lambda_2 \to \lambda_1$ in the Putzer formula for distinct eigenvalues gives Putzer's formula for real equal eigenvalues.

10. (Complex Eigenvalues) Assume $\lambda_1 = \overline{\lambda_2} = a + ib$ with $b > 0$. Then Putzer's first formula holds. Show the third formula details for $\Phi(t)$:

$$e^{at}\left(\cos(bt)\, I + (A - aI)\frac{\sin(bt)}{b}\right).$$

Illustrations

11. (Distinct Eigenvalues) Show the details for the solution of

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} -1 & 3 \\ -6 & 8 \end{pmatrix}\vec{u}.$$

12. (Complex Eigenvalues) Show the details for the solution of

$$\frac{d\vec{u}}{dt} = \begin{pmatrix} 2 & 5 \\ -5 & 2 \end{pmatrix}\vec{u}.$$

Isolated Equilibria

13. (Determinant Expansion) Verify that $|A - \lambda I|$ equals

$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

14. (Infinitely Many Equilibria) Explain why $A\vec{u} = \vec{0}$ has infinitely many solutions when $\det(A) = 0$.

Classification of Equilibria

15. (Rotating Figures) When sines and cosines appear in the Euler atoms, the phase portrait at $(0, 0)$ rotates around the origin. Explain precisely why this is true.

16. (Non-Rotating Figures) When sines and cosines do not appear in the Euler atoms, the phase portrait at $(0, 0)$ has no rotation. Give a precise explanation.

Attractor and Repeller

17. (Classification) Which of spiral, center, saddle, node can be an attractor or a repeller?

18. (Attractor) Prove that $(0,0)$ is an attractor if and only if the Euler atoms have limit zero at $t = \infty$.

19. (Repeller) Prove that $(0,0)$ is a repeller if and only if the Euler atoms have limit zero at $t = -\infty$.

20. (Center) A center is neither an attractor nor a repeller. Explain, using Euler atoms.

Phase Portrait Linear

Show the classification details for spiral, center, saddle, proper node, improper node. Include for saddle and node a drawing which shows eigenvector directions. Notation: $' = \frac{d}{dt}$.

21. (Spiral)

$$\begin{aligned} x' &= 2x + 3y, \\ y' &= -3x + 2y. \end{aligned}$$

22. (Center)

$$\begin{aligned} x' &= 3y, \\ y' &= -3x. \end{aligned}$$

23. (Saddle)

$$\begin{aligned} x' &= 3x, \\ y' &= -5y. \end{aligned}$$

24. (Proper Node)

$$\begin{aligned} x' &= 2x, \\ y' &= 2y. \end{aligned}$$

25. (Improper Node: Degenerate)

$$\begin{aligned} x' &= 2x + y, \\ y' &= 2y. \end{aligned}$$

26. (Improper Node: $\lambda_1 \neq \lambda_2$)

$$\begin{aligned} x' &= 2x + y, \\ y' &= 3y. \end{aligned}$$

10.3 Planar Almost Linear Systems

A nonlinear planar autonomous system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is called **almost linear** at equilibrium point $\vec{u} = \vec{u}_0$ if

$$\vec{F}(\vec{u}) = A(\vec{u} - \vec{u}_0) + \vec{G}(\vec{u}),$$
$$\lim_{\|\vec{u}-\vec{u}_0\|\to 0} \frac{\|\vec{G}(\vec{u})\|}{\|\vec{u} - \vec{u}_0\|} = 0.$$

The function $\vec{G}$ has the same smoothness as $\vec{F}$. We investigate the possibility that a local phase portrait at $\vec{u} = \vec{u}_0$ for the nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is graphically identical to the one for the linear system $\vec{v}'(t) = A\vec{v}(t)$ at $\vec{v} = 0$.

The results will apply to **all isolated equilibria** of $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$. This is accomplished by expanding F in a Taylor series about each equilibrium point, which implies that the ideas are applicable to different choices of A and G, depending upon which equilibrium point $\vec{u}_0$ was considered.

Define the **Jacobian matrix** of $\vec{F} = \begin{pmatrix} f \\ g \end{pmatrix}$ at equilibrium point $\vec{u}_0$ by the formula

$$J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}.$$

Taylor's theorem for functions of two variables says that

$$\vec{F}(\vec{u}) = J(\vec{u} - \vec{u}_0) + \vec{G}(\vec{u})$$

where $\vec{G}(\vec{u})/\|\vec{u} - \vec{u}_0\| \to 0$ as $\|\vec{u} - \vec{u}_0\| \to 0$. Therefore, for $\vec{F}$ continuously differentiable, we may always take $A = J$ to obtain from the almost linear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ its **linearization** $\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$.

Phase Portrait of an Almost Linear System

For planar almost linear systems $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$, phase portraits have been studied extensively, by Poincaré-Bendixson and a long list of researchers. It is known that only a finite number of local phase portraits are possible near each isolated equilibrium point of the nonlinear system, the library of figures being identical to those possibilities for a linear system $\vec{v}'(t) = A\vec{v}(t)$. A precise statement without proof appears below, followed by a summary that is easier to remember.

Theorem 10.4 (Paste Theorem: Almost Linear Phase Portrait)
Let the planar almost linear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ be given with $\vec{F}(\vec{u}) = A(\vec{u} - \vec{u}_0) + \vec{G}(\vec{u})$ near the isolated equilibrium point $\vec{u}_0$ (an isolated root of $\vec{F}(\vec{u}_0) = \vec{0}$ with $|A| \neq 0$). Let λ_1, λ_2 be the roots of $\det(A - \lambda I) = 0$. Then:

1. If $\lambda_1 = \lambda_2$, then the equilibrium $\vec{u}_0$ of the nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is either a node or a spiral. The equilibrium $\vec{u}_0$ is an asymptotically stable attractor if $\lambda_1 < 0$ and it is a repeller if $\lambda_1 > 0$. In short, the nonlinear system inherits stability from the linear system.

2. If $\lambda_1 = \overline{\lambda_2} = ib$ with $b > 0$, then the equilibrium $\vec{u}_0$ of the nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ is either a center or a spiral. The stability of the equilibrium $\vec{u}_0$ cannot be predicted from properties of A.

3. In all other cases, the isolated equilibrium $\vec{u}_0$ has graphically the same local phase portrait as the associated linear system $\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$ at $\vec{v} = \vec{0}$. In particular, local phase portraits of a saddle, spiral or node can be graphed from the linear system. The nonlinear system inherits locally the linearized system properties of stability and instability.

Paste Theorem Summary: The linearized phase portrait **locally pastes** onto the nonlinear phase portrait **with two exceptions**:
(1) Nodes from equal roots cause pasting of either a node or spiral.
(2) Centers (complex roots $\pm ib$) cause pasting a center or spiral.
Local **stability** and **instability** are inherited except for a center.

Classification of Almost Linear Equilibria

A system $\frac{d}{dt}\vec{u}(t) = A(\vec{u}(t) - \vec{u}_0) + \vec{G}(\vec{u}(t))$ has a local phase portrait determined by the linear system $\vec{v}'(t) = A\vec{v}(t)$, except in the case when the roots λ_1, λ_2 of the characteristic equation $\det(A - \lambda I) = 0$ are equal or purely imaginary (see Theorem 10.4). To summarize:

Table 3. Equilibria classification for almost linear systems

Eigenvalues of A	**Nonlinear Classification**
$\lambda_1 < 0 < \lambda_2$	Unstable saddle
$\lambda_1 < \lambda_2 < 0$	Stable improper node
$\lambda_1 > \lambda_2 > 0$	Unstable improper node
$\lambda_1 = \lambda_2 < 0$	Stable node or spiral
$\lambda_1 = \lambda_2 > 0$	Unstable node or spiral
$\lambda_1 = \overline{\lambda_2} = a + ib,\ a < 0,\ b > 0$	Stable spiral
$\lambda_1 = \overline{\lambda_2} = a + ib,\ a > 0,\ b > 0$	Unstable spiral
$\lambda_1 = \overline{\lambda_2} = ib,\ b > 0$	Stable or unstable, center or spiral

Almost Linear Equilibria Geometry

Applied literature may refer to an equilibrium point $\vec{u}_0$ of a nonlinear system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ as a spiral, center, saddle or node. The geometry of these classifications is explained below.

Spiral. To describe a **nonlinear spiral**, we require that an orbit starting on a given ray emanating from the equilibrium point must intersect that ray in infinitely many distinct points on $(-\infty, \infty)$.

Intuition. Basic understanding of a **nonlinear spiral** is obtained from a linear example, e.g.,

$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \vec{u}(t).$$

An orbit has component solutions

$$x(t) = e^{-t}(A\cos 2t + B\sin 2t), \quad y(t) = e^{-t}(-A\sin 2t + B\cos 2t)$$

which oscillate infinity often on $(-\infty, \infty)$, rotating around equilibrium point $(0, 0)$ with amplitude Ce^{-t}, for some constant $C > 0$.

Center. Local orbits are periodic solutions. Each local orbit is a closed curve which forms a planar region with boundary, having the equilibrium point interior. As the periodic orbits shrink, the planar region also shrinks, limiting as a planar set to the equilibrium point. Drawings often portray the periodic orbit as a convex figure, but this is not correct, in general, because the periodic orbit can have any shape. In particular, the linearized system may have phase portrait consisting of concentric circles, but the nonlinear phase portrait has no such exact geometric structure.

Saddle. The term implies that *locally* the phase portrait looks like a linear saddle. In nonlinear phase portraits, the straight lines to which orbits are asymptotic appear to be curves instead. These curves are called **separatrices**, which are generally unions of certain orbits and equilibria.

Node. Each orbit starting near the equilibrium is expected to limit to the equilibrium at either $t = \infty$ (stable attractor) or $t = -\infty$ (unstable repeller), in a fashion asymptotic to a direction $\vec{v}$. The terminology applies when the linearized system is a **proper node** (a.k.a. *star node*), in which case there is an orbit asymptotic to $\vec{v}$ for every direction $\vec{v}$. If there is only one direction $\vec{v}$ possible, or all orbits are asymptotic to just one separatrix, then the equilibrium is classified as an **improper node**. The term *degenerate node* applies to a subclass of improper nodes – see Example 10.4 page 776.

Pasting Figures to make a Nonlinear Phase Portrait

The plan provided by the theorem is to paste a library source figure, one of spiral, center, saddle or node, overlaying $(0, 0)$ in the source figure atop equilibrium point $\vec{u} = \vec{u}_0$ in the nonlinear phase portrait. Some observations follow, about what works and what fails.

1. The local paste is valid to graphical resolution near $\vec{u} = \vec{u}_0$, and invalid far away from the equilibrium point.

2. The pasted figure can mutate into a spiral, if the source figure is either a center, or else a node with $\lambda_1 = \lambda_2$. Otherwise, saddle, spiral and node locally paste into saddle, spiral, node.

3. Stability of the source figure is inherited by the nonlinear portrait, except when the source is a center. In this one exceptional case, no stability conclusion can be drawn. However, an attractor or repeller source figure always pastes into an attractor or a repeller.

