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Theorem (3.3.1) Pg 85

Assume that $F(x)$ has a continuous partial derivatives in R^n, where $n = 2$, which contains a closed ball.

$$\overline{B}(x_0; r) = \{x : \|x - x_0\| \leq r\}.$$

i.e. A closed ball in R^2 is a circle.

Then, there is a constant L such that:

$$(1) \quad \|F(x) - F(x')\| \leq L\|x - x'\|$$

for any two points x and x' in $\overline{B}(x_0; r)$.

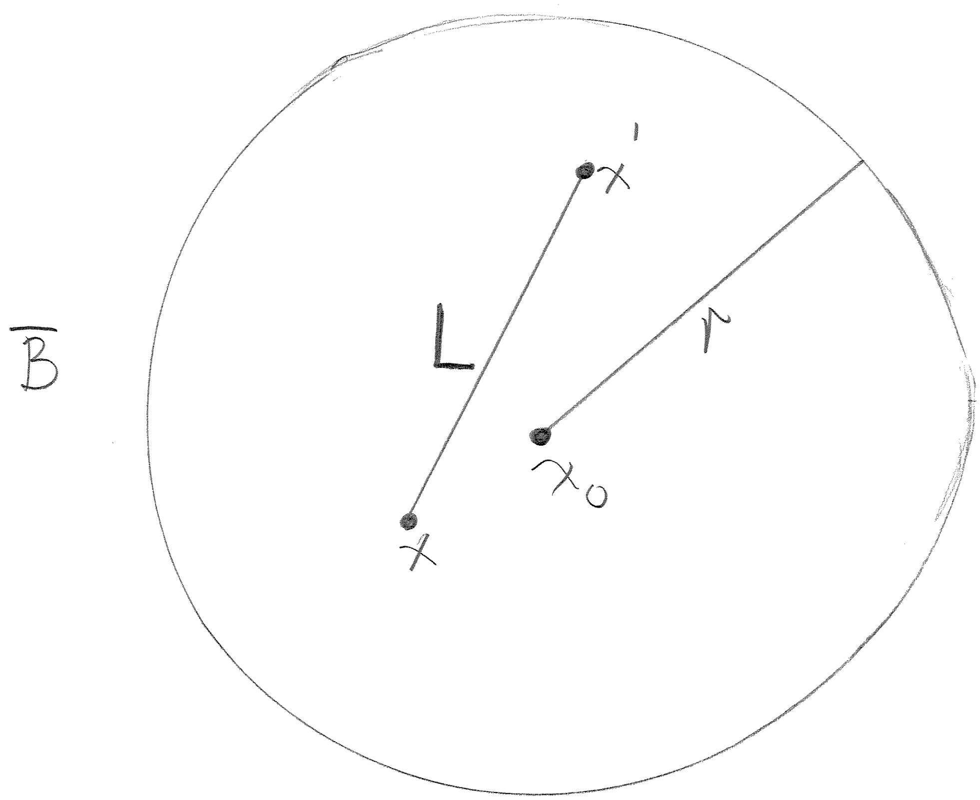
i.e. L is called the Lipschitz condition

Proof: Reducing (1) to th Mean Value Theorem for a function of one variable.

Parameterize the line segment from the points x to x'.

(2) $(1-t)x + tx' = x + tv$,

where $v = x' - x$ $\quad 0 \leq t \leq 1$.



Define the function $g(t)$ to be the values of F along (2):

$$g(t) = F((1-t)x + tx')$$
$$(3) \qquad = F(x+tv)$$

$$g'(t) = \lim_{h \to 0} \frac{g(t+h) - g(t)}{h}$$
$$= Dg(x+tv)\,v.$$

Solve by using the Chain Rule.

$g(0) = F(x)$ and $g(1) = F(x')$

We get the following:

$$F(x') - F(x) = g(1) - g(0)$$
$$(*) \qquad = \int_0^1 g'(t)\,dt$$
$$(4) \qquad = \int_0^1 Dg(x+tv)\,v\,dt$$

(*) Fundamental Theorem of Calculus.

Since the partial derivatives are continuous and F is continuous on compact set K then $F(x) = \max F(a)$ for some $x \in K$.

Take the norm of partial Derivatives of 4:

$$\|Dg(x+tv)v\| \leq \|Dg(x+tv)\| \cdot \|v\|$$

$$\|F(x') - F(x)\| \leq \int_0^1 \|Dg(x+tv)v\| dt$$

$$\leq \int_0^1 L\|v\| dt$$

$$\leq L\|v\|$$

$$= L\|x' - x\|. \quad \blacksquare$$