Name. ______________________

Differential Equations 5410-1
Midterm Exam 2, Due classtime 21-Oct-2002

Scores

__________ **Problem 1**. Cross bow.
__________ **Problem 2**. Periodic harvesting.
__________ **Problem 3**. Gronwall's Lemma.
__________ **Problem 4**. Variation of Parameters.
__________ **Average**.

Instructions. The four take-home problems constitute the entire exam. Answer checks are expected. If `maple` assist is used, then please attach the `maple` output.

1. **(Cross bow)** The height $y(t)$ of a crossbow bolt shot straight upward satisfies $v'(t) = -(0.0017)v(t)|v(t)| - 9.8$, $v(0) = 47$, $y(0) = 0$, where $v = dy/dt$, in `mks` units. (a) Find formulas for $y(t)$ on ascent and fall. (b) Plot the solution over its flight time. (c) Compute decimal approximations for the maximum height, the ascent time, the fall time and the impact speed.

2. **(Periodic harvesting)** The population equation $y' = y(1-y) - \sin(6.67t)$ appears to have a steady-state periodic solution that oscillates about $y = 1$. (a) Apply ideas from the example below to make a computer graphic that supports this conclusion. (b) Discuss the biological meaning. (c) Estimate the amplitude and pseudo-period of the oscillation.

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with(DEtools):
de:=diff(y(t),t)=y(t)*(2-y(t))-5*cos(2*Pi*t):
ic:=[y(0)=1.7],[y(0)=2],[y(0)=2.4],[y(0)=2.8]:
DEplot(de,y(t),t=0..4,y=1..3,[ic],stepsize=0.05);
```

3. **(Gronwall's Lemma)** Let $u(t)$ and $v(t)$ be continuous for $t \geq t_0$ and $u(t) \geq 0$. Assume that there is a constant v_0 such that $v(t) \leq v_0 + \int_{t_0}^{t} u(t)v(t)\,dt$, $t \geq t_0$. Prove

$$v(t) \leq v_0\, e^{\int_{t_0}^{t} u(x)dx}, \quad t \geq t_0.$$

Suggestion: Let $y(t) = v_0 + \int_{t_0}^{t} u(x)v(x)dx$. Verify $y'(t) \leq u(t)y(t)$ and $y(t_0) = v_0$. Apply identity $Y' + pY \equiv (QY)'/Q$ where $Q = e^{\int p}$.

4. **(Variation of Parameters)** Derive the formula $u(t) = u_h(t) + u_p(t)$, where

$$u_h(t) = u_0 e^{-\int_{t_0}^{t} p(x)dx}, \quad u_p(t) = \int_{t_0}^{t} e^{-\int_{x}^{t} p(r)dr} f(x)dx,$$

for the problem $y' + p(t)y = f(t)$, $y(t_0) = u_0$. State all assumptions made to derive the formula.