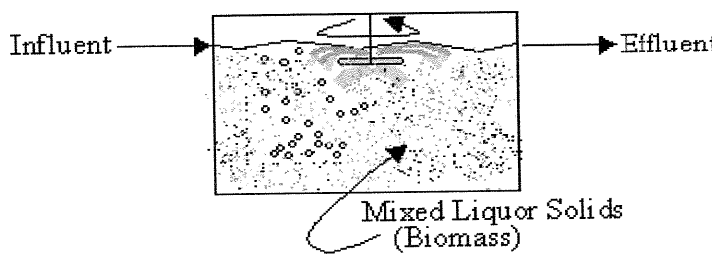
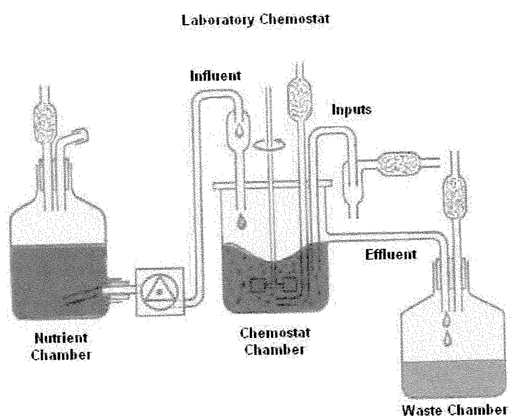
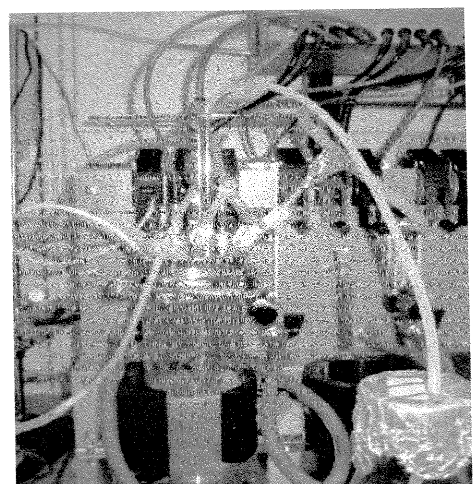
CHEMOSTATS

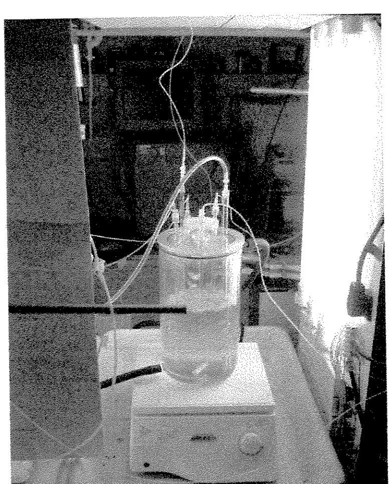
A chemostat is a laboratory apparatus used to grow microorganisms under controlled experimental conditions. Generally, a constant supply of nutrients is pumped into a vessel containing the microorganism and the same amount of fluid containing extra nutrients and byproducts flows out, keeping the volume in the chemostat constant.

