Chapter VI: Linear Equations

Let I be a real interval. Let $A : I \to \mathbf{L}(\mathbf{R}^N, \mathbf{R}^N)$ and $f : I \to \mathbf{R}^N$ be continuous functions. Consider the linear systems $u' = A(t)u + f(t)$, $t \in I$, and $u' = A(t)u$, $t \in I$.

Proposition 1. The initial value problem $u' = A(t)u + f(t)$, $u(t_0) = u_0$ is uniquely solvable for each $t_0 \in I$, $u_0 \in R^N$ and the solution $u(t)$ is defined on all of I.

If A and f are measurable on I and locally integrable there, then a parallel theory can be developed.

Proposition 2. The set of solutions of $u' = A(t)u$ is a vector space of dimension N.

Fundamental Solutions

Lemma 4 (Abel–Liouville). Let $\Phi(t)$ be an $N \times N$ matrix solution of $u' = A(t)u$. Then $g(t) = \det \Phi(t)$ satisfies the differential equation $g' = \text{trace}(A(t))g$. In particular, $\Phi(t)$ is nonsingular for all $t \in I$ if and only if $\Phi(t_0)$ is nonsingular for one $t_0 \in I$.

Definition. A nonsingular matrix $\Phi(t)$ whose columns are solutions of $u' = A(t)u$ is called a **fundamental matrix solution** or a **fundamental system**.

Proposition 5. Let Φ be a given fundamental matrix solution of $u' = A(t)u$. Then every other fundamental matrix solution Ψ has the form $\Psi = \Phi C$, where C is a constant nonsingular $N \times N$ matrix. Furthermore the set of all solutions of $u' = A(t)u$ is given by $\{\Phi c : c \in \mathbf{R}^N\}$, where Φ is a fundamental system.

Variation of Constants Formula

Proposition 6. Let Φ be a fundamental matrix solution of $u' = A(t)u$ and let $t_0 \in I$. Then $u_p(t) = \Phi(t) \int_{t_0}^t \Phi^{-1}(s) f(s) ds$ is a solution of $u' = A(t)u + f(t)$. Hence the set of all solutions of $u' = A(t)u + f(t)$ is given by

$$\left\{ \Phi(t) \left(c + \int_{t_0}^t \Phi^{-1}(s) f(s) ds \right) : c \in \mathbf{R}^N \right\},$$

where Φ is a fundamental system of $u' = A(t)u$.

Exponential Matrix

$$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}.$$

This series expansion is valid for constant matrices A. It converges uniformly on compact t-sets. The series represents a fundamental matrix for the equation $u' = Au$ which is the identity matrix at $t = 0$.

Real Jordan Form

The matrix formula $J = P^{-1}AP$ summarizes the real Jordan form of A. In this form, P is formed from the real and imaginary parts of the generalized eigenvectors of A, while $J = \text{diag}(J_1, \ldots, J_k)$; the matrices J_1, ..., J_k are called **Jordan blocks**. The structure of a Jordan block is as follows: the diagonal entries are either a real eigenvalue λ of A or else the 2×2 matrix $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$, which corresponds to the complex eigenvalue $\alpha + i\beta$. On the super-diagonal of the Jordan block there are ones (1) or 2×2 identity matrices.

Calculation of e^{AT}

If matrices E and N commute, then $e^{E+N} = e^E e^N$. A Jordan block C can be written as a sum $C = E + N$ where E is block-diagonal, N is **nilpotent** ($N^r = 0$ for some $r \geq 1$) and $EN = NE$. Therefore, $e^{Ct} = e^{ET} e^{Nt}$. The exponential e^{Et} is again block-diagonal, while the series e^{Nt} is a *finite sum*. There are two cases, corresponding to real or complex eigenvalues of A.

Proposition 7. Let A be an $N \times N$ constant matrix and consider the differential equation $u' = Au$. Then:

1. All solutions u of $u' = Au$ satisfy $u(t) \to 0$, as $t \to \infty$, if and only if Re$\lambda < 0$, for all eigenvalues λ of A.
2. All solutions u of $u' = Au$ are bounded on $[0, \infty)$, if and only if Re$\lambda \leq 0$, for all eigenvalues λ of A and those with zero real part are semisimple.

Floquet Theory

Let $A(t)$ be an $N \times N$ continuous matrix such that $A(t+T) = A(t)$, $-\infty < t < \infty$. Consider the differential equation $u' = A(t)u$.

Proposition 8. Let $\Phi(t)$ be a fundamental matrix solution of $u' = A(t)u$ with $A(t)$ T-periodic and continuous. Then $\Psi(t) = \Phi(t+T)$ is also a fundamental matrix.

Theorem 9 (Floquet). Let the $N \times N$ matrix $A(t)$ be T-periodic and continuous. There exists a constant matrix R and a T-periodic nonsingular matrix $C(t)$ such that the change of variable $u = C(t)y$ changes $u' = A(t)u$ into the constant-coefficient equation $y' = Ry$.

In particular, there is a fundamental matrix $\Phi(t)$ for $u' = A(t)u$ of the form $\Phi(t) = C(t)e^{Rt}$, where R is a constant matrix and $C(t)$ is a T-periodic nonsingular matrix of class C^1. If $\Phi(0) = I$, then $C(0) = C(T) = I$ and $e^{RT} = \Phi(T)$.

Corollary 10. Let the $N \times N$ matrix $A(t)$ be T-periodic and continuous. Let $\Phi(t)$ be a fundamental matrix for $u' = A(t)u$. Then, there exists a solution $u(t) \neq 0$ of period mT if and only if $\Phi^{-1}(0)\Phi(T)$ has an eigenvalue λ with $\lambda^m = 1$.

Solving $Q = e^X$ for X when $\det(Q) \neq 0$
The condition $\det(Q) \neq 0$ means that all eigenvalues of Q are nonzero. Write $Q = P^{-1}JP$ where J is a block-diagonal matrix whose diagonal entries are complex Jordan blocks.

It suffices to solve for X in $\lambda I + N = e^X$ when $\lambda \neq 0$, I is the identity and N is nilpotent, because this is the form of a complex Jordan block.

A candidate for the solution X in this special case is given by a formal logarithmic series $X = \lambda \ln(1 + N/\lambda) = \ln(\lambda)I - \sum_{k=1}^{p}(-N/\lambda)^k/k$ where $N^p = 0$. To verify that this solution X indeed satisfies $\lambda I + N = e^X$ is routine, because there is no issue of convergence.

Hill's equation $y'' + p(t)y = 0$

Write $y'' + p(t)y = 0$ as a system $u' = A(t)u$ where

$$A = \begin{bmatrix} 0 & 1 \\ -p(t) & 0 \end{bmatrix}, \quad u = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

Let $\Phi(t)$ be a fundamental matrix with $\Phi(0) = I$. Let $[y_1, y_2]$ be the first row of Φ and define $a = y_1(T) + y_2'(T)$. Corollary 10 says that Hill's equation has a periodic solution of period mT if and only if $\Phi(T)$ has an eigenvalue λ with $\lambda^m = 1$. An eigenvalue λ must satisfy $\lambda^2 - a\lambda + 1 = 0$, therefore the condition for an mT-periodic solution is

$$\left(\frac{a \pm \sqrt{a^2 - 4}}{2} \right)^m = 1.$$