Examples and Methods

Example 10.6 (Compute Isolated Equilibria)
Find all equilibria for the nonlinear system

$$x'(t) = x(t) + y(t), \quad y'(t) = 1 - x^2(t).$$

Solution: Equilibria are constant solutions, obtained formally by setting $x' = y' = 0$ in the two differential equations $x' = x + y, y' = 1 - x^2$. Then solve for constants x, y. The details:

Set $x' = 0$	$0 = x + y$
Set $y' = 0$	$0 = 1 - x^2$
Solve for x, y	$x = \pm 1$, $y = -x$.
Equilibria	$(1, -1)$ and $(-1, 1)$

Example 10.7 (Linearization at Equilibria)
Find the two linearizations at equilibria $(1, -1), (-1, 1)$ for the nonlinear system

$$x'(t) = x(t) + y(t), \quad y'(t) = 1 - x^2(t).$$

Solution: The system of differential equations is written with function notation in the form $x' = f(x, y), y' = g(x, y)$. Then

$$f(x, y) = x + y, \quad g(x, y) = 1 - x^2.$$

The Jacobian matrix

$$J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

is computed with symbols x, y, f, g as follows.

Partial derivative $f_x(x, y)$:	$f_x = \partial_x(x + y) = 1 + 0 = 1$
Partial derivative $g_x(x, y)$:	$g_x = \partial_x(1 - x^2) = 0 - 2x = -2x$
Partial derivative $f_y(x, y)$:	$f_y = \partial_y(x + y) = 0 + 1 = 1$
Partial derivative $g_y(x, y)$:	$g_y = \partial_y(1 - x^2) = 0 - 0 = 0$

Then

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -2x & 0 \end{pmatrix}.$$

The symbols x, y are used for the two substitutions: $x = 1, y = -1$ and $x = -1, y = 1$.

$$J(1,-1) = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}, \quad J(-1,1) = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}.$$

The two linearized problems are

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}\vec{u}, \quad \frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\vec{u}.$$

Example 10.8 (Classification of Linearized Problems)
Classify the two linear problems

$$\frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}\vec{u}, \quad \frac{d}{dt}\vec{u} = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}\vec{u}.$$

Solution:
The answers: $\begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$ is an unstable spiral; $\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$ is an unstable saddle.

The two characteristic equations are $\lambda^2 - \lambda + 2 = 0$ and $\lambda^2 + \lambda + 2 = 0$ with roots, respectively, $\frac{1}{2} \pm i\frac{\sqrt{7}}{2}$ and $2, -1$. According to the classification theory, page 769, the equilibrium $(0,0)$ is respectively an unstable spiral or an unstable saddle.

Example 10.9 (Pasting Linear Portraits onto Nonlinear Portraits)
Classify equilibria $(1,-1), (-1,1)$ for the nonlinear system

$$x'(t) = x(t) + y(t), \quad y'(t) = 1 - x^2(t),$$

as *nonlinear* spiral, center, saddle or node. Paste the linear portraits onto the nonlinear direction field portrait for Jacobians $J(-1,1), J(1,-1)$, if possible.

Solution: **Classifications**: $(-1,1)$ is a nonlinear unstable saddle; $(1,-1)$ is a nonlinear unstable spiral.

Previous examples show that for the linearized problems, $(-1,1)$ is an unstable saddle and $(-1,1)$ is an unstable spiral. Theorem 10.4 applies to conclude that the two linear phase portraits directly transfer onto the nonlinear phase portrait. This means that $(0,0)$ in each source figure can be pasted atop the corresponding equilibrium point in the nonlinear system, the pasted figure valid locally.

Computer phase portraits show the two pasted library figures with automatic fine tuning. Especially, the saddle will be tuned, because a library source figure usually has asymptotes parallel to the coordinate axes, whereas the computer graphic will show tuned asymptotes in eigenvector directions.

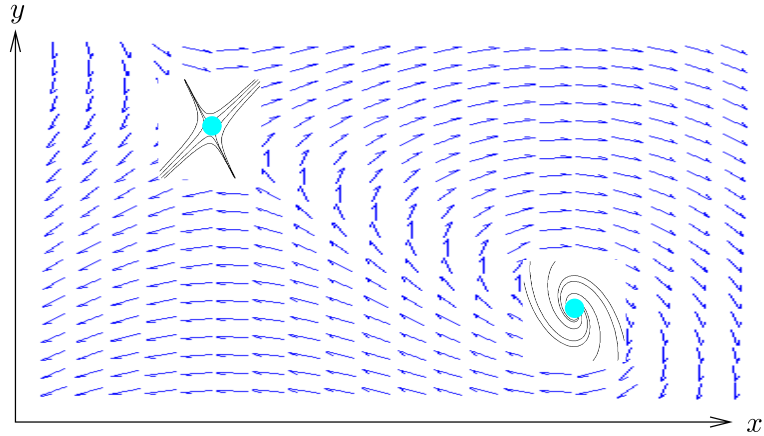


Figure 21. Pasting Source Figures onto a Nonlinear Phase portrait. Saddle at $(-1,1)$, spiral at $(1,-1)$. The saddle source uses a linear phase portrait for $\frac{d}{dt}\vec{v} = J(-1,1)\vec{v}$. The standard saddle source can be rotated to match the nonlinear direction field, with a similar result.

Example 10.10 (Trout System)

Consider a trout model for two species x, y:

$$\begin{aligned} x'(t) &= x(-2x - y + 180), \\ y'(t) &= y(-x - 2y + 120). \end{aligned}$$

The equilibria are $(0,0), (90,0), (0,60), (80,20)$. Find the linearized problem for each equilibrium, then make a tuned computer plot.

Solution:

System Form. Let $f(x,y) = x(-2x - y + 180), g(x,y) = y(-x - 2y + 120)$ to convert to system form $x' = f(x,y), y' = g(x,y)$.

Jacobian Matrix. Use symbols f, g, x, y to compute the Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$.

$f_x = \frac{\partial}{\partial x}\left(-2x^2 - xy + 180x\right) = -4x - y - 180$

$f_y = \frac{\partial}{\partial y}\left(-2x^2 - xy + 180x\right) = -x$

$g_x = \frac{\partial}{\partial x}\left(-xy - 2y^2 + 120y\right) = -y$

$g_y = \frac{\partial}{\partial y}\left(-xy - 2y^2 + 120y\right) = -x - 4y + 120$

$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -4x - y - 180 & -x \\ -y & -x - 4y + 120 \end{pmatrix}$

Equilibria. To find the equilibria, formally set $x' = y' = 0$. Details:

$x' = 0 = f(x,y)$ becomes $x(-2x - y + 180) = 0$

$y' = 0 = g(x,y)$ becomes $y(-x - 2y + 120) = 0$

Set the factors to zero, in four possible ways, to obtain the solutions

$$x = y = 0, \quad x = 0, y = 60, \quad x = 90, y = 0, \quad x = 80, y = 20.$$

Linearized Differential Equations. The linear problems $\frac{d}{dt}\vec{u} = J(x_0, y_0)\vec{u}$ at equilibria $(0,0), (0,60), (90,0), (80,20)$ are created from the four Jacobian matrices

$$J(0,0) = \begin{pmatrix} -180 & 0 \\ 0 & 120 \end{pmatrix}, \qquad J(0,60) = \begin{pmatrix} 120 & 0 \\ -60 & -120 \end{pmatrix},$$
$$J(90,0) = \begin{pmatrix} -180 & -90 \\ 0 & 30 \end{pmatrix}, \qquad J(80,20) = \begin{pmatrix} -160 & -80 \\ -20 & -40 \end{pmatrix}.$$

Eigenvalues. Answers for the four matrices are respectively:

$$120, 180; \quad 120, -120; \quad 30, -180; \quad -27.89, -172.11$$

Linear Classifications. Because there are no complex eigenvalues, then the possible linear phase portraits are either saddle or node. Checking limits of Euler atoms at $t = \infty$ reveals the classifications unstable node, saddle, saddle, stable node. No equal eigenvalues implies both nodes are **improper**.

Paste Theorem. All linear source figures paste directly onto the nonlinear phase portrait with stability properties inherited. See Theorem 10.4.

Eigenvectors help understanding of the phase portrait. In all four figures, asymptote directions are along an eigenvector. For instance, at $(80,20)$ the two eigenvector directions are $\vec{v}_1 = \begin{pmatrix} -0.6 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 6.6 \\ 1 \end{pmatrix}$.

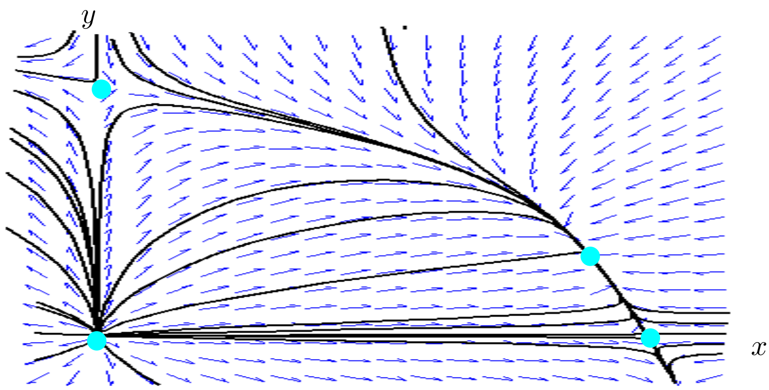


Figure 22. Trout System Phase portrait.
Saddles at $(0,60)$ and $(90,0)$. Improper nodes with unequal eigenvalues at $(0,0)$ and $(80,20)$. A separatrix can be visualized, which connects $(90,0)$ to $(0,0)$ to $(60,0)$ along the coordinate axes, and then to $(80,20)$.

Example 10.11 (Rabbit-Fox System)

Consider a predator-prey model for rabbits $x(t)$ and foxes $y(t)$:

$$\begin{aligned} x' &= \frac{1}{200}x(40-y), \\ y' &= \frac{1}{100}y(x-50). \end{aligned}$$

The equilibria are $(0,0), (50,40)$. Find the linearized problem for each equilibrium, then make a tuned computer plot.

Solution:

System Form. Let $f(x,y) = \frac{1}{200}x(40-y), g(x,y) = \frac{1}{100}y(x-50)$ to convert to system form $x' = f(x,y), y' = g(x,y)$.

Jacobian Matrix. Symbols f, g, x, y are used in the Jacobian $J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$.

$f_x = \frac{\partial}{\partial x}\left(x/5 - xy/200\right) = 1/5 - y/200$

$f_y = \frac{\partial}{\partial y}\left(x/5 - xy/200\right) = -x/200$

$g_x = \frac{\partial}{\partial x}\left(-y/2 + xy/100\right) = y/100$

$g_y = \frac{\partial}{\partial y}\left(-y/2 + xy/100\right) = -x - 4y + 120$

$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} -4x - y - 180 & -x \\ -y & -x - 4y + 120 \end{pmatrix}$

Equilibria. To find the equilibria $(0,0), (50,40)$, formally set $x' = y' = 0$. Details:

$0 = f(x,y)$ becomes $\frac{1}{200}x(40-y) = 0$

$0 = g(x,y)$ becomes $\frac{1}{100}y(x-50) = 0$

The solutions are $x = y = 0$ or else $x = 50, y = 40$.

Linearized Differential Equations. The linear problems $\frac{d}{dt}\vec{u} = J(x_0, y_0)\vec{u}$ at equilibria $(0,0), (50,40)$ are created from the two Jacobian matrices

$$J(0,0) = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad J(50,40) = \begin{pmatrix} 0 & -\frac{1}{4} \\ \frac{2}{5} & 0 \end{pmatrix}.$$

Eigenvalues. The answers are $\frac{1}{5}, -\frac{1}{2}$ and $\pm i/\sqrt{10}$, respectively.