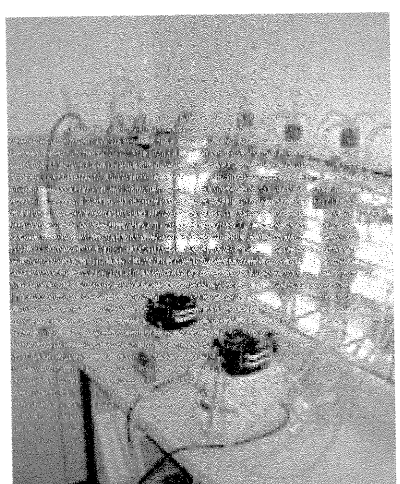
Basic Chemostat

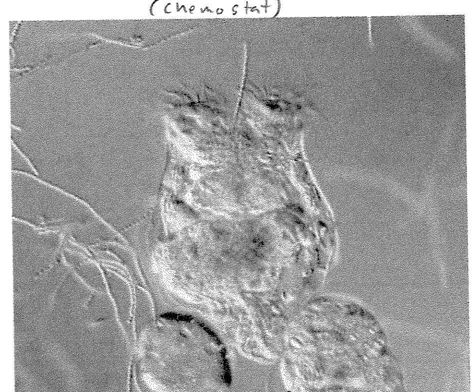


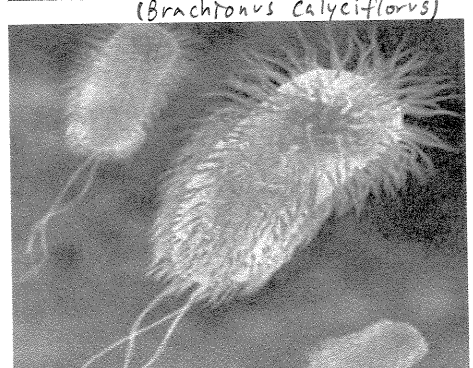


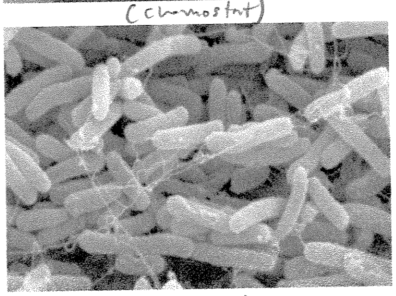

(chemostat)


(chemostat)

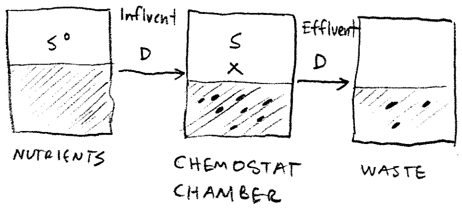

(chemostat)


(Brachronus Calyciflorus)


(E. coli)


(E. coli)

CHEMOSTATS   pg 128 - 131 ROBINSON



D = flow rate
S = nutrient concentration in the chamber
x = microorganism concentration

$r(s)$ = per capita growth rate $\Rightarrow r(s) = \dfrac{mS}{a+S}$ so $r(0)=0$, $\lim\limits_{S\to\infty} r(s) \to m$, $\dfrac{dr(s)}{dS} > 0$

m = maximal per capita growth rate / saturation level, $m>0$

a = half saturation constant (since $r(a) = m/2$), $a>0$

A SIMPLE CHEMOSTAT: ONE MICROORGANISM

NUTRIENT CONCENTRATION CHANGE

RATE OF CHANGE = INPUT RATE − OUTPUT RATE − CONSUMPTION RATE

$$S' = S^{(0)}D - SD - \beta x\left(\frac{mS}{a+S}\right)$$, β = amount of nutrient consumed per microorganism

or

$$S' = \left(S^{(0)} - S\right)D - \beta x\left(\frac{mS}{a+S}\right)$$

POPULATION CHANGE

RATE OF CHANGE = GROWTH RATE − OUTPUT RATE

$$x' = \left(\frac{mS}{a+S}\right)x - Dx$$

SYSTEM OF EQNS

$$\begin{cases} S' = \left(S - S^{(0)}\right)D - \beta x\left(\dfrac{mS}{a+S}\right) \\ x' = \left(\dfrac{mS}{a+S}\right)x - Dx \end{cases}$$

NULL CLINES

$$x = \frac{D(S^{(0)}-S)(a+S)}{\beta m S}$$

$$S = \frac{aD}{m-D}$$

TWO MICROORGANISMS COMPETING FOR ONE NUTRIENT (assume no direct interaction)

NUTRIENT CONCENTRATION CHANGE

RATE OF CHANGE = INPUT RATE − OUTPUT RATE − CONSUMPTION RATE OF x − CONSUMPTION RATE OF y

x and y are microorganism concentrations of the two different microorganisms.

$$S' = S^{(0)}D - SD - \frac{\beta_1 m_1 S x}{a_1+S} - \frac{\beta_2 m_2 S y}{a_2+S}$$

POPULATION CHANGE

RATE OF CHANGE = GROWTH RATE − OUTPUT RATE

$$x' = \frac{m_1 S x}{a_1+S} - Dx$$

$$y' = \frac{m_2 S y}{a_2+S} - Dy$$

WE ARE ONLY INTERESTED IN $S \geq 0$, $x \geq 0$, $y \geq 0$. RATHER THAN HAVING A SYSTEM OF THESE THREE EQUATIONS, ROBINSON (pg 130) REDUCES THE SYSTEM TO TWO EQUTIONS:

