Chapter XI: Sturm-Liouville Theory

This chapter studies a very classical problem in the theory of ordinary differential equations, namely linear second order differential equations which are parameter dependent and are subject to boundary conditions. While the existence of **eigenvalues** (parameter values for which nontrivial solutions exist) and **eigenfunctions** (corresponding nontrivial solutions) follows easily from the abstract **Riesz spectral theory** for compact linear operators, it is instructive to deduce the same conclusions using some of the results we have developed up to now for ordinary differential equations. The theory presented is for some rather specific cases, however, more general problems and various other cases may be considered and similar theorems may be established.

References: Coddington-Levinson, Cole and Hartman.

Linear Boundary Value Problems

Sturm-Liouville Problems.
Let $I = [a, b]$ be a compact interval and let $p, q, r \in C(I, \mathbf{R})$, with p, r positive on I. Let λ be a complex parameter. Consider the linear differential equation and boundary conditions

$$\begin{aligned}
&(p(t)x')' + (\lambda r(t) + q(t))x = 0, \quad t \in I, \\
&x(a)\cos\alpha - p(a)x'(a)\sin\alpha = 0, \\
&x(b)\cos\beta - p(b)x'(b)\sin\beta = 0.
\end{aligned}$$

The constants α and β are given and without loss in generality, $0 \le \alpha < \pi$, $0 < \beta \le \pi$.

Values λ (**eigenvalues**) are sought for which there is a nontrivial solution (**an eigenfunction**). The **spectrum** is the set of all eigenvalues. A pair (x, λ) with $x \neq 0$ satisfying the differential equation and the boundary conditions is called an **eigenpair**.

Lemma. Every eigenvalue of a Sturm-Liouville problem is real.

Prüfer transformation

Definition. Let $u(t) = u(t, \lambda)$ be the solution of $(p(t)u')' + (\lambda r(t) + q(t))u = 0$ which satisfies $u(a) = \sin\alpha$, $p(a)u'(a) = \cos\alpha$. Then $u \neq 0$ satisfies $x(a)\cos\alpha - p(a)x'(a)\sin\alpha = 0$. Let

$$\rho = \sqrt{u^2 + p^2(u')^2}, \qquad \phi = \arctan\frac{u}{pu'}.$$

Then $\phi(a) = \alpha$ and ρ, ϕ are solutions of the **Prüfer equations**

$$\begin{aligned} \rho' &= -\left[(\lambda r + q) - \tfrac{1}{p}\right]\rho\sin\phi\cos\phi, \\ \phi' &= \tfrac{1}{p}\cos^2\phi + (\lambda r + q)\sin^2\phi. \end{aligned}$$

The second differential equation depends only upon ϕ; once ϕ is known, ρ may be determined by integrating a linear equation, which in turn determines u.

Lemma. Let ϕ be the solution of Prüfer's equation such that $\phi(a) = \alpha$. Then ϕ satisfies the following conditions:

1. $\phi(b, \lambda)$ is a continuous strictly increasing function of λ;
2. $\lim_{\lambda\to\infty} \phi(b, \lambda) = \infty$;
3. $\lim_{\lambda\to-\infty} \phi(b, \lambda) = 0$.

Theorem. A Sturm–Liouville boundary value problem has an unbounded infinite increasing sequence of eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$ with $\lim_{n\to\infty} \lambda_n = \infty$. The eigenspace associated with each eigenvalue is one dimensional and the eigenfunctions associated with λ_k have precisely k simple zeros in (a, b).

Lemma. Let (u_j, λ_j) and (u_k, λ_k) be distinct eigenpairs of a Sturm–Liouville boundary value problem. Then u_j and u_k are orthogonal with respect to the weight function r, i.e.,

$$\langle u_j, u_k \rangle = \int_a^b u_j(t) u_k(t) r(t) dt = 0.$$

Nonhomogeneous Sturm–Liouville Problems

Denote by $\{(u_i, \lambda_i)\}_{i=0}^{\infty}$ the set of eigenpairs of a Sturm–Liouville problem, with the eigenfunctions normalized so that $\int_a^b r u_i^2 = 1$. Let $h \in L^2(a,b)$ be a given function.