Linear Classifications. Complex eigenvalues imply linear phase portraits of either center or node. Checking Euler atoms reveals the classification **center** at $(50,40)$. Real unequal eigenvalues at $(0,0)$ implies a saddle or node. Checking limits of the Euler atoms at $t = \infty$ implies $(0,0)$ is a **saddle**. Both linear source figures are **stable**.

Paste Theorem. The linear saddle source figure for $(0,0)$ pastes directly onto the nonlinear phase portrait at $(0,0)$ with stability properties inherited. The linear center source figure for $(50,40)$ pastes into a center or a spiral at $(50,40)$. The paste stability or instability is not decided. See Theorem 10.4.

The easiest path to deciding the nonlinear portrait at $(50,40)$ is a computer phase portrait, which shows a center structure.

Eigenvectors help understanding of the phase portrait. At $(0,0)$ the two eigenvector directions are $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

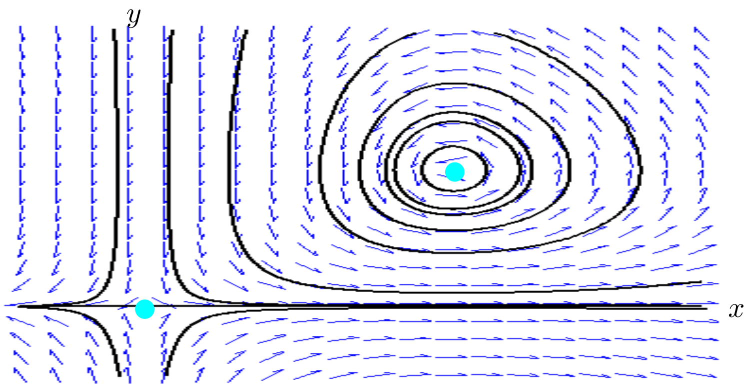


Figure 23. Rabbit-Fox System Phase portrait.
Eigenvector directions for the saddle at $(0, 0)$ are parallel to the coordinate axes. The linear center from $J(50, 40)$ happens to transfer to a nonlinear center at $(50, 40)$.

Exercises 10.3

Almost Linear Systems. Find all equilibria (x_0, y_0) of the given nonlinear system. Then compute the Jacobian matrix $A = J(x_0, y_0)$ for each equilibria.

1. (Spiral and Saddle)

$$\begin{aligned} \frac{d}{dt}x &= x + 2y, \\ \frac{d}{dt}y &= 1 - x^2. \end{aligned}$$

2. (Two Improper Nodes, Spiral)

$$\begin{aligned} \frac{d}{dt}x &= x - 3y + 2xy, \\ \frac{d}{dt}y &= 4x - 6y - xy - x^2. \end{aligned}$$

3. (Proper Node, Saddle)

$$\begin{aligned} \frac{d}{dt}x &= 3x - 2y - x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y. \end{aligned}$$

4. (Center and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y - xy. \end{aligned}$$

5. (Proper Node and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= y - xy. \end{aligned}$$

6. (Degenerate Node, Spiral and Two Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3 + y^3, \\ \frac{d}{dt}y &= y + 3xy. \end{aligned}$$

7. (Improper Node, Saddle)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

8. (Proper Node and a Saddle)

$$\begin{aligned} \frac{d}{dt}x &= 2x + y^3, \\ \frac{d}{dt}y &= 2y + 3xy. \end{aligned}$$

Phase Portrait Almost Linear. Linear library phase portraits can be locally pasted atop the equilibria of an almost linear system, with limitations. Apply the theory for the following examples. Complete the phase diagram by computer, thereby resolving the possible mutation of a center or node into a spiral. Label eigenvector directions where it makes sense.

9. (Center and Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= 2x - y - xy. \end{aligned}$$

10. (Degenerate Node, Three Saddles)

$$\begin{aligned} \frac{d}{dt}x &= x - y + x^2 - y^2, \\ \frac{d}{dt}y &= y - xy. \end{aligned}$$

11. (Degenerate Node, Spiral, Two Saddles)

$$\begin{aligned}\frac{d}{dt}x &= x - y + x^3 + y^3,\\ \frac{d}{dt}y &= y + 3xy.\end{aligned}$$

12. (Improper Node, Saddle)

$$\begin{aligned}\frac{d}{dt}x &= x - y + x^3,\\ \frac{d}{dt}y &= 2y + 3xy.\end{aligned}$$

13. (Proper Node, Saddle)

$$\begin{aligned}\frac{d}{dt}x &= 2x + y^3,\\ \frac{d}{dt}y &= 2y + 3xy.\end{aligned}$$

14. (Two Improper Nodes and Two Saddles)

$$\begin{aligned}\frac{d}{dt}x &= (120 - 4x - 2y)x,\\ \frac{d}{dt}y &= (60 - x - 2y)y\end{aligned}$$

Classification of Almost Linear Equilibria. With computer assist, find and classify the nonlinear equilibria.

15. (Co-existing Species)

$$\begin{aligned}x'(t) &= x(t)(24 - 2x(t) - y(t)),\\ y'(t) &= y(t)(30 - 2y(t) - x(t)).\end{aligned}$$

16. (Doomsday-Extinction)

$$\begin{aligned}x'(t) &= x(t)(x(t) - y(t) - 4),\\ y'(t) &= y(t)(x(t) + y(t) - 8).\end{aligned}$$

Almost Linear Geometry. A separatrix S is a union of curves and equilibria. Ideally, orbits limit to S. With computer assist, make a plot of threaded curves which identify one or more separatrices near the equilibrium.

17. (Saddle $(-1, 1)$**)**

$$\begin{aligned}\frac{d}{dt}x &= x + y,\\ \frac{d}{dt}y &= 1 - x^2.\end{aligned}$$

18. (Saddle $(-1/5, -2/5)$**)**

$$\begin{aligned}\frac{d}{dt}x &= 3x - 2y - x^2 - y^2,\\ \frac{d}{dt}y &= 2x - y.\end{aligned}$$

19. (Saddle $(-2/3, \sqrt[3]{4/3})$**)**

$$\begin{aligned}\frac{d}{dt}x &= 2x + y^3,\\ \frac{d}{dt}y &= 2y + 3xy.\end{aligned}$$

20. (Degenerate Improper Node $(0, 0)$**)**

$$\begin{aligned}\frac{d}{dt}x &= x - y + x^3 + y^3,\\ \frac{d}{dt}y &= y + 3xy.\end{aligned}$$

Rayleigh and van der Pol. Each example below has a unique periodic orbit surrounding an equilibrium point that is the limit at $t = \infty$ of any other orbit. Discuss the spiral repeller at $(0, 0)$ in the attached figure, from the linearized problem at $(0, 0)$ and **Paste Theorem** 10.4. Create a phase portrait with computer assist for the nonlinear problem.

21. (Lord Rayleigh 1877, Clarinet Reed Model)

$$\begin{aligned}\frac{d}{dt}x &= y,\\ \frac{d}{dt}y &= -x + y - y^3.\end{aligned}$$

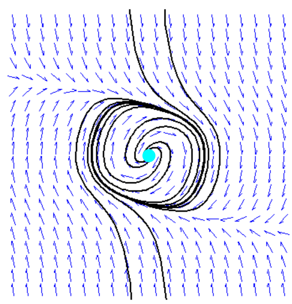


Figure 24. Clarinet Reed.

22. (van der Pol 1924, Radio Oscillator Circuit Model)

$$\begin{aligned}\frac{d}{dt}x &= y,\\ \frac{d}{dt}y &= -x + (1 - x^2)y.\end{aligned}$$

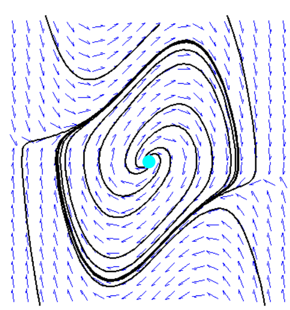


Figure 25. Oscillator Circuit.

10.4 Biological Models

Studied here are **predator-prey models** and **competition models** for two populations. Assumed as background from Malthus' Law (Chapter 1 Section 1) are the one-dimensional Malthusian model $\frac{d}{dt}P = kP$ and the one-dimensional Verhulst model $\frac{d}{dt}P = (a - bP)P$.

Predator-Prey Models

One species called the **Predator** feeds on the other species called the **Prey**. The prey feeds on some constantly available food supply, e.g., rabbits eat plants and foxes eat rabbits.

Credited with the classical predator-prey model is the Italian mathematician **Vito Volterra** (1860-1940), who worked on cyclic variations in shark and prey-fish populations in the Adriatic sea. The following biological assumptions apply to model a predator-prey system.

Malthusian Growth	The prey population grows according to the growth equation $x'(t) = a\,x(t)$, $a > 0$, in the absence of predators.
Malthusian Decay	The predator population decays according to the decay equation $y'(t) = -b\,y(t)$, $b > 0$, in the absence of prey.
Chance Encounters	The prey decrease population at a rate $-pxy$, $p > 0$, due to chance encounters of predators y with prey x. Predators increase population due to these chance interactions at a rate qxy, $q > 0$.

The interaction terms qxy and $-pxy$ are justified by arguing that the frequency of chance encounters is proportional to the product xy. Biologists explain the proportionality by saying that doubling either population should double the frequency of chance encounters. Adding the Malthusian rates and the chance encounter rates gives the **Volterra predator-prey system**[2]

$$(1) \qquad \begin{aligned} x'(t) &= (a - p\,y(t))x(t), \\ y'(t) &= (q\,x(t) - b)y(t). \end{aligned}$$

The differential equations are displayed in this form in order to emphasize that each of $x(t)$ and $y(t)$ satisfy a scalar first order differential equation $u'(t) = r(t)u(t)$ in which the rate function $r(t)$ depends on time. For initial population sizes near zero, the two differential equations behave very much like the Malthusian growth model $u'(t) = a\,u(t)$ and the Malthusian decay model $u'(t) = -b\,u(t)$. This basic growth/decay property allows us to identify the predator variable y, or the prey variable x, regardless of the order in which the differential equations are

[2]The system is written with prey x and predator y. Alphabetical order **predator-prey** would suggest the variables to be reversed, y and then x. History is otherwise.

written. As viewed from Malthus' law $u' = ru$, the prey population has growth rate $r = a - py$ which gets smaller as the number y of predators grows, resulting in fewer prey. Likewise, the predator population has decay rate $r = -b + qx$, which gets larger as the number x of prey grows, causing increased predation. These are the basic ideas of Verhulst, applied to the individual populations x and y.

System Variables

The system of two differential equations (1) can be written as a planar vector autonomous system

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u})$$

where vector functions $\vec{F}$ and $\vec{u}$ are defined by

$$(2) \qquad \vec{F}(\vec{u}) = \begin{pmatrix} (a-py)x \\ (qx-b)y) \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

The vector function $\vec{F}$ is everywhere defined and continuously differentiable. The Picard–Lindelöf theorem provides existence-uniqueness.