REDUCE 3 TO 2

DEFINE $z = S^{(0)} - S - \beta_1 x - \beta_2 y$

$$z' = S^{(0)\prime} - S' - \beta_1 x' - \beta_2 y'$$

$$= \left(-(S^{(0)} - S)D + \frac{\beta_1 m_1 S x}{a_1 + S} + \frac{\beta_2 m_2 S y}{a_2 + S} \right) - \beta_1 \left(\frac{m_1 S x}{a_1 + S} - Dx \right) - \beta_2 \left(\frac{m_2 S y}{a_2 + S} - Dy \right)$$

$$= -(S^{(0)} - S)D + \beta_1 D x + \beta_2 D y$$

$$= -D\left(S^{(0)} - S - \beta_1 x - \beta_2 y\right)$$

$$= -Dz$$

so

$$z' = Dz, \quad z(t) = z_0 e^{-Dt}$$

THIS TERM, HOWEVER, APPROACHES ZERO AS $t \to \infty$. POPULATION BIOLOGISTS, LOOKING AT LONG TERM BEHAVIOR, TAKE $z = 0$.

$$z = S^{(0)} - S - \beta_1 x - \beta_2 y$$

$$\Rightarrow \quad 0 = S^{(0)} - S - \beta_1 x - \beta_2 y$$

$$\Leftrightarrow \quad S = S^{(0)} - \beta_1 x - \beta_2 y$$

So 2 equations result:

$$\begin{cases} x' = \dfrac{m_1 (S^{(0)} - \beta_1 x - \beta_2 y)}{a_1 + (S^{(0)} - \beta_1 x - \beta_2 y)} x - Dx \\[2ex] y' = \dfrac{m_2 (S^{(0)} - \beta_1 x - \beta_2 y)}{a_2 + (S^{(0)} - \beta_1 x - \beta_2 y)} y - Dy \end{cases}$$

SIMPLE CHEMOSTAT $S^{(0)} = 1$ (SOME NUTRIENT FLOW)

```
> restart:
  with(plots):
  with(DEtools):

  eqnx := diff(x(t),t)  = m*S(t)*x(t)/(a+S(t))-1*d*x(t):
  eqnS := diff(S(t),t)  = (S0-S(t))*d-B*m*S(t)*x(t)/(a+S(t)):

  x_null := S(t) = solve(rhs(eqnx)=0,S(t));
  S_null := x(t) = factor(solve(rhs(eqnS)=0,x(t)));

  S0 := 1; m := 1; d := .50; a := .50; B := .10;
  x_null := S(t) = solve(rhs(eqnx)=0,S(t));
  S_null := x(t) = factor(solve(rhs(eqnS)=0,x(t)));

  DEplot([eqnS,eqnx],[S(t),x(t)], t=0..40, stepsize = .05,
        S=-1..10, x=-1..10,[[x(0)=.5,S(0)=.5],[x(0)=.5,S(0)=8]],
        dirgrid = [30,20], arrows=small,thickness=1,
  linecolor=black);

  DEplot([eqnS,eqnx],[S(t),x(t)], t=0..40, stepsize = .05,
        S=0..1, x=-1..6,[[x(0)=.5,S(0)=.5],[x(0)=.5,S(0)=8]],
        dirgrid = [30,20], arrows=small,thickness=1,
  linecolor=black);
Warning, the name changecoords has been redefined
```

$$x\_null := \mathrm{S}(t) = -\frac{d\,a}{-m+d}$$

$$S\_null := \mathrm{x}(t) = \frac{(S0 - \mathrm{S}(t))\,d\,(a + \mathrm{S}(t))}{B\,m\,\mathrm{S}(t)}$$