A nonhomogeneous Sturm–Liouville boundary value problem uses the same boundary conditions but replaces the differential equation by

$$\Big(p(t)x'\Big)' + (\lambda r(t) + q(t))x = rh.$$

Solutions are interpreted in the Carathéodory sense.

Lemma. For $\lambda = \lambda_k$, a nonhomogeneous Sturm–Liouville problem has a solution u if and only if $\int_a^b r u_k h = 0$. If w is another solution, then $u - w = c u_k$ for some constant c.

Completeness of the Eigenfunctions

Lemma. A Sturm–Liouville problem has a solution for every λ_k, $k = 0, 1, 2, \cdots$, if and only if $\int_a^b r u_k h = 0$ for $k = 0, 1, 2, \cdots$.

Lemma. The set $\{u_i\}_{i=0}^{\infty}$ forms an orthonormal system for the Hilbert space $L_r^2(a, b)$ of all $f \in L^2(a, b)$ with inner product $\langle f, g \rangle = \int_a^b f(t)g(t)r(t)dt$.

Completeness Criterion

The set $\{u_i\}_{i=0}^{\infty}$ will be a **complete orthonormal system** provided it is shown that (see Rudin's text)

$$\int_a^b u_k h = 0 \text{ for } k = 0, 1, 2, \cdots \text{ implies } h = 0.$$

Lemma. If $\lambda \neq \lambda_k,\ k = 0, 1, \cdots$, then the Sturm–Liouville problem has a solution for every $h \in L^2(a, b)$.

Proof: For $\lambda \neq \lambda_k,\ k = 0, 1, \cdots$ we let u be a nontrivial solution which satisfies the first boundary condition and let v be a nontrivial solution of which satisfies the second boundary condition. Then the Wronskian of u and v is $uv' - u'v = \frac{c}{p}$ for some nonzero constant c. Define the **Green's function**

$$G(t, s) = \frac{1}{c} \begin{cases} v(t)u(s), & a \leq s \leq t \\ v(s)u(t), & t \leq s \leq b. \end{cases}$$

Then $w(t) = \int_a^b G(t, s)r(s)h(s)ds$ is the unique solution of the Sturm–Liouville boundary value problem.

Corollary. The Green's function G defines a continuous mapping $G : L^2(a, b) \to C^1[a, b]$, called **Green's operator**, by the formula $h \mapsto G(h) = w$. Further, $\langle Gh, w \rangle = \langle h, Gw \rangle$.

Definition (weakly closed). A set $S \subset L^2(a,b)$ is called **weakly closed** if $\{x_n\} \subset S$ and $\langle x_n, h\rangle \to \langle x, h\rangle$ for all $h \in L^2(a,b)$ implies $x \in S$.

Definition (the set S). Define

$$S = \{w \in L^2(a,b) : \langle u_i, h\rangle = 0,\ i = 0, 1, 2, \cdots\}.$$

Lemma. The set S is a weakly closed linear manifold in $L^2(a,b)$ and the Green's operator G maps S into S.

Lemma. If S contains a nonzero element, then there exists $x \in S$ such that $\langle G(x), x\rangle \neq 0$.

Proof: If $\langle G(x), x\rangle = 0$ for all $x \in S$, then linearity implies that for all $x, y \in S$ and $\alpha \in \mathbf{R}$

$$\begin{aligned} 0 &= \langle G(x+\alpha y), x+\alpha y\rangle \\ &= 2\alpha\langle G(y), x\rangle. \end{aligned}$$

Choose $x = G(y)$ to obtain a contradiction, since $y \neq 0$ implies $G(y) \neq 0$.

Lemma. If S contains a nonzero element, then there exists a nonzero $x \in S$ and $\mu \neq 0$ such that $G(x) = \mu x$.