A planar vector autonomous system $\frac{d}{dt}\vec{u} = \vec{F}(\vec{u})$ can be written in standard scalar system form

$$x' = f(x,y), \quad y' = g(x,y)$$

by providing definitions for $f(x,y)$ and $g(x,y)$. For predator-prey system (1), the definitions are

$$f(x,y) = (a - p\,y)x, \quad g(x,y) = (q\,x - b)y.$$

Equilibria

The equilibrium points $\vec{u} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ satisfy $\vec{F}(\vec{u}) = \vec{0}$. For predator-prey system (1), the equilibria are $(0,0)$ and $(b/q, a/p)$, found by solving for x_0, y_0 in the equations $(a - p\,y_0)x_0 = 0$, $(q\,x_0 - b)y_0 = 0$.

Linearized Predator-Prey System

The linearized system at equilibrium (x_0, y_0) is the vector-matrix system $\frac{d}{dt}\vec{v}(t) = A\vec{v}(t)$, where A is the Jacobian matrix $J(x,y)$ evaluated at point $x = x_0, y = y_0$, briefly $A = J(x_0, y_0)$. In terms of system variables[3],

$$J(x_0, y_0) = \begin{pmatrix} f_x(x_0,y_0) & f_y(x_0,y_0) \\ g_x(x_0,y_0) & g_y(x_0,y_0) \end{pmatrix}.$$

[3]Notation f_x means $\partial f/\partial x$, the calculus x-derivative with all other variables held constant.

For the predator-prey system, we start by computing

$$f_x = \frac{\partial}{\partial x}(a\,x - p\,xy) = a - p\,y, \quad f_y = \frac{\partial}{\partial y}(a\,x - p\,xy) = 0 - p\,x,$$
$$g_x = \frac{\partial}{\partial x}(q\,xy - b\,y) = q\,y - 0, \quad g_y = \frac{\partial}{\partial y}(q\,xy - b\,y) = q\,x - b.$$

The Jacobian matrix is given explicitly by

$$J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} a - p\,y & -p\,x \\ q\,y & q\,x - b \end{pmatrix}. \tag{3}$$

The matrix J is evaluated at equilibrium points $(0,0), (b/q, a/p)$ to obtain the 2×2 matrices for the linearized systems:

$$J(0,0) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}, \quad J(b/q, a/p) = \begin{pmatrix} 0 & -bp/q \\ aq/p & 0 \end{pmatrix}.$$

The linearized systems $\vec{v}'(t) = A\vec{v}(t)$ are:

Equilibrium $(0,0)$ $\quad \frac{d}{dt}\vec{u}(t) = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix} \vec{u}(t)$

Equilibrium $(b/q, a/p)$ $\quad \frac{d}{dt}\vec{u}(t) = \begin{pmatrix} 0 & -bp/q \\ aq/p & 0 \end{pmatrix} \vec{u}(t)$

Saddle $J(0,0)$. Matrix $\begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}$ has unequal real eigenvalues $a, -b$ and associated Euler atoms e^{at}, e^{-bt}. No rotation implies a saddle or node, but limits at infinity imply a linear **saddle**. The **Paste Theorem** implies system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ has a saddle at equilibrium $(0,0)$.

Center $J(b/q, a/p)$. Matrix $\begin{pmatrix} 0 & -bp/q \\ aq/p & 0 \end{pmatrix}$ has eigenvalues $\lambda = \pm i\sqrt{ab}$ and associated Euler atoms $\cos(t\sqrt{ab}), \cos(t\sqrt{ab})$. Pure rotation (no exponential factor) implies a linear **center**. The **Paste Theorem** implies system $\frac{d}{dt}\vec{u}(t) = \vec{F}(\vec{u}(t))$ has either a center or a spiral at equilibrium $(b/q, a/p)$.

Shown below in Theorem 10.5 is that **the spiral case does not happen**. The proof of Lemma 10.2 is in the exercises.

Lemma 10.2 (Predator-Prey Implicit Solution)
Let $(x(t), y(t))$ be an orbit of the predator-prey system (1) with $x(0) > 0$ and $y(0) > 0$. Then for some constant C,

$$a\,\ln|y(t)| + b\,\ln|x(t)| - q\,x(t) - p\,y(t) = C. \tag{4}$$

Theorem 10.5 (Spiral Case Eliminated)

Equilibrium $(b/q, a/p)$ of predator-prey system (1) cannot be a spiral.

Proof: Assume the equilibrium $(b/q, a/p)$ is a spiral point and some orbit touches the line $x = b/q$ in points $(b/q, u_1)$, $(b/q, u_2)$ with $u_1 \neq u_2$, $u_1 > a/p$, $u_2 > a/p$. Consider the energy function $E(u) = a \ln|u| - p\,u$. Due to relation (4), $E(u_1) = E(u_2) = E_0$, where $E_0 \equiv C + b - b \ln|b/q|$. By the Mean Value Theorem of calculus, $dE/du = 0$ at some u between u_1 and u_2. This is a contradiction, because $dE/du = (a - pu)/u$ is strictly negative for $a/p < u < \infty$. Therefore, equilibrium $(b/q, a/p)$ is **not a spiral**. ■

Rabbits and Foxes

An instance of predator-prey theory is a Volterra population model for x rabbits and y foxes given by the system of differential equations

$$(5) \qquad \begin{aligned} x'(t) &= \frac{1}{250}\,x(t)(40 - y(t)), \\ y'(t) &= \frac{1}{50}\,y(t)(x(t) - 60). \end{aligned}$$

The equilibria of system (5) are $(0, 0)$ and $(60, 40)$. A phase portrait for system (5) appears in Figure 26.

The linearized system at $(60, 40)$ is

$$\begin{aligned} x'(t) &= -\frac{6}{25}y(t), \\ y'(t) &= \frac{4}{5}x(t). \end{aligned}$$

This system has eigenvalues $\pm i\sqrt{24/125}$. The Euler atoms are $\sin(t\sqrt{24/125})$ and $\cos(t\sqrt{24/125})$, which have period $2\pi/\sqrt{24/125} \approx 14.33934302$. The linear classification is a center.

The nonlinear classification at $(60, 40)$ is then a **center**, because of Theorem 10.5. Intuition dictates that the period of smaller and smaller nonlinear orbits enclosing the equilibrium $(60, 40)$ must approach a value that is approximately 14.3.

The fluctuations in population size $x(t)$ are measured graphically by the maximum and minimum values of x in the phase portrait, or more simply, by graphing t versus $x(t)$ in a planar graphic. To illustrate, the orbit for $x(0) = 60$, $y(0) = 100$ is graphed in Figure 27, from which it is determined that the rabbit population $x(t)$ fluctuates between 39 and 87. Similar remarks apply to foxes $y(t)$.

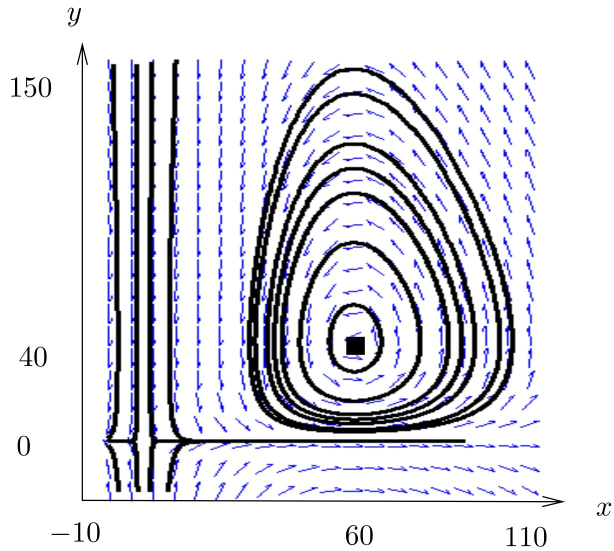


Figure 26. Rabbit and Fox System (5). Equilibria $(0, 0)$ and $(60, 40)$ are respectively a saddle and a center. The oscillation period is about 17 for the largest orbit and 14.5 for the smallest orbit.

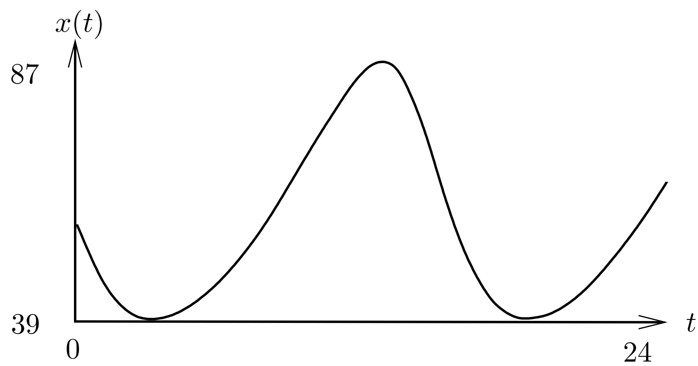


Figure 27. Scene Plot of $x(t)$ Rabbits.
An initial rabbit population of 60 and fox population of 100 causes the rabbit population $x(t)$ to fluctuate from 39 to 87. The plot uses nonlinear equations (5) with $x(0) = 60$, $y(0) = 100$.

Pesticides, Aphids and Ladybugs

The classical predator-prey equations apply for prey *Aphid* $x(t)$ and predator *Ladybug* $y(t)$, which for simplicity are assumed to be

$$\begin{aligned} x'(t) &= (1 - y(t))x(t), \\ y'(t) &= (x(t) - 1)y(t), \end{aligned} \tag{6}$$

with units in millions.

Consider deployment of an indiscriminate pesticide which kills a certain percentage of each insect. Typically available pesticide strengths are $s = 0.5$, $s = 0.75$. Strength $s = 0$ is no pesticide. We will assume hereafter that $0 \le s < 1$. The predator-prey equations mutate by adding terms for pesticide-caused death rates, resulting in the **Pesticide Model**

$$(7) \qquad \begin{aligned} x'(t) &= (1 - y(t))x(t) - s\,x(t), \\ y'(t) &= (x(t) - 1)y(t) - s\,y(t). \end{aligned}$$

Explained below in Figures 28, 29 and 30 are the results in the following table.

Table 4. Effects of Pesticide on Aphids and Ladybugs

The aphids increase and the ladybugs decrease.
The insecticide had a counterproductive effect. Aphid damage to the garden plants increased by using a pesticide.

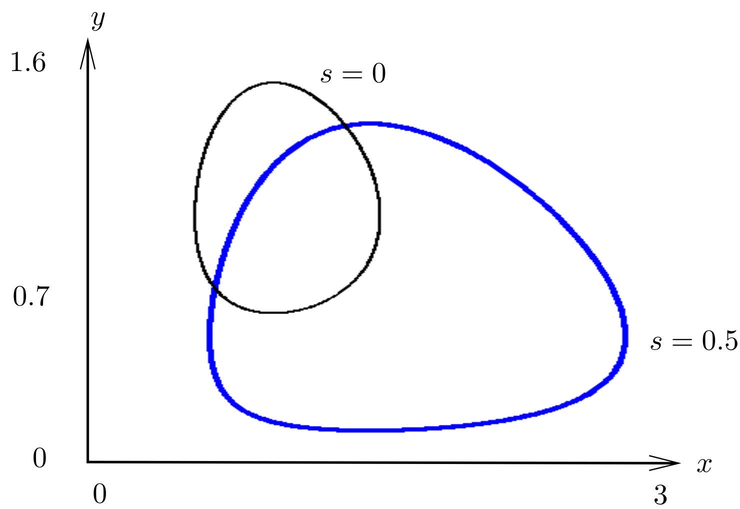


Figure 28. Aphid-Ladybug Portraits $s = 0$, $s = 0.5$.
Aphid population max and min are measured by the orbit width. Ladybug population max and min are measured by the orbit height. Both orbits use $x(0) = y(0) = 0.7$. Details appear in the x and y scene plots, *infra*.