$$S0 := 1$$

$$m := 1$$

$$d := 0.50$$

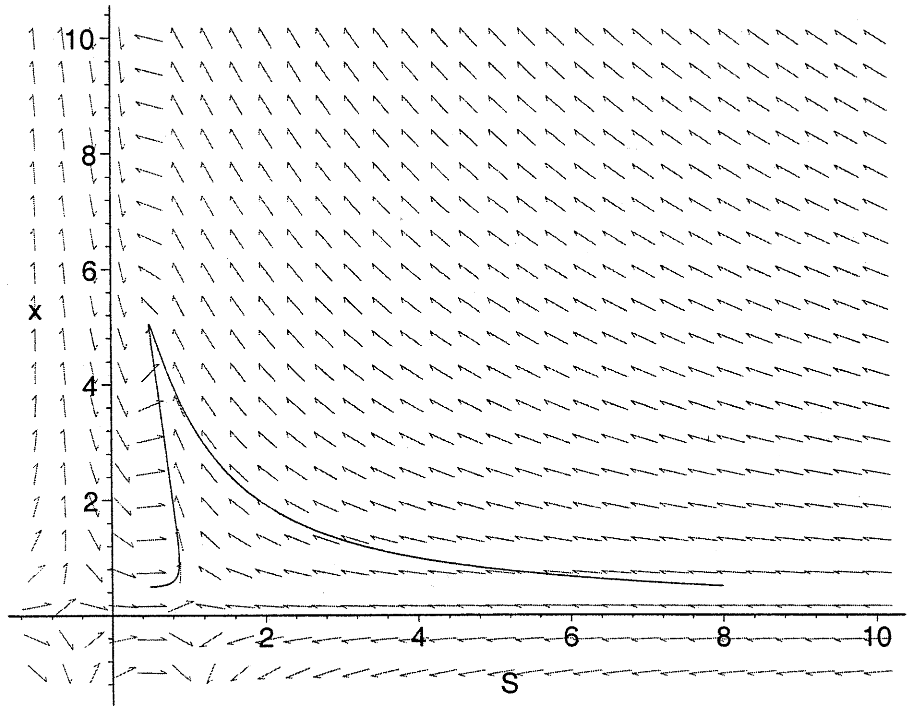
$$a := 0.50$$

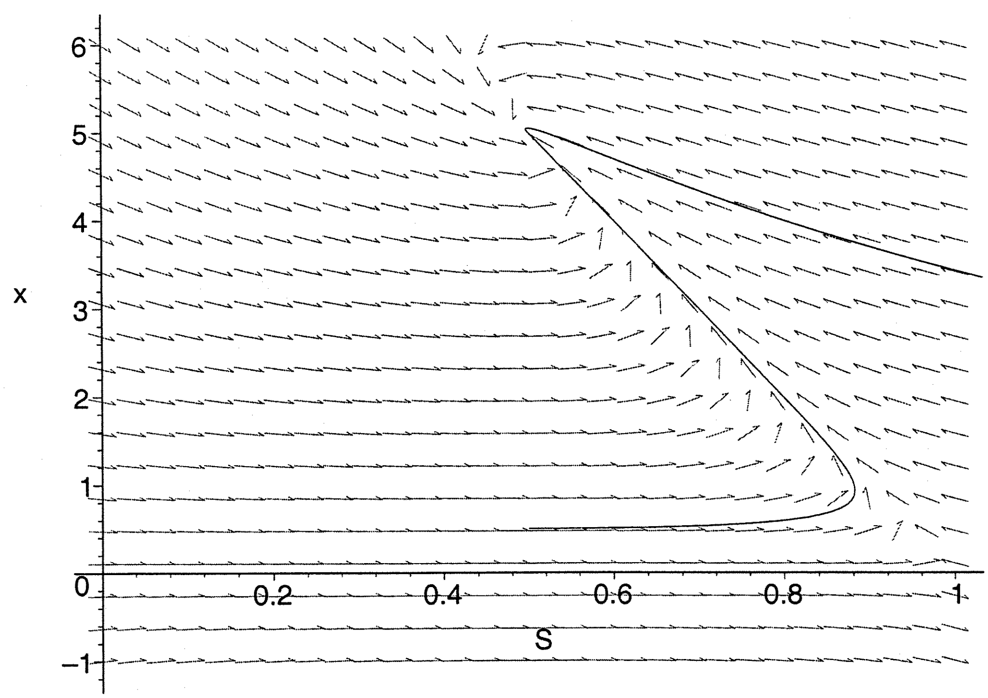
$$B := 0.10$$

$$x\_null := \mathrm{S}(t) = 0.5000000000$$

$$S\_null := \mathrm{x}(t) = -\frac{5.000000000\,(\mathrm{S}(t) + 0.5000000000)\,(\mathrm{S}(t) - 1.000000000)}{\mathrm{S}(t)}$$

SIMPLE CHEMOSTAT WITH $S^{(0)} = 1$ (SOME NUTRIENT FLOW)





