Al Tanner
5420

Solve $\vec{x}' = A\vec{x}$ (1) where $(\lambda, \vec{v})$ is the only eigenpair of A. (pg 35-36 Robinson)

$\boxed{\vec{x}_1 = e^{\lambda t}\vec{V} \text{ (definition of eigenpair)}}$ For X_2, the following method is used.

$e^{At}\vec{w}$ is a solution for any vector $\vec{w}$ (pg 17, Robinson)

choose a $\vec{w}$ such that $(A - \lambda I)\vec{w} = \vec{v}$, $(\lambda, \vec{v})$ the only eigenpair of A

We know $A\vec{v} = \lambda\vec{v}$ (definition of eigenpair)

$$\Rightarrow A\vec{v} - \lambda\vec{v} = 0$$

$$\Rightarrow (A - \lambda I)\vec{v} = 0$$, now substitute $\vec{v} = (A - \lambda I)\vec{w}$

$$\Rightarrow (A - \lambda I)(A - \lambda I)\vec{w} = 0$$

$$\Rightarrow (A - \lambda I)^2\vec{w} = 0$$

and

$$(A - \lambda I)^n\vec{w} = (A - \lambda I)^{n-2}(A - \lambda I)^2\vec{w}$$

$$= (A - \lambda I)^{n-2} \cdot 0$$

$$= 0$$

so $(A - \lambda I)^n\vec{w} = 0$ for $n \geq 2$

If two matrices B and C commute ($BC = CB$), then

$e^{(B+C)t} = e^{Bt}e^{Ct}$ (pg 56-57, Robinson)

In our case, $(A - \lambda I)(\lambda I) = (\lambda I)(A - \lambda I)$ since λI is a scalar multiple of the identity.

$$e^{At}\vec{w} = e^{(A - \lambda I + \lambda I)t}\vec{w}$$

$$= e^{[(A - \lambda I) + (\lambda I)]t}\vec{w}$$

$$= e^{(A - \lambda I)t}e^{\lambda I t}\vec{w}$$

$$= e^{\lambda t} I\, e^{(A - \lambda I)}\vec{w}$$

$$= e^{\lambda t} I \left(\vec{w} + t(A - \lambda I)\vec{w} + \frac{t^2}{2!}(A - \lambda I)^2\vec{w} + \cdots\right)$$

but $(A - \lambda I)^n\vec{w} = 0$ for $n \geq 2$, so

$$= e^{\lambda t} I (\vec{w} + t(A - \lambda I)\vec{w}) \Rightarrow$$

$$\boxed{X_2 = e^{\lambda t}(\vec{w} + t\vec{v})}$$

Al Tanner
5420

Solve $(A - \lambda I)\vec{w} = \vec{v}$ where $(\lambda, \vec{v})$ is the only eigenpair of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$(A - \lambda I)\vec{w} = \vec{v}$$

$$\Rightarrow \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Solve for w_1, w_2 by Gaussian Elimination

create augmented matrix:

$$\left[\begin{array}{cc|c} a-\lambda & b & v_1 \\ c & d-\lambda & v_2 \end{array}\right]$$

solve to rref to find w_1, w_2

$$\left[\begin{array}{cc|c} a-\lambda & b & v_1 \\ c & d-\lambda & v_2 \end{array}\right] \div a-\lambda \text{ (normalize)} \qquad (a-\lambda) \neq 0$$

$$= \left[\begin{array}{cc|c} 1 & \frac{b}{(a-\lambda)} & \frac{v_1}{(a-\lambda)} \\ c & d-\lambda & v_2 \end{array}\right] \mathrm{II} - c\,\mathrm{I}$$

$$= \left[\begin{array}{cc|c} 1 & \frac{b}{(a-\lambda)} & \frac{v_1}{(a-\lambda)} \\ 0 & (d-\lambda) - \frac{cb}{(a-\lambda)} & v_2 - \frac{cv_1}{(a-\lambda)} \end{array}\right] \div \left[(d-\lambda) - \frac{cb}{(a-\lambda)}\right] \text{ (normalize) } * \qquad \left[(d-\lambda) - \frac{cb}{(a-\lambda)}\right] \neq 0$$

$$= \left[\begin{array}{cc|c} 1 & \frac{b}{(a-\lambda)} & \frac{v_1}{(a-\lambda)} \\ 0 & 1 & \frac{v_2(a-\lambda) - cv_1}{(a-\lambda)(d-\lambda) - cb} \end{array}\right] \mathrm{I} - \frac{b}{(a-\lambda)}\mathrm{II}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & \frac{v_1}{(a-\lambda)} - \frac{b}{(a-\lambda)} \cdot \frac{v_2(a-\lambda) - cv_1}{(a-\lambda)(d-\lambda) - cb} \\ 0 & 1 & \frac{v_2(a-\lambda) - cv_1}{(a-\lambda)(d-\lambda) - cb} \end{array}\right]$$

which gives

$$w_1 = \frac{v_1}{(a-\lambda)} - \frac{b}{(a-\lambda)} \cdot \frac{v_2(a-\lambda) - cv_1}{(a-\lambda)(d-\lambda) - cb}$$

$$w_2 = \frac{v_2(a-\lambda) - cv_1}{(a-\lambda)(d-\lambda) - cb}$$

$*$ details:

$$v_2 - \frac{cv_1}{a-\lambda} \div (d-\lambda) - \frac{cb}{(a-\lambda)}$$

$$= \frac{v_2(a-\lambda) - cv_1}{(a-\lambda)} \div \frac{(d-\lambda)(a-\lambda) - cb}{(a-\lambda)}$$

$$= \frac{v_2(a-\lambda) - cv_1}{\cancel{(a-\lambda)}} \cdot \frac{\cancel{(a-\lambda)}}{(d-\lambda)(a-\lambda) - cb}$$

$$= \frac{v_2(a-\lambda) - cv_1}{(a-\lambda)(d-\lambda) - cb}$$