Pesticide model (7) is equivalent to the classical predator-prey system (1) with replacements $a = 1 - s$, $b = 1 + s$. The nonlinear phase portrait for the pesticide model has according to predator-prey theory a saddle at $(0, 0)$ and a center at $(1 + s, 1 - s)$.

The scene plots in Figures 29 and 30 show that the aphids increase and the ladybugs decrease, for the two populations, $x(t)$ **aphids**, $y(t)$ **ladybugs** in pesticide system (7), with pesticide strengths $s = 0$ and $s = 0.5$ and initial populations $x(0) = 0.7$, $y(0) = 0.7$ (in millions).

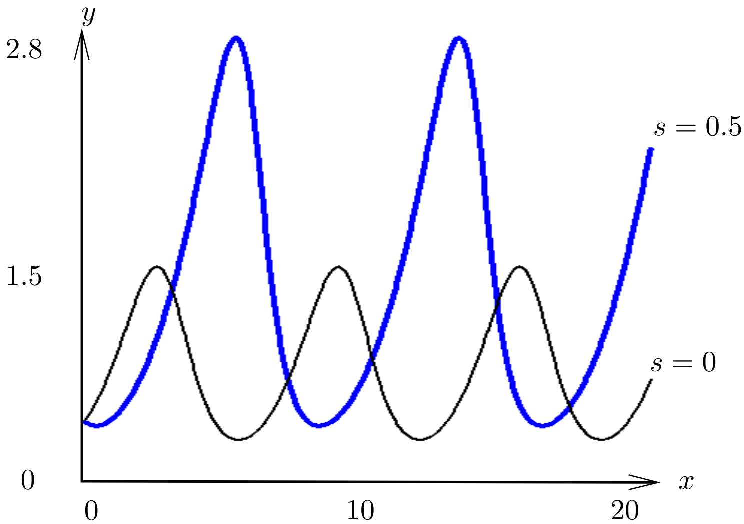


Figure 29. Aphid Scene $x(t)$.
Aphids increase when pesticide strength $s = 0.5$ is applied.

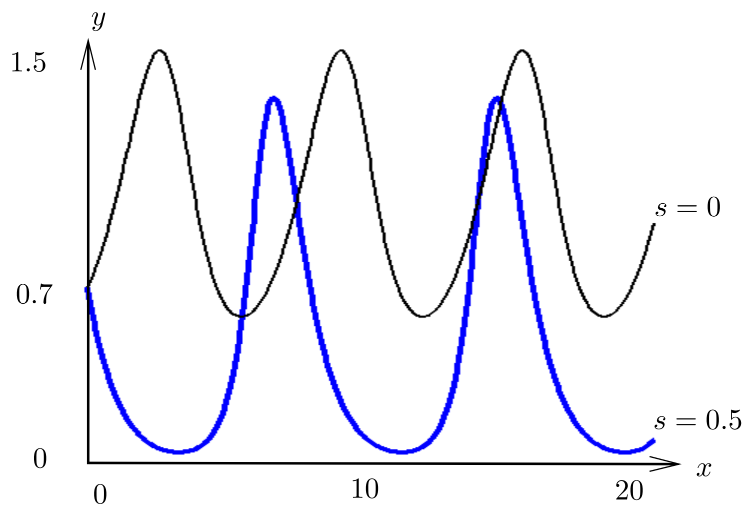


Figure 30. Ladybug Scene $y(t)$.
Ladybugs decrease when pesticide strength $s = 0.5$ is applied.

Competition Models

Two populations **1** and **2** feed on some constantly available food supply, e.g., two kinds of insects feed on fallen fruit. The following biological assumptions apply to model a two-population competition system.

Verhulst model **1**	Population **1** grows or decays according to the logistic equation $x'(t) = (a - bx(t))x(t)$, in the absence of population **2**.
Verhulst model **2**	Population **2** grows or decays according to the logistic equation $y'(t) = (c - dy(t))y(t)$, in the absence of population **1**.

Chance encounters	Population **1** decays at a rate $-pxy$, $p > 0$, due to chance encounters with population **2**. Population **2** decays at a rate $-qxy$, $q > 0$, due to chance encounters with population **1**.

Adding the Verhulst rates and the chance encounter rates gives the **Volterra competition system**

$$(8) \qquad \begin{aligned} x'(t) &= (a - bx(t) - py(t))x(t), \\ y'(t) &= (c - dy(t) - qx(t))y(t). \end{aligned}$$

The equations show that each population satisfies a time-varying first order differential equation $u'(t) = r(t)u(t)$ in which the rate function $r(t)$ depends on time. For initial population sizes near zero, the two differential equations essentially reduce to the Malthusian growth models $x'(t) = ax(t)$ and $y'(t) = cy(t)$. As viewed from Malthus' law $u' = ru$, population **1** has growth rate $r = a - bx - py$ which decreases if population **2** grows, resulting in a reduction of population **1**. Likewise, population **2** has growth rate $r = c - dy - qx$, which reduces population **2** as population **1** grows. While a, c are Malthusian growth rates, constants b, d measure **inhibition** (due to lack of food or space) and constants p, q measure **competition**.

Equilibria

The equilibrium points $\vec{u}$ satisfy $\vec{F}(\vec{u}) = \vec{0}$ where $\vec{F}$ is defined by

$$(9) \qquad \vec{F}(\vec{u}) = \begin{pmatrix} (a - bx - py)x \\ (c - dy - qx)y \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To isolate the most important applications, the assumption will be made of exactly four roots in population quadrant I. This is equivalent to the condition $bd - qp \neq 0$ plus all equilibria have nonnegative coordinates.

Three of the four equilibria are found to be $(0, 0)$, $(a/b, 0)$, $(0, c/d)$. The last two represent the carrying capacities of the Verhulst models in the absence of the second population. The fourth equilibrium (x_0, y_0) is found as the *unique root* $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ of the linear system

$$\begin{pmatrix} b & p \\ q & d \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix},$$

which according to Cramer's rule is

$$x_0 = \frac{ad - pc}{bd - qp}, \quad y_0 = \frac{bc - qa}{bd - qp}.$$

Linearized Competition System

The Jacobian matrix $J(x, y)$ is computed from the partial derivatives of system variables f, g, which are found as follows.

$$
\begin{array}{lllll}
f(x,y) & = & (a - b\,x - p\,y)\,x, & = & a\,x - b\,x^2 - p\,xy \\
g(x,y) & = & (c - d\,y - q\,x)\,y & = & c\,y - d\,y^2 - q\,xy \\
f_x & = & \frac{\partial}{\partial x}(a\,x - b\,x^2 - p\,xy) & = & a - 2b\,x - p\,y \\
f_y & = & \frac{\partial}{\partial y}(a\,x - b\,x^2 - p\,xy) & = & -p\,x \\
g_x & = & \frac{\partial}{\partial x}(c\,y - d\,y^2 - q\,xy) & = & -q\,y \\
g_y & = & \frac{\partial}{\partial y}(c\,y - d\,y^2 - q\,xy) & = & c - 2d\,y - q\,x
\end{array}
$$

The Jacobian matrix is given explicitly by

$$
(10) \qquad J(x,y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} a - 2bx - py & -px \\ -qy & c - 2dy - qx \end{pmatrix}.
$$

The matrix J is evaluated at an equilibrium point (a root of $\vec{F}(\vec{u}) = \vec{0}$) to obtain a 2×2 matrix A for the linearized system $\frac{d}{dt}\vec{v}(t) = A\,\vec{v}(t)$. The four linearized systems are:

Equilibrium $(0, 0)$
Nodal Repeller

$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \vec{u}(t)$$

Equilibrium $(a/b, 0)$
Saddle or Nodal Attractor

$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} -a & -ap/b \\ 0 & c - qa/b \end{pmatrix} \vec{u}(t)$$

Equilibrium $(0, c/d)$
Saddle or Nodal Attractor

$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} a - cp/d & 0 \\ -qc/d & -c \end{pmatrix} \vec{u}(t)$$

Equilibrium (x_0, y_0)
Saddle or Nodal Attractor

$$\frac{d}{dt}\vec{u}(t) = \begin{pmatrix} -bx_0 & -px_0 \\ -qy_0 & -dy_0 \end{pmatrix} \vec{u}(t)$$

Equilibria $(a/b, 0)$ and $(0, c/d)$ are either both saddles or both nodal attractors, accordingly as $bd - qp > 0$ or $bd - qp < 0$, because of the requirement that a, b, c, d, p, q, x_0, y_0 be positive.

The analysis of equilibrium (x_0, y_0) is made by computing the eigenvalues λ of the linearized system, from characteristic equation $\lambda^2 + (bx_0 + dy_0)\lambda + (bd - pq)x_0y_0 = 0$, giving

$$\lambda = \frac{1}{2}\left(-(bx_0 + dy_0) \pm \sqrt{D}\right), \quad \text{where} \quad D = (bx_0 - dy_0)^2 + 4pqx_0y_0.$$

Because $D > 0$, the equilibrium is a saddle when the roots have opposite sign, and it is a nodal attractor when both roots are negative. The saddle case is $D > (bx_0 + dy_0)^2$ or equivalently $4x_0y_0(pq - bd) > 0$, which reduces to $bd - qp < 0$. In summary:

If $bd - qp > 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, (x_0, y_0) are respectively a saddle, saddle, nodal attractor.

If $bd - qp < 0$, then equilibria $(a/b, 0)$, $(0, c/d)$, (x_0, y_0) are respectively a nodal attractor, nodal attractor, saddle.

Biological Meaning of $bd - qp$ Negative or Positive

The quantities bd and qp are measures of inhibition and competition.

Survival-Extinction	The inequality $bd - qp < 0$ means that competition qp is large compared with inhibition bd. The equilibrium point (x_0, y_0) is unstable in this case, which biologically means that the two species cannot coexist: **Survival** for one species and **Extinction** for the other species.
Co-existence	The inequality $bd - qp > 0$ means that competition qp is small compared with inhibition bd. The equilibrium point (x_0, y_0) is asymptotically stable in this case, which biologically means the two species **Co-exist**.

Survival of One Species

Consider populations $x(t)$ and $y(t)$ that satisfy the competition model

$$\begin{aligned} x'(t) &= x(t)(24 - x(t) - 2y(t)), \\ y'(t) &= y(t)(30 - y(t) - 2x(t)). \end{aligned} \tag{11}$$

We apply the general competition theory with $a = 24$, $b = 1$, $p = 2$, $c = 30$, $d = 1$, $q = 2$. The equilibrium points are $(0,0)$, $(0,30)$, $(24,0)$, $(12,6)$, shown in Figure 31 as solid circles and squares. Eigenvalues are computed from Jacobian matrix $J(x,y) = \begin{pmatrix} 24 - 2x - 2y & -2x \\ -2y & 30 - 2y - 2x \end{pmatrix}$ evaluated at the four equilibria. The answers:

Equilibrium $(0,0)$**:** $\lambda = 24, 30$, nodal repeller.

Equilibrium $(0,30)$**:** $\lambda = -36, -30$, nodal attractor.

Equilibrium $(24,0)$**:** $\lambda = -24, -18$, nodal attractor.