SIMPLE CHEMOSTAT WITH $S^{(0)} = 0$ (NO NUTRIENT FLOW

```
> restart:
  with(plots):
  with(DEtools):

  eqnx := diff(x(t),t) = m*S(t)*x(t)/(a+S(t))-1*d*x(t):
  eqnS := diff(S(t),t) = (S0-S(t))*d-B*m*S(t)*x(t)/(a+S(t)):

  x_null := S(t) = solve(rhs(eqnx)=0,S(t));
  S_null := x(t) = factor(solve(rhs(eqnS)=0,x(t)));

  S0 := 0; m := 1; d := .50; a := .50; B := .10;
  x_null := S(t) = solve(rhs(eqnx)=0,S(t));
  S_null := x(t) = factor(solve(rhs(eqnS)=0,x(t)));

  DEplot([eqnS,eqnx],[S(t),x(t)], t=0..10, stepsize = .05,
        S=-1..10, x=-1..10, [[x(0)=.5,S(0)=10],[x(0)=1,S(0)=1]],
        dirgrid = [30,20], arrows=small, thickness=1,
        linecolor=black);

  DEplot([eqnS,eqnx],[S(t),x(t)], t=0..10, stepsize = .05,
        S=-1..3, x=-1..3,[[x(0)=.5,S(0)=10],[x(0)=1,S(0)=1]],
        dirgrid = [30,20], arrows=small,thickness=1,
  linecolor=black);
```

Warning, the name changecoords has been redefined

$$x\_null := S(t) = -\frac{d\,a}{-m+d}$$

$$S\_null := x(t) = \frac{(S0 - S(t))\,d\,(a+S(t))}{B\,m\,S(t)}$$

$$S0 := 0$$

$$m := 1$$

$$d := 0.50$$

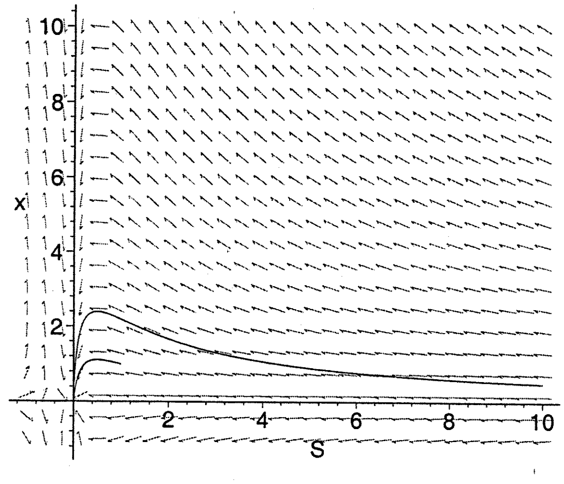
$$a := 0.50$$

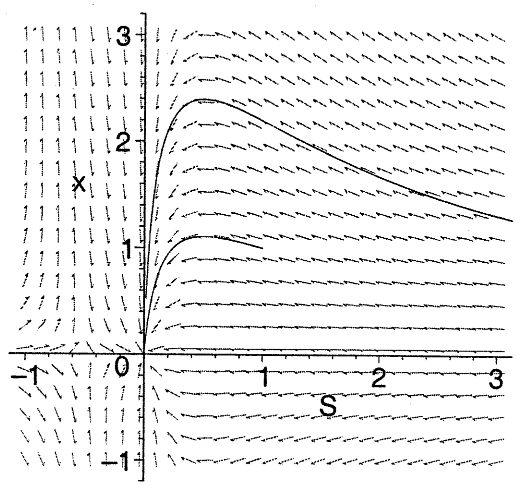
$$B := 0.10$$

$$x\_null := S(t) = 0.5000000000$$

$$S\_null := x(t) = -2.500000000 - 5.\,S(t)$$

SIMPLE CHEMOSTAT WITH $S(0) = 0$ (NO NUTRIENT FLOW)





[>

Al Tanner

CHEMOSTAT WITH 2 ORGANISMS

```
> restart:
  with(plots):
  with(DEtools):

  eqnx := diff(x(t),t) = (m1*(S0-B1*x(t)-B2*y(t))/
                          (a1+S0-B1*x(t)-B2*y(t))-d)*x(t):
  eqny := diff(y(t),t) = (m2*(S0-B1*x(t)-B2*y(t))/
                          (a2+S0-B1*x(t)-B2*y(t))-d)*y(t):

  x_null := y(t) = factor(solve(rhs(eqnx)=0,y(t)));
  y_null := x(t) = factor(solve(rhs(eqny)=0,x(t)));

  S0 := 1; m1 := 1; m2 :=1.2; d := .50; a1 := .25; a2 :=.50;
  B1 := .10; B2 := .10;

  x_null := y(t) = solve(rhs(eqnx)=0,y(t));
  y_null := x(t) = solve(rhs(eqny)=0,x(t));

  deplots := DEplot([eqnx,eqny],[x(t),y(t)], t=0..25,
             stepsize = .01, x=-1..10, y=-1..10, [[x(0)=.1, y(0)=1],
             [x(0)=.1,y(0)=2],[x(0)=.1,y(0)=3],[x(0)=.1,y(0)=4],
             [x(0)=10,y(0)=2],[x(0)=2, y(0)=10],[x(0)=4,y(0)=8],
             [x(0)=6,y(0)=6],[x(0)=8,y(0)=4]],dirgrid = [50,30],
             arrows=small,thickness=1,linecolor=black):


  display([plot(7.5-x, x=0..8),plot(6.428571429-y,
  y=0..8),deplots]);
```

