Notes on Robinson's Proof of the Abel-Liouville Theorem

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Robinson's Proof can be found on page 58. The first step is to write the matrix $\mathbf{M}(t)\mathbf{M}^{-1}(t_0)$ as the augmentation of its column vectors. Robinson's notation is

$$(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n) = aug(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$$

The column vectors are obtained by multiplying by the unit vectors, $\hat{u}^j$, as usual.

The next step is to take the derivative of the determinant using

$$\frac{d}{dt}det(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n) = \sum_{j=1}^{n} det(\vec{v}_1, ..., \vec{v}_{j-1}, \frac{d\vec{v}_j}{dt}, \vec{v}_{j+1}, ..., \vec{v}_n)$$

Basically, you can take the derivative inside the determinant if you only differentiate a column at a time and then sum. The proof of this identity is based on the multilinearity of the determinant in its columns, but it is too big for inclusion here. This step is completed by evaluating the derivative at t_0, which was arbitrarily chosen in the beginning. Note that

$$\frac{d}{dt}(\mathbf{M}(t)\mathbf{M}^{-1}(t_0)\hat{u}^j) = [\frac{d}{dt}\mathbf{M}(t)]\mathbf{M}^{-1}(t_0)\hat{u}^j$$

since matrix multiplication is a linear operation.

At this point there are a lot of terms of the form $\mathbf{M}(t_0)\mathbf{M}^{-1}(t_0)\hat{u}^j$ and we can replace them by $\hat{u}^j$ since $\mathbf{M}(t_0)\mathbf{M}^{-1}(t_0) = \mathbf{I}$. In the next step we again make this substitution.

The next step uses $\mathbf{A}(t_0)\hat{u}^j = \sum_{i=1}^{n} a_{i,j}(t_0)\hat{u}^i$ where $a_{i,j}(t_0)$ is the element of $\mathbf{A}(t_0)$ in the i^{th} row and the j^{th} column. It is evident by inspection that the right hand side of this last identity is equal to the j^{th} column of $\mathbf{A}(t_0)$.

For the second-to-last step note each determinant in the sum is of a matrix equal to the identity matrix with its j^{th} column replaced by the

j^{th} column of $\mathbf{A}(t_0)$. There is only one possible nonzero pattern and it consists of $a_{j,j}(t_0)$ and all the $1's$ from off the j^{th} column. Therefore each determinant equals $a_{j,j}(t_0)$. The last equality is just the definition of trace.

The conclusion of the argument thus far is that

$$\frac{d}{dt} det(\mathbf{M}(t)\mathbf{M}^{-1}(t_0))|_{t=t_0} = tr(\mathbf{A}(t_0)) \qquad \forall t_0$$

and this implies

$$\frac{d}{dt} det(\mathbf{M}(t)) = tr(\mathbf{A}(t))\, det(\mathbf{M}(t))$$

when we apply the properties of multiplication of determinants and determinants of inverses.

The Wronskian is the determinant of the solution matrix so we have shown that the Wronskian satisfies the required differential equation.