Equilibrium $(12,6)$**:** $\lambda = 8.23, -26.23$, saddle.

The **Paste Theorem** says that the linear portraits can be pasted atop the four equilibria in the nonlinear phase portrait. The tuned portrait appears in Figure 31, clipped to the population quadrant $x \geq 0, y \geq 0$.

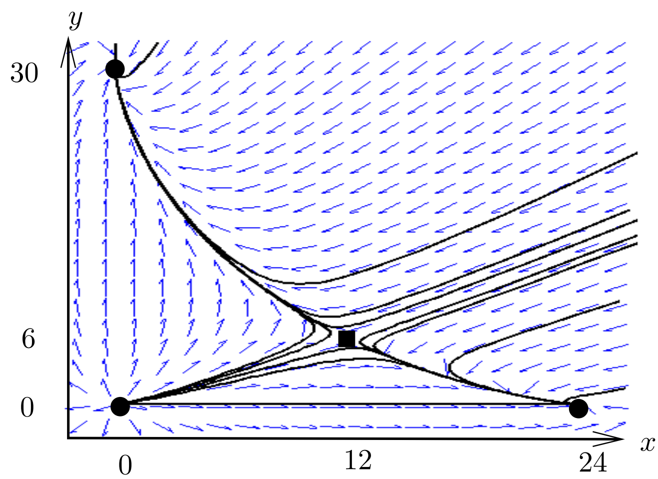


Figure 31. Survival of One Species. Portrait for system (11). Equilibria are $(0,0)$, $(0,30)$, $(24,0)$ and $(12,6)$, classified respectively as nodal repeller, nodal attractor, nodal attractor and saddle. The population with initial advantage survives, while the other dies out.

Co-existence

Consider populations $x(t)$ and $y(t)$ that satisfy the competition model

$$
\begin{aligned}
x'(t) &= x(t)(24 - 2x(t) - y(t)), \\
y'(t) &= y(t)(30 - 2y(t) - x(t)).
\end{aligned}
\tag{12}
$$

We apply the general competition theory with $a = 24$, $b = 2$, $p = 1$, $c = 30$, $d = 2$, $q = 1$. The equilibrium points are $(0,0)$, $(0,15)$, $(12,0)$ and $(6,12)$, shown in Figure 32 as solid circles and squares. Eigenvalues are computed from Jacobian matrix $J(x,y) = \begin{pmatrix} 24-4x-y & -x \\ -y & 30-4y-x \end{pmatrix}$ evaluated at the four equilibria. The answers:

Equilibrium $(0,0)$**:** $\lambda = 24, 30$, nodal repeller.

Equilibrium $(0,30)$**:** $\lambda = 18, -24$, saddle.

Equilibrium $(24,0)$**:** $\lambda = 9, -30$, saddle.

Equilibrium $(12,6)$**:** $\lambda = -7.61, -28.39$, nodal attractor.

The linear portraits can be pasted atop the four equilibria in the nonlinear phase portrait, according to the **Paste Theorem**. Figure 32 is the tuned portrait.

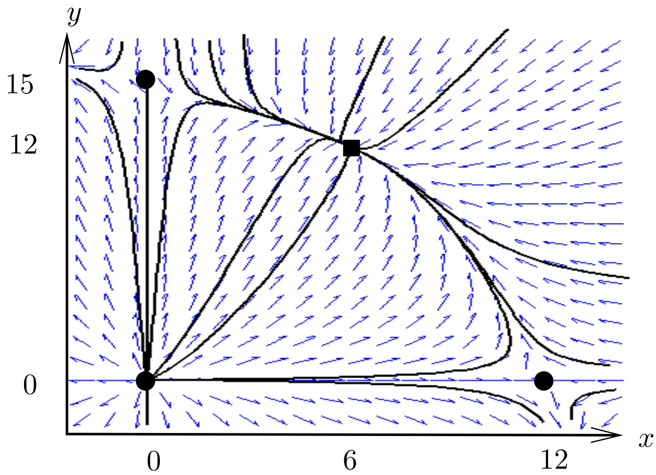


Figure 32. Coexistence.
Phase portrait of system (12). The equilibria are $(0,0)$, $(0,15)$, $(12,0)$ and $(6,12)$, classified respectively as nodal repeller, saddle, saddle, nodal attractor. A solution with $x(0) > 0$, $y(0) > 0$ limits at $t = \infty$ to the solid square $(6,12)$. **Coexistence states** are $x = 6, y = 12$.

Alligators, Explosion and Extinction

Let us assume a competition-type model (8) in which the Verhulst dynamics has explosion-extinction type. Accordingly, the signs of a, b, c, d in (8) are assumed to be negative, but p, q are still positive. The populations $x(t)$ and $y(t)$ are unsophisticated in the sense that each population in the absence of the other is subject to only the possibilities of population explosion or population extinction.

It can be verified for this general setting, although we shall not attempt to do so here, that the population quadrant $x(0) > 0$, $y(0) > 0$ is separated into two regions I and II, whose common boundary is a separatrix consisting of three equilibria and two orbits. An orbit starting in region I will have (a) $x(\infty) = 0$, $y(\infty) = \infty$, or (b) $x(\infty) = \infty$, $y(\infty) = 0$, or (c) $x(\infty) = \infty$, $y(\infty) = \infty$. Orbits starting in region II will satisfy (d) $x(\infty) = 0$, $y(\infty) = 0$. The biological conclusion is that either population explosion or extinction occurs for each population.

Consider the instance

$$\begin{aligned} x'(t) &= x(t)(x(t) - y(t) - 4), \\ y'(t) &= y(t)(x(t) + y(t) - 8). \end{aligned} \tag{13}$$

Let's apply the general competition theory with $a = 24$, $b = 2$, $p = 1$, $c = 30$, $d = 2$, $q = 1$. The equilibria are $(0,0)$, $(0,8)$, $(4,0)$ and $(6,2)$, shown in Figure 33 as solid circles and a square. Eigenvalues λ are computed from Jacobian matrix $J(x,y) = \begin{pmatrix} 2x - y - 4 & -x \\ -y & x + 2y - 8 \end{pmatrix}$ evaluated at the four equilibria. The answers below and the **Paste Theorem** predict the tuned portrait in Figure 33.

Equilibrium $(0,0)$**:** $\lambda = -4, -8$, nodal attractor.
Equilibrium $(0,30)$**:** $\lambda = 8, -12$, saddle.
Equilibrium $(24,0)$**:** $\lambda = 4, -4$, saddle.
Equilibrium $(12,6)$**:** $\lambda = 4 \pm 2.83\,i$, spiral repeller.

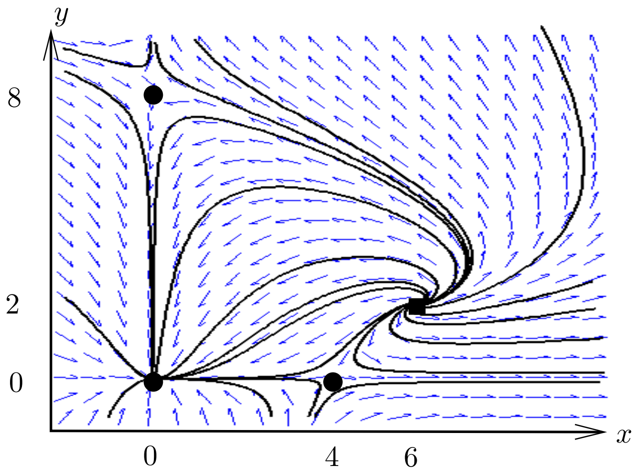


Figure 33. Population Explosion or Extinction.
Phase portrait of system (13). The equilibria are $(0,0)$, $(0,8)$, $(4,0)$ and $(6,2)$, classified respectively as nodal attractor, saddle, saddle and spiral repeller. The node and two saddles are marked with a solid disk and the spiral repeller is marked with a solid square.

Exercises 10.4

Predator-Prey Models.

Consider the system

$$\begin{aligned} x'(t) &= \frac{1}{250}(1-2y(t))x(t), \\ y'(t) &= \frac{3}{500}(2x(t)-1)y(t). \end{aligned}$$

1. (System Variables) The system has vector-matrix form

$$\frac{d}{dt}\vec{u} = \vec{F}(\vec{u}(t)).$$

Display formulas for $\vec{u}$ and $\vec{F}$.

2. (System Parameters) Identify the values of a, b, c, d, p, q, as used in the textbook's predator-prey system.

3. (Identify Predator and Prey) Which of $x(t), y(t)$ is the predator?

4. (Switching Predator and Prey) Give an example of a predator-prey system in which $x(t)$ is the predator and $y(t)$ is the prey.

Implicit Solution Predator-Prey.

These exercises prove equation

$$a\,\ln|y| + b\,\ln|x| - q\,x - p\,y = C$$

for predator-prey system

$$\begin{aligned} x'(t) &= (a - p\,y(t))x(t), \\ y'(t) &= (q\,x(t) - b)y(t). \end{aligned}$$

5. (First Order Equation) Verify from the chain rule of calculus the first order equation

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{y}{x}\frac{qx-b}{a-py}.$$

6. (Separated Variables) Verify

$$\left(\frac{a}{y} - p\right)dy = \left(q - \frac{b}{x}\right)dx.$$

7. (Quadrature) Integrate the equation of Exercise 6 to obtain

$$a\,\ln|y| - p\,y = q\,x - b\,\ln|x| = C.$$

Then re-arrange to obtain the reported implicit solution.

8. (Energy Function) Define $E(t) = a\ln|u| - pu$. Show that $dE/du = (a - pu)/u$. Then show that $dE/du < 0$ for $a > 0$, $p > 0$ and $a/p < u < \infty$.

Linearized Predator-Prey System.

Consider

$$\begin{aligned} x'(t) &= (100 - 2y(t))x(t), \\ y'(t) &= (2x(t) - 160)y(t). \end{aligned}$$

9. (Find Equilibria) Verify equilibria $(0,0), (80,50)$.

10. (Jacobian Matrix) Compute $J(x,y)$ for each x, y. Then find $J(0,0)$ and $J(80,50)$.

11. **(Transit Time)** Find the transit time of an orbit for one loop about $(0,0)$ for system $\frac{d}{dt}\vec{v} = \begin{pmatrix} 0 & -160 \\ 100 & 0 \end{pmatrix}\vec{v}$, the linearization about $(80,50)$.

12. **(Paste Theorem)** Describe the local figures expected near equilibria in the nonlinear phase portrait.

Rabbits and Foxes. Consider

$$\begin{aligned} x'(t) &= \frac{1}{200}\,x(t)(50-y(t)), \\ y'(t) &= \frac{1}{100}\,y(t)(x(t)-40). \end{aligned}$$

13. **(Equilibria)** Verify equilibria $(0,0)$, $(40,50)$, showing all details.

14. **(Jacobian)** Compute Jacobian $J(x,y)$, then $J(0,0)$ and $J(40,50)$.

15. **(Rabbit Oscillation)** Find a graphical estimate for the period of oscillation of the rabbit population $x(t)$ for the nonlinear system, given $x(0)=100$, $y(0)=60$ and t is in weeks. Answer: about 23 weeks.

16. **(Rabbit-Gerbil Competing Species)** Consider system

$$\begin{aligned} x' &= \left(\tfrac{5}{4} - \tfrac{x}{160} - \tfrac{3y}{1000}\right)x, \\ y' &= \left(3 - \tfrac{3y}{500} - \tfrac{3x}{160}\right)y. \end{aligned}$$

Verify equilibria $(0,0)$, $(0,500)$, $(200,0)$, $(80,250)$. Show the first three are nodes and the last is a saddle.