Warning, the name changecoords has been redefined

$$x\_null := \mathrm{y}(t) = \frac{m1\ S0 - m1\ B1\ \mathrm{x}(t) - d\ a1 - d\ S0 + d\ B1\ \mathrm{x}(t)}{B2\ (m1 - d)}$$

$$y\_null := \mathrm{x}(t) = -\frac{-m2\ S0 + m2\ B2\ \mathrm{y}(t) + d\ a2 + d\ S0 - d\ B2\ \mathrm{y}(t)}{B1\ (m2 - d)}$$

$$S0 := 1$$

$$m1 := 1$$

$$m2 := 1.2$$

$$d := 0.50$$

$$a1 := 0.25$$

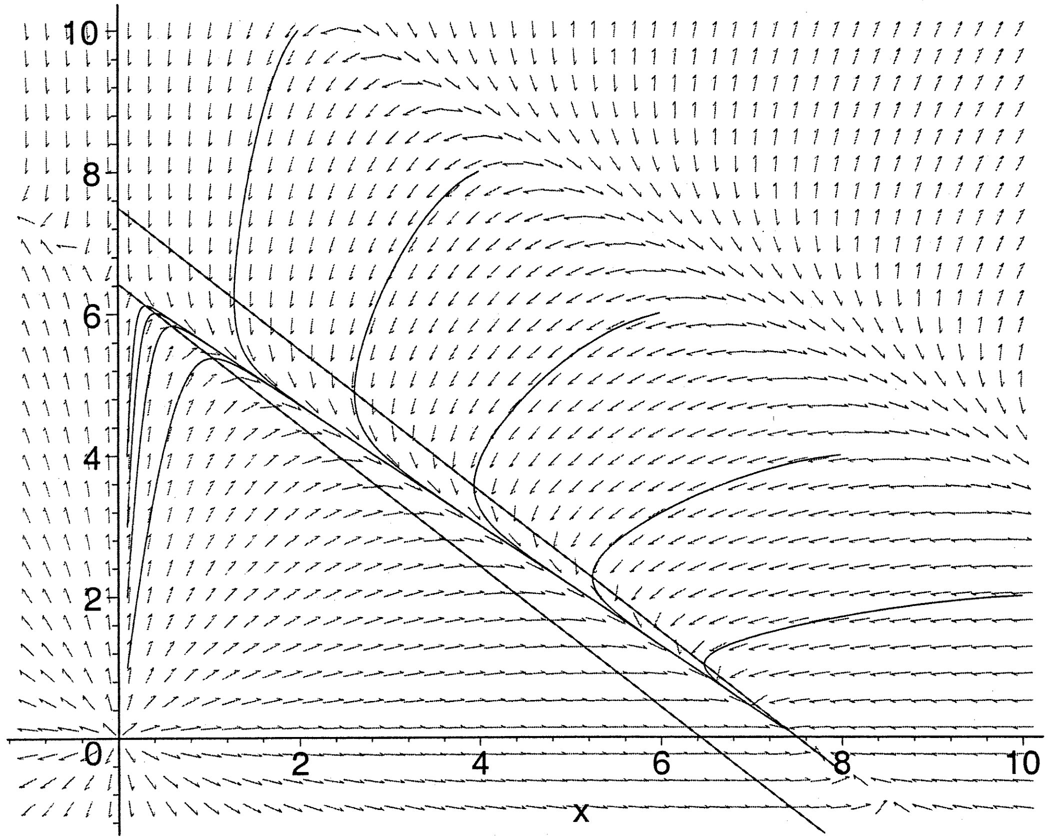
$$a2 := 0.50$$

$$B1 := 0.10$$

$$B2 := 0.10$$

$$x\_null := \mathrm{y}(t) = 7.500000000 - 1.\ \mathrm{x}(t)$$

$$y\_null := \mathrm{x}(t) = 6.428571429 - 1.\ \mathrm{y}(t)$$


10
8
6
4
2
0
2
4
6
8
10
x

>