Proof: Let $x \in S$ such that $\langle G(x), x\rangle \neq 0$. If $\langle G(u), u\rangle \leq 0$ for all $u \in S$, then define $\mu = \inf\{\langle G(u), u\rangle : u \in S,\ \|u\| = 1\}$, otherwise define $\mu = \sup\{\langle G(u), u\rangle : u \in S,\ \|u\| = 1\}$. There exists an $x_0 \in S$ with $\|x_0\| = 1$ such that $\langle G(x_0), x_0\rangle = \mu \neq 0$. If S is one dimensional, then $G(x_0) = \mu x_0$. If S has dimension greater than 1, then there exists a nonzero $y \in S$ such that $\langle y, x_0\rangle = 0$. Let $z = \frac{x_0+\epsilon y}{\sqrt{1+\epsilon^2}}$, then $\langle G(z), z\rangle$ has an extremum at $\epsilon = 0$, which implies $\langle G(x_0), y\rangle = 0$ for any $y \in S$ with $\langle y, x_0\rangle = 0$. Hence $\langle G(x_0) - \mu x_0, x_0\rangle = 0$ implies that $\langle G(x_0), G(x_0) - \mu x_0\rangle = 0$ and thus $\langle G(x_0) - \mu x_0, G(x_0) - \mu x_0\rangle = 0$. This proves that μ is an eigenvalue.

Theorem (Completeness). The set of eigenfunctions $\{u_i\}_{i=0}^{\infty}$ forms a complete orthonormal system in the Hilbert space $L_r^2(a, b)$.

Proof: It will be shown that $S = \{0\}$. If this is not the case, a previous lemma implies there exists a nonzero element $h \in S$ and a nonzero number μ such that $G(h) = \mu h$. Properties of the Green's operator imply that $w = G(h)$ satisfies the boundary conditions and solves the equation

$$\Big(p(t)h'\Big)' + (\lambda r(t) + q(t))h = \frac{r}{\mu}h.$$

Therefore, $\lambda - \frac{1}{\mu} = \lambda_j$ for some j. Hence $h = cu_j$ for some nonzero constant c, contradicting $h \in S$.

Exercises

1. Find all eigenpairs for

$$\begin{aligned} &x'' + \lambda x = 0 \\ &x(0) = 0 = x'(0). \end{aligned}$$

2. Supply the details for the proof of Lemma 2.

3. Prove Lemma 4.

4. Prove that the Green's function given by (10) is continuous on the square $[a,b]^2$ and that $\frac{\partial G(t,s)}{\partial t}$ is continuous for $t \neq s$. Discuss the behavior of this derivative as $t \to s$.

5. Provide the details of the proof of Corollary 7. Also prove that $G : L^2(a,b) \to L^2(a,b)$ is a compact mapping.

6. Let $G(t,s)$ be defined by equation (10). Show that

$$G(t,s) = \sum_{i=0}^{\infty} \frac{u_i(t)u_i(s)}{\lambda - \lambda_i},$$

where the convergence is in the L^2 norm.

7. Replace the boundary conditions (2) by the periodic boundary conditions $x(a) = x(b)$, $x'(a) = x'(b)$. Prove that the existence and completeness part of the above theory may be established provided the functions satisfy $p(a) = p(b)$, $q(a) = q(b)$, $r(a) = r(b)$.

8. Apply the previous exercise to the problem

$$\begin{aligned} &x'' + \lambda x = 0, \\ &x(0) = x(2\pi) \\ &x'(0) = x'(2\pi). \end{aligned}$$

9. Let the differential operator L be given by $L(x) = (tx')' + \frac{m^2}{t}x$, $0 < t < 1$, and consider the eigenvalue problem $L(x) = -\lambda t x$. In this case the hypotheses imposed earlier are not applicable and other types of boundary conditions than those given by (3) must be sought in order that a development parallel to that given in Section 2 may be made. Establish such a theory for this *singular* problem. Extend this to more general singular problems.