Differential Equations 5410

Sample Midterm Exam 3

Tuesday, 7 December 2004

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected.

1. (Variation of parameters)

(a) [50%] State and prove the variation of parameters formula for a second order linear differential equation.

(b) [50%] Solve by variation of parameters $y'' - y = xe^x$.

(c) [50%] Solve by variation of parameters $\mathbf{u}' = A\mathbf{u} + \mathbf{F}(t)$, given $A = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$ and $\mathbf{F}(t) = \begin{pmatrix} t \\ 1+t \end{pmatrix}$. The answer contains two arbitrary constants c_1, c_2.

2. ()

(a) [50%] Find the linearized equation at each equilibrium point: $x' = xy - y$, $y' = x^2 - x^3 y$.

(b) [50%] Classify the equilibria as stable or unstable: $x' = 3x$, $y' = x(y-1)$, $z' = x + z$.

(c) [50%] Prove that a matrix equation $\mathbf{x}' = A\mathbf{x}$ is asymptotically stable at $\mathbf{x} = \mathbf{0}$, if the real part of each eigenvalue of A is negative.

(d) [50%] Classify as a center, spiral, saddle or node: $\mathbf{x}' = A\mathbf{x}$, $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, when $b > 0$, for all possible choices of a.

3. (Linear 3×3 systems)

(a) [50%] Solve $x' = x - 2y$, $y' = x + y$, $z' = z$ by Putzer's spectral recipe.

(b) [50%] Solve $x' = x - z$, $y' = y - x$, $z' = z + y$ by eigenanalysis.

(c) [50%] Solve $x' = x - z$, $y' = y - x$, $z' = z + y$ by a spectral formula.

4. (Resonance)

(a) [50%] Define pure resonance. Find a periodic solution in the non-resonant case for $x'' + 16x = \cos(\omega t)$.

(b) [50%] Define practical resonance. Find the unique periodic solution of $x'' + 2x' + x = \cos(\omega t)$.

(c) [50%] Prove that practical resonance occurs exactly when $\omega = \sqrt{k/m - c^2/(2m^2)}$ is positive, for the equation $mx'' + cx' + kx = F_0 \cos(\omega t)$.

5. (Theory of linear systems)

(a) [50%] State the existence-uniqueness theorem for $\mathbf{x}' = A(t)\mathbf{x} + \mathbf{F}(t)$. Include a statement about the domain of the unique solution $\mathbf{x}(t)$.

(b) [50%] State and prove the superposition principle for $\mathbf{x}' = A\mathbf{x} + \mathbf{f}(t)$.

(c) State and prove Abel's formula for $\mathbf{x}' = A\mathbf{x}$.