Pesticides. Consider the system

$$\begin{aligned} x'(t) &= (10-y(t))x(t) - s_1x(t), \\ y'(t) &= (x(t)-20)y(t) - s_2y(t). \end{aligned}$$

17. **(Average Populations)** Explain: A field biologist should count, on the average, populations of about $20+s_2$ prey and $10-s_1$ predators.

18. **(Equilibria)** Show details for computing the pesticide system equilibria $(0,0)$, $(20+s_2, 10-s_1)$, where s_1, s_2 are the pesticide death rates.

Survival of One Species. Consider

$$\begin{aligned} x'(t) &= x(t)(24-x(t)-2y(t)), \\ y'(t) &= y(t)(30-y(t)-2x(t)). \end{aligned}$$

19. **(Equilibria)** Find all equilibria.

20. **(Interactions)** Show that doubling either x or y causes the interaction term $2xy$ to double.

21. **(Nonlinear Classification)** Classify each equilibrium point (x_0,y_0) as center, spiral, node, saddle, using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

22. **(Extinction and Competing Species)** Equilibria for which either $x=0$ or $y=0$ signal extinction states. Discuss how the phase portrait of the nonlinear system shows extinction of one species but not both.

Co-existence

Find the equilibria, then classify them as node, saddle, spiral, center using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

23. **(Node, Saddle, Saddle, Node)**

$$\begin{aligned} x' &= (144-2x-3y)x, \\ y' &= (90-6y-x)y. \end{aligned}$$

24. **(Node, Saddle, Saddle, Node)**

$$\begin{aligned} x' &= (120-4x-2y)x, \\ y' &= (60-x-2y)y. \end{aligned}$$

Explosion and Extinction

Find the equilibria, then classify them as node, saddle, spiral, center using the **Paste Theorem**. Determine stability for node and spiral. Make a computer phase portrait to confirm the classifications.

25. **(Node, Saddle, Saddle, Spiral)**

$$\begin{aligned} x' &= x(x-2y-4), \\ y' &= y(x+2y-8). \end{aligned}$$

26. **(Node, Saddle, Saddle, Spiral)**

$$\begin{aligned} x' &= x(x-y-4), \\ y' &= y(x+y-6). \end{aligned}$$

10.5 Mechanical Models

Nonlinear Spring-Mass System

The classical linear undamped spring-mass system is modeled by the equation $mx''(t) + kx(t) = 0$. This equation describes the excursion $x(t)$ from equilibrium $x = 0$ of a mass m attached to a spring of Hooke's constant k, with no damping and no external forces.

In the nonlinear theory, the Hooke's force term $-kx$ is replaced by a **Restoring Force** $F(x)$ which satisfies these four requirements:

Equilibrium 0**.** The equation $F(0) = 0$ is assumed, which gives $x = 0$ the status of a rest position.

Oddness. The equation $F(-x) = -F(x)$ is assumed, which says that the force F depends only upon the magnitude of the excursion from equilibrium, and not upon its direction. Then force F acts to **restore** the mass to its equilibrium position, like a Hooke's force $x \to kx$.

Zero damping. The damping effects always present in a real physical system are ignored. In linear approximations, it would be usual to assume a viscous damping effect $-cx'(t)$; from this viewpoint we assume $c = 0$.

Zero external force. There is no external force acting on the system. In short, only two forces act on the mass, (1) Newton's second law and (2) restoring force F.

The competition method applies to model the nonlinear spring-mass system via the two competing forces $mx''(t)$ and $F(x(t))$. The dynamical equation:

$$mx''(t) + F(x(t)) = 0. \tag{1}$$

Soft and Hard Springs

A restoring force F modeled upon Hooke's law is given by the equation $F(x) = kx$. With this force, the nonlinear spring-mass equation (1) becomes the undamped linear spring-mass system

$$mx''(t) + kx(t) = 0. \tag{2}$$

The linear equation can be thought to originate by replacing the actual spring force F by the first nonzero term of its Taylor series

$$F(x) = F(0) + F'(0)x + F''(0)\frac{x^2}{2!} + \cdots.$$

The assumptions $F(-x) = -F(x)$ and $F(0) = 0$ imply that $F(x)$ is a function of the form $F(x) = xG(x^2)$, hence all even terms in the Taylor series of F are zero.

Linear approximations to the force F drop the quadratic terms and higher from the Taylor series. More accurate nonlinear approximations are obtained by retaining extra Taylor series terms.

A restoring force F is called **hard** or **soft** provided it is given by a truncated Taylor series as follows.

Hard spring	$F(x) = kx + \beta x^3$, $\beta > 0$.
Soft spring	$F(x) = kx - \beta x^3$, $\beta > 0$.

For small excursions from equilibrium $x = 0$, a hard or soft spring force has magnitude approximately the same as the linear Hooke's force $F(x) = kx$.

Energy Conservation

Given nonlinear spring-mass equation $mx''(t) + F(x(t)) = 0$, each solution $x(t)$ satisfies on its domain of existence the **Conservation Law**

$$(3) \qquad \frac{m}{2}(x'(t))^2 + \int_{x(0)}^{x(t)} F(u)\,du = C, \quad C \equiv \frac{m}{2}(x'(0))^2.$$

To prove the law, multiply the nonlinear differential equation by $x'(t)$ to obtain $mx''(t)x'(t) + F(x(t))x'(t) = 0$, then apply quadrature to obtain (3).

Kinetic and Potential Energy

Using $v = x'(t)$, the term $mv^2/2$ in (3) is called the **Kinetic energy** (KE) and the term $\int_{x_0}^{x} F(u)du$ is called the **Potential energy** (PE). Equation (3) says that $KE + PE = C$ or that *energy is constant* along trajectories.

The conservation laws for the soft and hard nonlinear spring-mass systems, using position-velocity notation $x = x(t)$ and $y = x'(t)$, are therefore given by the equations

$$(4) \qquad my^2 + kx^2 + \frac{1}{2}\beta x^4 = C_1, \quad C_1 = \text{constant} > 0,$$

$$(5) \qquad my^2 + kx^2 - \frac{1}{2}\beta x^4 = C_2, \quad C_2 = \text{constant}.$$

Phase Plane and Scenes

Nonlinear behavior is commonly graphed in the **phase plane**, in which $x = x(t)$ and $y = x'(t)$ are the position and velocity of the mechanical system. The plots of t versus $x(t)$ or $x'(t)$ are called **Scenes**; these plots are invaluable for verifying periodic behavior and stability properties.

Hard spring

The only equilibrium for a hard spring $x' = y$, $my' = -kx - \beta x^3$ is the origin $x = y = 0$. Conservation law (4) describes a closed curve in the phase plane, which implies that trajectories are periodic orbits that encircle the equilibrium point $(0, 0)$. The classification of **center** applies. See Figures 34 and 35.

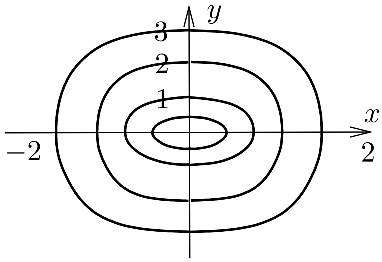


Figure 34. Hard spring $x''(t) + x(t) + 2x^3(t) = 0$**.**
Phase portrait for $x' = y$, $y' = -2x^3 - x$ on $|x| \le 2$, $|y| \le 3.5$. Initial data: $x(0) = 0$ and $y(0) = 1/2, 1, 2, 3$.

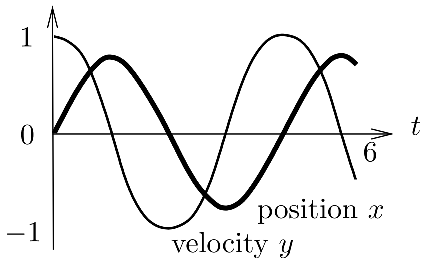


Figure 35. Hard spring $x''(t) + x(t) + 2x^3(t) = 0$**.**
Coordinate scenes for $x' = y$, $y' = -2x^3 - x$, $x(0) = 0$, $y(0) = 1$.

More intuition about the orbits can be obtained by finding the energy C_1 for each orbit. The value of C_1 decreases to zero as orbits close down upon the origin. Otherwise stated, the xyz-plot with $z = C_1$ has a minimum at the origin, which physically means that the equilibrium state $x = y = 0$ minimizes the energy. See Figure 36.

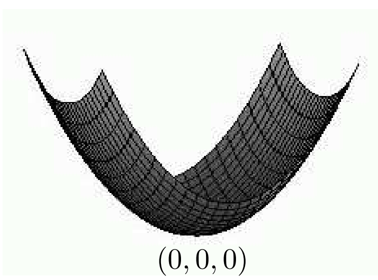


Figure 36. Hard spring energy minimization.
Plot for $x''(t) + x(t) + 2x^3(t) = 0$, using $z = y^2 + x^2 + x^4$ on $|x| \le 1/2$, $|y| \le 1$. The minimum is realized at $x = y = 0$.

Soft Spring

There are three equilibria for a soft spring

$$\begin{aligned} x' &= y, \\ my' &= -kx + \beta x^3. \end{aligned}$$

They are $(-\alpha, 0)$, $(0, 0)$, $(\alpha, 0)$, where $\alpha = \sqrt{k/\beta}$. If $(x(0), y(0))$ is given not at these points, then the mass undergoes motion. In short, the stationary mass positions are at the equilibria.

Linearization at the equilibria reveals part of the phase portrait. The linearized system at the origin is the system $x' = y$, $my' = -kx$, equivalent to the equation $mx'' + kx = 0$. It has a center at the origin. This implies the origin for the soft spring is either a center or a spiral. The other two equilibria have linearized systems equivalent to the equation $mx'' - 2kx = 0$; they are saddles.

The phase plot in Figure 37 shows separatrices, which are unions of solution curves and equilibrium points. Orbits in the phase plane, on either side of a separatrix, have physically different behavior. Shown is a center behavior interior to the union of the separatrices, while outside all orbits are unbounded.

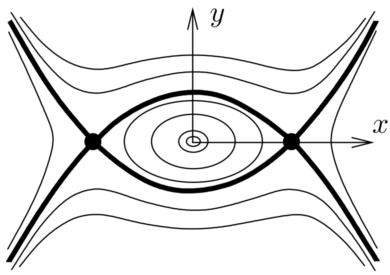


Figure 37. Soft spring $x''(t) + x(t) - 2x^3(t) = 0$.
A phase portrait for $x' = y$, $y' = 2x^3 - x$ on $|x| \le 1.2$, $|y| \le 1.2$. The 8 separatrices are the 6 bold curves plus the two equilibria $(\sqrt{0.5}, 0)$, $(-\sqrt{0.5}, 0)$.

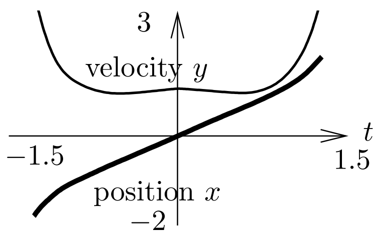


Figure 38. Soft spring $x''(t)+x(t)-2x^3(t) = 0$.
Coordinate scenes for $x' = y$, $y' = 2x^3 - x$, $x(0) = 0$, $y(0) = 4$.

Nonlinear Pendulum

Consider a nonlinear undamped pendulum of length L making angle $\theta(t)$ with the gravity vector. The **nonlinear pendulum equation** is given by

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) = 0 \tag{6}$$

and its linearization at $\theta = 0$, called the **linearized pendulum equation**, is

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\theta(t) = 0. \tag{7}$$

The linearized equation is valid only for small values of $\theta(t)$, because of the assumption $\sin\theta \approx \theta$ used to obtain (7) from (6).

Damped Pendulum

Physical pendulums are subject to friction forces, which we shall assume proportional to the velocity of the pendulum. The corresponding model which includes frictional forces is called the **damped pendulum equation**:

$$\frac{d^2\theta(t)}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{L}\sin(\theta(t)) = 0. \tag{8}$$

It can be written as a first order system by setting $x(t) = \theta(t)$ and $y(t) = \theta'(t)$:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\sin(x(t)) - cy(t). \end{aligned} \tag{9}$$

Undamped Pendulum

The position-velocity differential equations for the undamped pendulum are obtained by setting $x(t) = \theta(t)$ and $y(t) = \theta'(t)$:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\sin(x(t)). \end{aligned} \tag{10}$$

Equilibrium points of nonlinear system (10) are at $y = 0$, $x = n\pi$, $n = 0$, ± 1, ± 2, ... with corresponding linearized system (see the exercises)

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= -\frac{g}{L}\cos(n\pi)x(t). \end{aligned} \tag{11}$$

The characteristic equation of linear system (11) is $r^2 - \frac{g}{L}(-1)^n = 0$, because $\cos(n\pi) = (-1)^n$. The roots have different character depending on whether or not n is odd or even.

Even $n = 2m$. Then $r^2 + g/L = 0$ and the linearized system (11) is a **Center**. The orbits of (11) are concentric circles surrounding $x = n\pi$, $y = 0$.

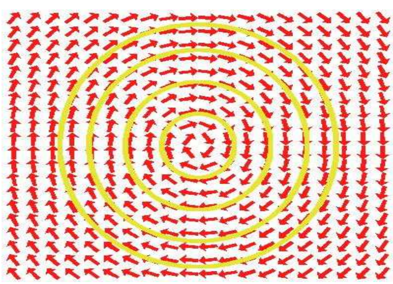


Figure 39. Linearized pendulum at equilibrium $x = 2m\pi$, $y = 0$.
Orbits are concentric circles.

Odd $n = 2m + 1$. Then $r^2 - g/L = 0$ and the linearized system (11) is classified as a **Saddle**. The orbits of (11) are hyperbolas with center $x = n\pi$, $y = 0$.

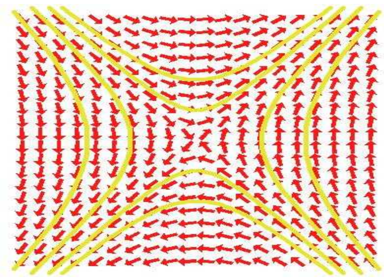


Figure 40. Linearized pendulum at $x = (2m+1)\pi$, $y = 0$.
Orbits are hyperbolas.

Drawing the Nonlinear Phase Diagram. The idea of the plot is to paste the linearized phase diagram onto the local region centered at the equilibrium point, when possible. The copying is guaranteed to be correct for the saddle case, but a center must be copied either as a spiral or a center. Extra analysis is needed to determine the figure to copy in the case of the center. The result appears in Figure 41.

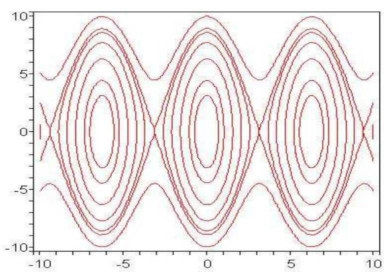


Figure 41. Nonlinear Pendulum.
Centers at $(-2\pi, 0)$, $(0, 0)$, $(2\pi, 0)$. Saddles at $(-3\pi, 0)$, $(-pi, 0)$, $(\pi, 0)$, $(3\pi, 0)$. Separatrices are unions of equilibria and conservation law curves $y^2 + \frac{4g}{L}\sin^2(x/2) = 2E$, with $E = 2\frac{g}{L}$ and $\frac{g}{L} = 10$.

We document the analysis used to produce Figure 41. The orbits trace an xy-curve given by integrating the separable equation

$$\frac{dy}{dx} = \frac{-g}{L}\frac{\sin x}{y}.$$

Then the conservation law for the mechanical system is

$$\frac{1}{2}y^2 + \frac{g}{L}(1 - \cos x) = E$$

where E is a constant of integration. This equation is arranged so that E is the sum of the kinetic energy $y^2/2$ and the potential energy $g(1 - \cos x)/L$, therefore E is the total mechanical energy. Using the double angle identity $\cos 2\phi = 1 - 2\sin^2\phi$ the conservation law can be written in the shorter form

$$y^2 + \frac{4g}{L}\sin^2(x/2) = 2E$$

When the energy E is small, $E < 2g/L$, then the pendulum never reaches the vertical position and it undergoes sustained periodic oscillation: the stable equilibria $(0, 2k\pi)$ have a local center structure.

When the energy E is large, $E > 2g/L$, then the pendulum reaches the vertical position and goes over the top repeatedly, represented by a saddle structure. The statement is verified from the two explicit solutions $y = \pm\sqrt{2E - 4g\sin^2(x/2)/L}$.

The energy equation $y^2 + \frac{4g}{L}\sin^2(x/2) = 4\frac{g}{L}$ (equivalent to $E = 2g/L$) produces the separatrix curves. **Separatrices** consist of equilibrium points plus solution curves which limit to the equilibria as $t \to \pm\infty$.

Exercises 10.5

Linear Mechanical Models

Consider the unforced linear model $mx'' + cx' + kx = 0$, where m, c, k are positive constants: m=mass, c=dashpot constant, k=Hooke's constant.

1. **(Dynamical System Form)** Write the scalar problem as $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$. Explicit definitions of $\vec{\mathbf{u}}\,(t)$ and A are expected.

2. **(Attractor to $\vec{\mathbf{u}} = \vec{\mathbf{0}}$)** Explain why $\lim_{t\to\infty} \vec{\mathbf{u}}\,(t) = \vec{\mathbf{0}}$, giving citations to theorems in this book.

3. **(Isolated Equilibrium)** Prove that $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$ has a unique equilibrium at $\vec{\mathbf{u}} = \vec{\mathbf{0}}$. Then explain why the equilibrium is isolated.

4. **(Phase Plots)** Classify the cases of **over-damped** and **under-damped** as a stable node or a stable spiral for $\vec{\mathbf{u}}\,' = A\vec{\mathbf{u}}$ at equilibrium $\vec{\mathbf{u}} = \vec{\mathbf{0}}$. Why are classifications *center* and *saddle* impossible?

Nonlinear Spring-Mass System

Consider the general model $x'' + F(x) = 0$ with the assumptions on page 804.

5. **(Harmonic Oscillator)** Let $F(x) = \omega^2 x$ with $\omega > 0$. Show F is odd and $F(0) = 0$. Then find the general solution $x(t)$ for $x'' + F(x) = 0$.

6. **(Taylor Series)** Show that an odd function $F(x)$ with Maclaurin series $\sum_{n=0}^{\infty} a_n x^n$ has all even order terms zero, that is, $a_n = 0$ for n even.

Soft and Hard Springs

Classify as a hard or soft spring. Then write the conservation law for the equation.

7. $x'' + x + x^3 = 0$

8. $x'' + x - x^3 = 0$

Hard spring

9. Prove that a hard spring has exactly one equilibrium $x = y = 0$.

10. Substitute $x = x(t), y = x'(t)$ into $z = y^2 + x^2 + x^4$ to obtain $z(t)$. Function $z(t)$ has a minimum when $\frac{dz}{dt} = 0$. Reduce this equation to $x'' + x + 2x^3 = 0$.

Soft Spring

Consider soft spring $x'' + kx - \beta x^3 = 0$, $k > 0$, $\beta > 0$.

11. **(Equilibria)** Verify the three equilibria $(0,0), (0, \sqrt{k}\beta), (0, -\sqrt{k}\beta)$.

12. **(Saddles)** Verify by linearization and the **Paste Theorem** that nonlinear equilibria $(0, \sqrt{k}\beta), (0, -\sqrt{k}\beta)$ are saddles.

13. **(Center or Spiral)** The **Paste Theorem** says that equilibrium $(0,0)$ of the nonlinear system is a center or spiral. Verify by computer phase portrait $m = k = 1$ and $\beta = 2$ Figure 37, page 807.

14. **(Mass at Rest)** Verify that the only solutions with the mass at rest are the equilibria. **Mass at rest** means velocity zero: $\vec{\mathbf{u}}\,'(t_0) = \vec{\mathbf{0}}$ for some t_0, vector notation from Exercise 1.

15. **(Phase Portrait)** Solve for the equilibria of $x'' + 4x - x^3 = 0$. Draw a phase portrait similar to Figure 37, page 807.

16. **(Separatrix)** The energy equation for $x'' + 4x - x^3 = 0$ is $\frac{1}{2}y^2 + 2x^2 - \frac{1}{4}x^4 = E$. Substitute the saddle equilibria to find $E = 4$. Plot implicitly the energy equation curve. A separatrix is the union of the two saddle equilibria and this implicit curve.

Damped Nonlinear Pendulum

Consider $\frac{d^2\theta(t)}{dt^2} + c\frac{d\theta}{dt} + \frac{g}{L}\sin(\theta(t)) = 0$, which has vector-matrix form $\vec{\mathbf{u}}' = \vec{\mathbf{G}}(\vec{\mathbf{u}}(t))$.

17. Display both $\vec{\mathbf{u}}$ and $\vec{\mathbf{G}}$.

18. Find the Jacobian matrix of $\vec{\mathbf{G}}$ with respect to $\vec{\mathbf{u}}$.

Undamped Nonlinear Pendulum

Consider $\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) = 0$, having vector-matrix form $\vec{\mathbf{u}}' = \vec{\mathbf{F}}(\vec{\mathbf{u}}(t))$.

19. Find the Jacobian matrix of $\vec{\mathbf{F}}$ with respect to $\vec{\mathbf{u}}$.

20. Solve $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \vec{\mathbf{0}}$ for $\vec{\mathbf{u}}$, showing all details.

21. Evaluate the Jacobian matrix at the roots of $\vec{\mathbf{F}}(\vec{\mathbf{u}}) = \vec{\mathbf{0}}$.

22. Plot $y^2 + \frac{4g}{L}\sin^2(x/2) = 4\frac{g}{L}$ implicitly for $\frac{g}{L} = 10$. The separatrix is this curve plus equilibria.

PDF Sources

Text, Solutions and Corrections

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Paperback Textbook: There are 12 chapters on differential equations and linear algebra, book format 7 x 10 inches, 1077 pages. Copies of the textbook are available in two volumes at **Amazon** Kindle Direct Publishing for Amazon's cost of printing and shipping. No author profit. Volume I chapters 1-7, ISBN 9798705491124, 661 pages. Volume II chapters 8-12, ISBN 9798711123651, 479 pages. Both paperbacks have extra pages of backmatter: background topics Chapter A, the whole book index and the bibliography.

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